Nonlinear control with bounded signals

Georgia Kaliora* and Alessandro Astolfi
Department of Electrical and Electronic Engineering
Imperial College of Science, Technology and Medicine
London, SW7 2BT, United Kingdom
Tel: +44 20 75946289
Fax: +44 20 75946282
{g.kaliora,a.astolfi}@ic.ac.uk

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Abstract

The stabilization problem for a class of nonlinear cascaded systems is solved using bounded control. It is shown that when the lower subsystem of the cascade is *input to state stable* and the upper subsystem simply stable, global asymptotic stability can be achieved via a simple static feedback having bounded amplitude that requires knowledge of the "upper" part of the state only. This is made possible by invoking the bounded real lemma and a generalized version of the small gain theorem. Thus, stabilization is achieved with typical saturation functions, saturations of constant sign, or quantized control. Moreover, the problem of asymptotic stabilization of a stable linear system with bounded outputs is solved by means of dynamic feedback. Finally, a new class of stabilizing control laws for a chain of integrators with input saturation is proposed. Some robustness issues are also addressed and the theory is illustrated with an example on the stabilization of a mechanical system.

Keywords: Bounded control, Bounded outputs, Feedforward systems, $\mathcal{L}_2$-stability.

1 Introduction

In this paper we study the problem of asymptotic stabilization with bounded control of stable cascades described by equations of the form

$$\begin{array}{rcl} \dot{z} & = & Jz + h(\xi) + r(\xi)u \\ \dot{\xi} & = & f(\xi) + g(\xi)u \end{array} \tag{1}$$

and some related problems. See Section 2 for the precise formulation and the standing assumptions.

Nonlinear control with saturated signals is a problem that although well studied [7, 27, 28], still gathers a lot of interest [6, 14, 15, 18]. Limitations on available energy impose bounded actuator input while it is also very common that due to sensors limitations the outputs of the system are bounded.

System (1) belongs to the family of systems in feedforward form. This class of systems can be stabilized using the *forwarding approach* or one of its modifications, see *e.g.* [7, 24, 8, 17, 21]. On the other hand *nonlinear small gain theorem* based approaches have also

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been used for the stabilization of these systems [29, 27, 14]. Finally, system (1) can (under some special assumptions) also be studied from an *absolute stability* [31] point of view.

Forwarding is a systematic tool for the stabilization of general cascades, a special case of which is the form described by the equations (1). This methodology requires, in general, the solution of a partial differential equation and tends to generate complex control laws. In most of the cases these control laws require full state feedback and, though implicitly, they are chosen in a way that exploits some passivity properties [1]. On the other hand, although forwarding tackles successfully saturated inputs, it is not a low amplitude design, so it does not impose restrictions on the control amplitude.

Low amplitude designs are typically based on small gain considerations. They also require full state feedback, and whether linear [27, 13] or nonlinear [14, 6] systems are considered, the control laws consist of a generalization of the nested saturations scheme of [28] or linear combinations of saturations [27]. These designs also make use of passivity, in the sense that, at each step of the procedure, the feedback consists of functions of the state for which the system is always of relative degree one.

On the other hand, when linear versions of system (1) are considered, an analysis based on absolute stability can be easily implemented and can lead to simple control laws. This way of thinking also provides flexibility and robustness against some class of perturbations. This is made possible, because unlike with passivity based designs, no phase restrictions are imposed.

The results of this paper are motivated by the observation that under the assumption that the lower subsystem of (1) is *input to state stable*, an absolute stability point of view can be used in the design of stabilizing saturated controllers for linear as well as nonlinear systems. In a more precise formulation, the results stem from a general result on the $\mathcal{L}_2$ *stability of feedback interconnections* found in the recent paper [22] and from the *linear bounded real lemma* [5]. The proposed design requires partial state feedback only, and it bears no connection to passivity arguments. As a matter of fact, it will be shown that, in a very clear and natural framework, a number of stabilization issues for system (1), such as global asymptotic and input to state stabilization, robust stabilization, stabilization with bounded outputs and others, can be addressed.

More specifically, the contribution of this paper can be described as follows.

- A new class of bounded control laws for system (1) is presented. Notably, the boundness of the input is not only a system requirement, but at the same time it is a design limitation. When applying the proposed result in the general case of nonlinear cascades, the input is required to be small enough to keep the lower subsystem in an area close enough to the origin, otherwise the tools used are not valid. The result can be applied iteratively.

- The linear bounded real lemma and the generalized small gain theorem of [22] are used to solve in a unified way the following problems.

  - Robust stabilization of system (1) with partial state feedback in the presence of time delays. Under the present framework, these perturbations can be accommodated in a natural way, unlike the case where passivity based controllers are used. See for example [16] where some robustness issues of the nested saturations scheme have been studied.
  - Asymptotic stabilization with control of constant sign.
  - *Practical stabilization* with quantized control, *i.e.* it will be shown that a control input taking values in a discrete set can drive the state of the closed loop system

in an arbitrarily small neighbourhood of the origin.

- Global asymptotic stabilization of stable linear systems with bounded output is solved via dynamic linear feedback. Stabilization with feedback of perturbed and bounded outputs was achieved via time varying control in [4, 11, 19] and via dynamic control, that includes state observation, in [12, 18]. The dynamic law presented here is not based on state estimation and it is applicable to minimum and nonminimum phase systems, contradicting the remark in [12] about the output stabilizability of systems with unstable zeros, in the presence of saturated outputs.

- A new globally asymptotically stabilizing control law for a chain of integrators is obtained with recursive application of the main result. This is considerably different from the ones of [28, 27, 14, 6].

- Output feedback stabilization of a simple mechanical system is addressed, as an application. The system in consideration is the *Translational Oscillator with a Rotational Actuator*, commonly referred to as TORA [2]. Various constructive nonlinear control methodologies have been tested on this system, see for example [9, 18, 24], while in [20] the problem was set in an Euler-Lagrange framework and a passivity based output feedback controller was proposed. With the exception of this last reference, all proposed controllers either required the whole state, or utilized some kind of state observation, to achieve global asymptotic stability. In this paper it will be shown that a much simpler dynamic output feedback controller of dimension one can globally asymptotically stabilize the TORA.

In what follows the construction of the bounded control signals - or the mathematical description of a bounded output - will be achieved with the use of saturation functions. More specifically we shall use three different types of such nonlinearities, all belonging to the sector $[0, 1]$[1], the simplest of which are shown in Table 1. For the rest of the paper we use the general symbol $\sigma(\cdot)$ to denote any of the functions in Table 1.

The paper is organized as follows. In Section 2 we present a preliminary result on the stabilization of a cascade consisting of an ISS system driving an integrator. In the same section we formulate precisely the two main problems that are addressed and solved. In Section 3 we present and prove two useful Lemmas about the solvability of a matrix inequality. In Section 4 we elaborate on our main result on the stabilization of a nonlinear feedforward system with a bounded, partial state feedback control law, and we also present some interesting extensions and implications. Motivated by the results in Sections 3 and 4, a dynamic control law that solves the problem of the stabilization of a linear SISO system with bounded output is presented in Section 5. In Section 6 we give the applications of our main result, namely in Section 6.1 we show how the result of Section 4 can be applied iteratively, for the stabilization of a chain of integrators, in Section 6.2 we present a more general result concerning the stabilization with control of a constant sigh of all asymptotically null controllable systems with bounded control and in Section 6.3 we design a stabilizing output feedback controller for the TORA. Finally, in Section 7 we provide some conclusions.

[1] A nonlinear function $\sigma(y)$ is said to belong to the sector $[k_1, k_2]$ if for all y, $k_1 \leq \frac{\sigma(y)}{y} \leq k_2$.

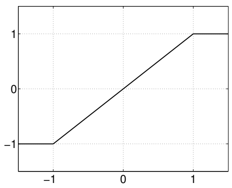
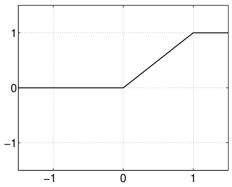
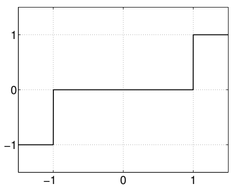
Plot	Mathematical description	Properties
	$\sigma_s(y) = sign(y) \min\{\lvert y \rvert, 1\}$	Symmetric, one to one close to the origin
	$\sigma_+(y) = \max\{0, \sigma_s(y)\}$	Non-symmetric, non-negative for all y
	$\sigma_q(y) = \begin{cases} 0 \text{ for } \lvert y \rvert < 1 \\ sign(y) \text{ elsewhere} \end{cases}$	Quantized, symmetric

Table 1: Typical saturation functions in the sector $[0, 1]$.

2 A motivating result and problem formulation

In this section we show how stabilization of a simple cascade can be obtained using bounded partial state feedback, and we state formally the problems studied in the paper.

Consider a system described by equations of the form[2]

$$\begin{aligned} \dot{z} &= h(\xi) \\ \dot{\xi} &= f(\xi) + g(\xi)u, \end{aligned} \tag{2}$$

with $z \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $u \in \mathbb{R}$ and assume that the lower subsystem is locally exponentially stable (LES) and input to state stable (ISS) with respect to u. We now show that this cascaded system can be stabilized with a simple bounded feedback law that requires knowledge of z only. The rationale behind this result is straight forward. To begin with note that if ϵ is sufficiently small, by LES and ISS of the ξ–subsystem, any trajectory of the closed loop system will converge to the slice $|\xi| < \delta_\epsilon$, where δ_ϵ can be made arbitrarily small reducing ϵ. Note also that $z(t)$ is bounded for all bounded t. On the slice $\|\xi\| < \delta_\epsilon$ the system can be approximated by a linear time invariant system given by the equations

$$\begin{bmatrix} \dot{z} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} 0 & H \\ 0 & F \end{bmatrix} \begin{bmatrix} z \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} u,$$

where

$$H \triangleq \left. \frac{\partial h(\xi)}{\partial \xi} \right|_{\xi=0}, \; F \triangleq \left. \frac{\partial f(\xi)}{\partial \xi} \right|_{\xi=0}, \; G \triangleq g(0).$$

Then the following result which is a simple application of the *circle criterion* (see *e.g.* [10]), can be established.

[2]Throughout the paper it is assumed that all functions and mappings are at least $\mathcal{C}^1$.

Proposition 1 *Consider system (2) with $f(0)=0$ and $h(0)=0$ and a nonlinearity $\sigma(\cdot)$ belonging to the sector $[0,1]$. Suppose that*

(H1) the system $\dot\xi = f(\xi)+g(\xi)u$ is ISS with respect to the input u, and $\dot\xi = f(\xi)$ is LES;

(H2) $HF^{-1}G<0$;

(H3) $\sigma(s)=0 \;\;\Rightarrow\;\; s=0$.

Then there exist $\kappa^\star>0$ and $\epsilon^\star>0$, such that for any $\kappa\in(0,\kappa^\star)$ and $\epsilon\in(0,\epsilon^\star)$

$$\mathbf{spec}(\begin{bmatrix} 0 & H \\ -\kappa G & F \end{bmatrix}) \subset \mathbf{C}^- \tag{3}$$

and the closed loop system

$$\begin{array}{rcl} \dot z & = & h(\xi) \\ \dot\xi & = & f(\xi) - g(\xi)\epsilon\sigma(\frac{\kappa}{\epsilon}z) \end{array} \tag{4}$$

is GAS (LES).

Remark 1 It is worth noting that GAS (LES) of the system

$$\begin{array}{rcl} \dot z & = & H\xi \\ \dot\xi & = & F\xi - G\epsilon\sigma(\frac{\kappa}{\epsilon}z) \end{array} \tag{5}$$

implies condition (3), but the converse is (in general) not true. However, for some special cases it can be proven that local exponential stability of (5) implies the absolute stability of (5). This is the case, for example, when the transfer function

$$\mathcal{G}_o(s) = H(sI-F)^{-1}G$$

has no zeros and only real poles and κ is chosen such that

$$\mathbf{spec}(\begin{bmatrix} 0 & H \\ -\kappa G & F \end{bmatrix}) \subset \mathbb{R}^-.$$

Remark 2 Note that, if the pair $\{F,G\}$ is controllable, system (2) is controllable if and only if $HF^{-1}G\neq 0$. Moreover, Hypothesis (H2) is not restrictive. In fact, if $HF^{-1}G>0$ the result of Proposition 1 holds with $\kappa^\star<0$.

We remark that ϵ is the level of saturation, whereas κ is the feedback gain, or in other words, κz is the appropriate output that needs to be fed back. An interesting extension of Proposition 1 would be the iterative application of the methodology proposed. Indeed this is possible, as it will be discussed in the following sections, where, it will be proven that the closed loop system (4) is also *ISS with restrictions* with respect to a new external input.

The fact that the proof of Proposition 1 is based on the circle criterion facilitates the handling of a series of system uncertainties, such as time delays. While it is known that passivity based designs may be inadequate in the presence of delays, the result in Proposition 1 is robust against (constant) time delays in the input or output path, as summarised in the following Corollary.

Corollary 1 *Consider system (2) and a positive constant τ. Under the assumptions of Proposition 1, there exists a positive $\kappa_1^\star = \kappa_1^\star(\tau)$ and an $\epsilon^\star > 0$ such that for all $\kappa_1 \in (0, \kappa_1^\star)$ and $\epsilon \in (0, \epsilon^\star)$ the control law*

$$u = -\epsilon\sigma(\frac{\kappa_1}{\epsilon} z(t-\tau)) \tag{6}$$

globally asymptotically (locally exponentially) stabilizes system (2).

Proof. Note that, as before, $|\sigma(\frac{\kappa_1}{\epsilon} z(t-\tau))| \leq 1$ for any positive constant τ, thus if ϵ is small enough ξ will eventually enter a small region around $\xi = 0$. Therein we consider the system

$$\begin{array}{rcl} \dot{z}(t) & = & H\xi(t) \\ \dot{\xi}(t) & = & F\xi(t) - G\epsilon\sigma(\frac{\kappa_1}{\epsilon} z(t-\tau)). \end{array} \tag{7}$$

If $\mathcal{G}_\xi(s)$ is the transfer function of the open loop ξ-subsystem with output $H\xi$, then the transfer function of the system

$$\begin{array}{rcl} \dot{z}(t) & = & H\xi(t) \\ \dot{\xi}(t) & = & F\xi(t) + Gu \\ \eta & = & z(t-\tau) \end{array}$$

is $\frac{e^{-s\tau}\mathcal{G}_\xi(s)}{s}$. Note now that the Nyquist plot of $\frac{\mathcal{G}_\xi(s)}{s}$ is bounded from the left by a vertical line, say through the point $(0, -\frac{1}{\kappa})$. Then there exists a positive number $\kappa_1 < \kappa$ such that the Nyquist plot of $\frac{e^{-s\tau}\mathcal{G}_\xi(s)}{s}$ is also bounded from the left by a vertical line through the point $(0, -\frac{1}{\kappa_1})$. An example is shown on Figure 1. The conclusion, therefore, follows as an application of the circle criterion. $\triangleleft$

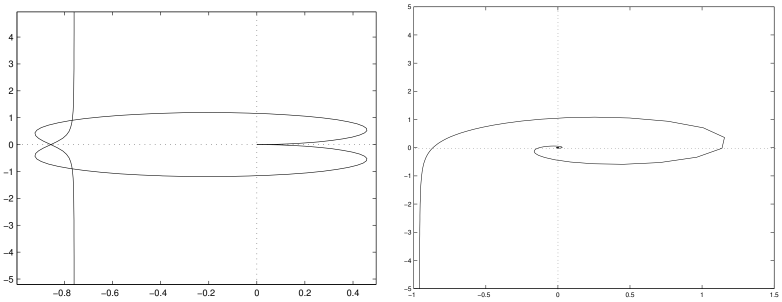


Figure 1: Nyquist diagram of the transfer function of the cascade of an asymptotically stable system and an integrator $(\frac{\mathcal{G}_\xi(s)}{s})$ (left) and Nyquist diagram of the same system with a delay in the output $(\frac{e^{-s\tau}\mathcal{G}_\xi(s)}{s})$ (right).

Remark 3 Corollary 1 provides a *'delay dependent'* stability result, *i.e.* the closed loop system is not asymptotically stable for any τ, but only for $0 \leq \tau \leq \tau^\star$. However, unlike other delay dependent criteria, the result in Corollary 1 is constructive, *i.e.* for any delay τ an appropriate stabilizing feedback (6) can be found.

We are now ready to state formally the stabilization problems that will be dealt with in this paper.

Partial state feedback stabilization problem. Consider a nonlinearity $\sigma(\cdot)$, a system described by equations of the form

$$\begin{array}{rcl} \dot{z} & = & Jz + h(\xi) + r(\xi)u \\ \dot{\xi} & = & f(\xi) + g(\xi)u, \end{array} \tag{8}$$

where $z \in \mathbb{R}^p$, $\xi \in \mathbb{R}^n$ and $u \in \mathbb{R}$ and suppose the following.

(A1) The system $\dot{\xi} = f(\xi) + g(\xi)u$ is ISS with respect to u, and $\dot{\xi} = f(\xi)$ is LES;

(A2) $J + J' \leq 0$[3].

Find (if possible) a positive constant ϵ and an output

$$\eta = Kz \tag{9}$$

such that (8) in closed loop with the control law

$$u = -\epsilon\sigma(\frac{1}{\epsilon}\eta) + v \tag{10}$$

is LES and ISS with restrictions with respect to v.

Regarding this problem we define the following matrices

$$\begin{array}{c} H \triangleq \left.\dfrac{\partial h(\xi)}{\partial \xi}\right|_{\xi=0} \in \mathbb{R}^{p\times n}, \quad F \triangleq \left.\dfrac{\partial f(\xi)}{\partial \xi}\right|_{\xi=0} \in \mathbb{R}^{n\times n}, \\ R \triangleq r(0) \; \in \mathbb{R}^{p\times 1}, \quad G \triangleq g(0) \; \in \mathbb{R}^{n\times 1}, \end{array} \tag{11}$$

$$A \triangleq \begin{bmatrix} J & H \\ \mathbf{0} & F \end{bmatrix}, \quad B \triangleq \begin{bmatrix} R \\ G \end{bmatrix}, \quad C \triangleq [I \; \mathbf{0}]. \tag{12}$$

and the approximation of (8) for small ξ, given by

$$\begin{bmatrix} \dot{z} \\ \dot{\xi} \end{bmatrix} = A \begin{bmatrix} z \\ \xi \end{bmatrix} + Bu. \tag{13}$$

The second problem that will be solved in the paper is the problem of the stabilization of a linear SISO system $\dot{z} = Jz + \bar{H}w$ when the available output is subject to saturation. This is formally stated as follows.

Bounded output stabilization problem. Consider a nonlinearity $\sigma(\cdot)$ and a system described by equations of the form

$$\begin{array}{c} \dot{z} = Jz + \bar{H}w \\ y = \sigma(Kz). \end{array} \tag{14}$$

Suppose (A2) holds. Find (if possible) a dynamic control law

$$\begin{array}{c} \dot{\xi} = F\xi - Gy \\ w = \Gamma\xi \end{array} \tag{15}$$

such that the closed loop system (14)-(15) is GAS (LES).

[3] Assumption (A2) can be replaced by $J'S + SJ \leq 0$ for some $S = S' > 0$.

Remark 4 The two stabilization problems stated here are different to each other, in the sense that they can appear in different situations. However, as it will be clear from the following, their solutions are based on the same theoretical tools and therefore it is logical for both of them to be addressed in the same framework.

3 Two useful lemmas

In this section we present two lemmas that are instrumental to prove the main results of the paper. They are both related to the existence of solutions for a special *matrix inequality*. Note that the proofs of both these lemmas are constructive, *i.e.* we provide a family of solutions of the considered matrix inequality.

Lemma 1 *Let A, B, and C be defined as in (12) and suppose $\{A, B\}$ is controllable, $F \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix and $J \in \mathbb{R}^{p\times p}$ is such that (A2) holds. Then, there exist $P \in \mathbb{R}^{(n+p)\times(n+p)}$ and $K \in \mathbb{R}^{1\times p}$ such that*

$$\begin{array}{c} (KC - B'P)'(KC - B'P) + PA + A'P \leq 0 \\ P = P' > 0, \end{array} \tag{16}$$

and $A_{cl} = A - BKC$ is Hurwitz.

Proof. Let P be defined as

$$P \triangleq \begin{bmatrix} \chi I & Y \\ Y' & P_\xi \end{bmatrix} \tag{17}$$

with χ a positive constant and $P_\xi = P_\xi' > 0$ to be selected. As a result, the inequality (16) rewrites

$$\begin{array}{l} \begin{bmatrix} \chi(J + J') & \chi H + YF + J'Y \\ \chi H' + F'Y' + Y'J & Y'H + H'Y + P_\xi F + F'P_\xi \end{bmatrix} + \\ \begin{bmatrix} K' - \chi R - YG \\ -Y'R - P_\xi G \end{bmatrix} \begin{bmatrix} K - \chi R' - G'Y' & -R'Y - G'P_\xi \end{bmatrix} \leq 0. \end{array} \tag{18}$$

Setting

$$K = \chi R' + G'Y' \tag{19}$$

the problem is translated into finding matrices Y and P_ξ such that (18) holds. To this end, note that Y can always be selected such that

$$\chi H + YF + J'Y = 0, \tag{20}$$

for all H, F, J and all positive constants χ. With K and Y defined by (19) and (20), (18) reduces to

$$\begin{bmatrix} \chi(J + J') & 0 \\ 0 & T + P_\xi F + F'P_\xi + P_\xi GG'P_\xi \end{bmatrix} \leq 0,$$

where

$$\begin{array}{rcl} T & = & Y'H + H'Y + Y'RR'Y + Y'RG'P_\xi + P_\xi GR'Y \\ & = & Y'(H + RG'P_\xi) + (H' + P_\xi GR')Y + Y'RR'Y. \end{array}$$

Since $\chi(J + J') \leq 0$, the problem is reduced to finding a P_ξ such that

$$T + P_\xi F + F'P_\xi + P_\xi GG'P_\xi \leq 0. \tag{21}$$

To solve this problem, let $\bar{P}_\xi$ be such that

$$\bar{P}_\xi F' + F\bar{P}_\xi = -(GG' + I).$$

Then setting $P_\xi = \bar{P}_\xi^{-1}$ yields

$$P_\xi F + F' P_\xi + P_\xi G G' P_\xi = -P_\xi P_\xi < 0. \tag{22}$$

Hence, it is sufficient to show that T can be made arbitrarily small. To this end notice that the solution of (20) is

$$Y = \chi \bar{Y} \tag{23}$$

where $\bar{Y}$ is the solution of

$$H + \bar{Y}F + J'\bar{Y} = 0.$$

Therefore

$$T = \chi(\bar{Y}'(H + RG'P_\xi) + (H' + P_\xi G R')\bar{Y}) + \chi^2(\bar{Y}'RR'\bar{Y})$$

and this can be made arbitrarily small by a proper selection of $\chi > 0$.

Besides, P constructed as above is positive definite. For, note that the Schur complement of P is $P_\xi - \chi\bar{Y}'\bar{Y}$ which is positive definite for a positive definite P_ξ and small enough χ. Therefore there exists a positive χ such that condition (16) holds.

To complete the proof we need to show that $A_{cl} = A - BKC$ is Hurwitz. To this end observe that inequality (16) is equivalent to

$$A_{cl}'P + PA_{cl} + PBB'P + C'K'KC \leq 0, \tag{24}$$

which yields

$$P^{-1}A_{cl}' + A_{cl}P^{-1} \leq -BB' \leq 0. \tag{25}$$

On the other hand, it is trivial to check that if $\{A, B\}$ is controllable $\{B', A_{cl}\}$ is observable. From that and from inequality (25), according to [32, Lemma 12.2], it is concluded that A_{cl} is Hurwitz. $\triangleleft$

Remark 5 Inequality (24) arises in the non-standard $\mathcal{H}_\infty$ control problem [5] described by the equations

$$\begin{aligned} \dot{x} &= Ax + Bu + Bw \\ z &= u \\ y &= Cx, \end{aligned} \tag{26}$$

where A, B and C are as in (12), u is the control input, w is the exogenous input, z is the penalty variable and y is the measurement. Lemma 1 expresses the fact that there exists a static output feedback control law $u = -Ky$ rendering system (26) asymptotically stable and with a $\mathcal{L}_2$-gain from w to z less than or equal to one.

Note that if J has eigenvalues on the $j\omega$ axis then $\gamma = 1$ is the smallest achievable $\mathcal{L}_2$-gain for system (26), *i.e.* any static or dynamic output feedback stabilizing controller yields a closed loop system with $\mathcal{L}_2$-gain larger or equal to one.

Lemma 2 *Let $J \in \mathbb{R}^{p\times p}$, $\bar{H} \in \mathbb{R}^{p\times 1}$ and $K \in \mathbb{R}^{1\times p}$ be known matrices such that (A2) holds, $\{J', \bar{H}\}$ is controllable and $\{K, J\}$ is observable. Then, there exist $P \in \mathbb{R}^{(2p)\times(2p)}$, $G \in \mathbb{R}^{p\times 1}$, $\Gamma \in \mathbb{R}^{1\times p}$ and a Hurwitz matrix $F \in \mathbb{R}^{p\times p}$ such that (16) holds, with $R = 0$, $H = \bar{H}\Gamma$ and A, B, and C as in (12), and the matrix $A_{cl} = A - BKC$ is Hurwitz.*

Proof. Partition P as in (17) and repeat the first steps of the proof of Lemma 1. However, note that we are looking now for F, G and Γ.
Let F be a Hurwitz matrix with distinct eigenvalues, and L be such that

$$\mathbf{spec}(J' + \bar{H}L) = \mathbf{spec}(-F).$$

Note that such an L exists because of controllability of the pair $\{J', \bar{H}\}$. Then there exists a nonsingular matrix X such that

$$J'X + \bar{H}LX + XF = 0.$$

Therefore, setting $Y = \chi X$, for some positive χ, solves the Sylvester equation (20) with $H = \bar{H}LX$. Next, set $\Gamma = LX$, $G = Y^{-1}K'$ and let $\bar{P}_\xi$ be the positive definite matrix that solves the Lyapunov equation

$$\bar{P}_\xi F' + F\bar{P}_\xi = -\left(X^{-1}K'K(X^{-1})' + I\right).$$

Choosing $P_\xi = \chi^2 \bar{P}_\xi^{-1}$, it is easy to verify that the first of inequalities (16) holds for a large enough $\chi > 0$. On the other hand, with the above selections for P_ξ and Y, the matrix P, is positive definite for a large enough $\chi > 0$.
Observe, now, that inequality (16), or the equivalent inequality (24), yields

$$A_{cl}'P + PA_{cl} \leq -\left(PBB'P + C'K'KC\right) \leq -C'K'KC \leq 0, \tag{27}$$

and that observability of the pair $\{K, J\}$ implies detectability of the pair $\{KC, A_{cl}\}$. As a result, by [32, Lemma 12.2], A_{cl} is Hurwitz. $\triangleleft$

Remark 6 Dual to what stated in Remark 5, consider the non-standard $\mathcal{H}_\infty$ control problem described by the equations

$$\begin{array}{rcl} \dot{x} &=& Jx + \bar{H}u \\ z &=& Kx \\ y &=& Kx + w, \end{array} \tag{28}$$

where J is such that (A2) holds, u is the control input, w is the exogenous input, z is the penalty variable and y is the measurement. Lemma 2 expresses the fact that there exists a dynamic output feedback control law, of the same dimension as system (28), described by equations of the form

$$\begin{array}{rcl} u &=& \Gamma\xi \\ \dot{\xi} &=& F\xi - Gy, \end{array} \tag{29}$$

such that the closed loop system (28)-(29) is asymptotically stable and with an $\mathcal{L}_2$-gain from w to z less than or equal to one. Note that if J has eigenvalues on the $j\omega$ axis then an $\mathcal{L}_2$-gain equal to one is the smallest achievable gain, with any output feedback.

4 Stabilization with bounded partial state feedback

In this section we state our main result on the stabilization, with partial state feedback bounded control, of the system described by the equations (8). Following the discussion in Section 2, the key point to what follows is that if u is bounded by a sufficiently small number, *i.e.* if ϵ is chosen small enough, then in finite time $\|\xi\|$ will be small, hence, the functions $f(\xi)$, $r(\xi)$, $h(\xi)$ and $g(\xi)$ can be replaced by their approximations for small $\|\xi\|$. Therefore, it is sufficient to study the stabilization with bounded control problem for system (13).

Proposition 2 *Consider a nonlinearity $\sigma(\cdot)$ belonging to the sector $[0,1]$ and the system described by the equations*

$$\begin{aligned} \dot{z} &= Jz + h(\xi) + r(\xi)u \\ \dot{\xi} &= f(\xi) + g(\xi)u, \end{aligned} \tag{30}$$

with $z \in \mathbb{R}^p$, $\xi \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $f(0) = 0$, $h(0) = 0$. Suppose (A1) and (A2) hold and moreover, assume the following

(C1) the linear approximation of (30) is controllable.

Then there exists $\epsilon^\star > 0$ and a matrix $K \in \mathbb{R}^{1\times p}$ such that if $\epsilon \in (0, \epsilon^\star)$, the static partial state feedback control law

$$u = -\epsilon\sigma(\frac{1}{\epsilon}Kz) \tag{31}$$

globally stabilizes *system (30). If, moreover*

(C2) the only trajectory $z(t)$ of $\dot{z}(t) = Jz(t)$ such that $\sigma(Kz(t)) = 0$, for all t, is $z(t) = 0$,

then (31) globally asymptotically (locally exponentially) stabilizes *system (30) and furthermore, the system*

$$\begin{aligned} \dot{z} &= Jz + h(\xi) - r(\xi)\epsilon\sigma(\tfrac{1}{\epsilon}Kz) + r(\xi)w \\ \dot{\xi} &= f(\xi) - g(\xi)\epsilon\sigma(\tfrac{1}{\epsilon}Kz) + g(\xi)w, \end{aligned} \tag{32}$$

is ISS with respect to the new input w, with the restriction $|w| \le \mu$, with $\mu < \epsilon$.

Remark 7 A similar result has been proven in [7, Theorem 14.3.3] on the basis of the results in [29]. Note, however that the result of [7, Theorem 14.3.3] requires, in general, full state feedback, and that the result in Proposition 2 is based on a different construction. As a result Proposition 2 can also be used in the design of output feedback control laws (see Section 5), in the design of quantized or constant sign controllers (see Corollary 2 and Corollary 3) and when dealing with some robustness problems (see Corollary 1 and Corollary 5).

In fact, the assumptions on the system in [7, Theorem 14.3.3] are different to the assumptions in Proposition 2. There, the construction depends on the pair $\{J, R\}$ being stabilizable, while a cross-term corresponding to $h(\xi)$ of system (30) is assumed to be of order at least two. Under this formulation the feedback used in [7, Theorem 14.3.3] is of the form (31), but this time K is such that $J - RK$ is Hurwitz. From what will become clear from the proof of Proposition 2 and the examples presented in the rest of the paper, it is obvious that the two results are not addressing the same problem. For example, Proposition 2 also covers the case where J is skew symmetric and $R = 0$, therefore the pair $\{J, R\}$ is not stabilizable, and the upper subsystem is driven entirely by ξ.

Proof. As discussed earlier, because of (A1), there exists $\epsilon^\star > 0$ such that if $\epsilon \in (0, \epsilon^\star)$, the state of the closed loop system (30)-(31) will in finite time enter a small enough *'slice'* where $\|\xi\| < \delta_\epsilon$, for an arbitrarily small $\delta_\epsilon > 0$. There, we can consider the linear approximation of (30).

Denoting $x = [z' \ \ \xi']'$, the state space equations of the cascade (8) and the output described in the partial state feedback stabilization problem (equation (9)) are written as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ \eta &= KCx. \end{aligned} \tag{33}$$

At this point we assume that there exists a matrix K such that the linear feedback

$$u_l = -KCx$$

exponentially stabilizes system (13). This assumption will turn out to be automatically satisfied. The proposed control law (31) can be written

$$\begin{aligned} u = -\epsilon\sigma(\tfrac{1}{\epsilon}\eta) &= -\left[KCx + \left(\epsilon\sigma(\tfrac{1}{\epsilon}\eta) - KCx\right)\right] \\ &= -KCx - \left(\epsilon\sigma(\tfrac{1}{\epsilon}\eta) - \eta\right) = -KCx - \psi(\eta), \end{aligned}$$

where $\psi(\eta)$ is a new nonlinearity restricted to the sector $[-1, 0]$.

Note that up to now, K is some matrix that sets A_{cl} to be Hurwitz. However, to prove stability in the presence of the nonlinearity $\psi(\eta)$ a special "stabilizing" K has to be selected. For, note that system (33)-(31) can be regarded as the feedback interconnection of the system

$$\begin{aligned} \dot{x} &= A_{cl}x + Bv \\ \eta &= KCx, \end{aligned} \tag{34}$$

where $A_{cl} \triangleq A - BKC$, with

$$v = \psi(\eta).$$

Moreover, the $\mathcal{L}_2$-gain of $\psi(\eta)$ is not larger than one, hence, selecting K satisfying inequality (16) for some $P > 0$, yields[4], by the *generalized small gain Theorem* in [22], a closed loop asymptotically stable system. Moreover, A_{cl} is a Hurwitz matrix, from Lemma 1.

To complete the proof of Proposition 2, we need to prove the ISS property of system (32). First notice, that if $\epsilon \in (0, \epsilon^\star)$, for any w such that $|w| < \epsilon$, the state will enter the slice $\|\xi\| < \delta_\epsilon$.

We now prove that the system

$$\dot{x} = Ax - B\epsilon\sigma(\frac{\eta}{\epsilon}) + Bw, \tag{35}$$

which is the approximation of (32) for $\|\xi\|$ small, is ISS with some restriction on w. For, consider the positive definite function

$$V = x'Px, \tag{36}$$

with P as defined in (17). Along the trajectories of (35) one has

$$\dot{V} = x'(A'P + PA)x - 2x'PB\left(\epsilon\sigma(\frac{\eta}{\epsilon}) - w\right). \tag{37}$$

With simple calculations, using (20) and (23) it is easy to see that

$$x'(A'P + PA)x = \chi z'(J + J')z - \xi'\left(Q_\xi - \chi(H'\bar{Y} + \bar{Y}'H)\right)\xi, \tag{38}$$

where Q_ξ is a positive definite matrix and P_ξ is the positive definite solution of the Lyapunov equation $F'P_\xi + P_\xi F = -Q_\xi$. Note also that, by assumption (A2), $J + J'$ is negative semi-definite and that

$$PB = \begin{bmatrix} \chi R + YG \\ Y'R + P_\xi G \end{bmatrix} = \begin{bmatrix} K' \\ Y'R + P_\xi G \end{bmatrix}. \tag{39}$$

[4]Recall that, by Remark 5 and Lemma 1, system (34) has a $\mathcal{L}_2$-gain ($\mathcal{H}_\infty$-norm) less than or equal to one.

As a result, (37) becomes:

$$\begin{array}{rcl} \dot{V} & = & \chi z'(J+J')z - \xi'\left(Q_\xi - \chi(H'\bar{Y}+\bar{Y}'H)\right)\xi - 2\eta\left(\epsilon\sigma(\frac{\eta}{\epsilon}) - w\right) \\ & & -2\xi'(Y'R+P_\xi G)\epsilon\sigma(\frac{\eta}{\epsilon}) + 2\xi'(Y'R+P_\xi G)w \\ & \leq & -\alpha\|\xi\|^2 + 2\epsilon\beta\|\xi\| + 2\beta\|\xi\||w| - 2\eta\left(\epsilon\sigma(\frac{\eta}{\epsilon}) - w\right), \end{array} \tag{40}$$

for some positive constants α and β. Considering that the term $\epsilon\sigma(\frac{\eta}{\epsilon})$ has the same sign as η, then for large η the term $-2\eta\left(\epsilon\sigma(\frac{\eta}{\epsilon}) - w\right)$ is not positive, provided w is kept strictly less than ϵ. This means that the system (35) with output η is *Input to Output Stable* with some (non-zero) restriction on w. Using the result in [26], to prove ISS it is sufficient to show that the pair $\{K, J\}$ is detectable.

Detectability of the pair $\{K, J\}$ can be established by contradiction. Consider without loss of generality that J is described by

$$J = \begin{bmatrix} J_o & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & & 0 \\ \vdots & & & & \vdots \\ & & J_1 & & \\ \vdots & & & \vdots & \\ & & & & J_j \end{bmatrix}, \tag{41}$$

with J_o Hurwitz and $J_\nu + J'_\nu = 0$, $\nu = 1, \cdots j$, *i.e.* suppose the cascade has an asymptotically stable part, described by J_o, i integrators and j pairs of pure imaginary eigenvalues. K can be partitioned as $K = [K_o, K_1, \cdots, K_i,\ [k_{11}\ k_{21}], \cdots, [k_{1j}\ k_{2j}]]$. Note first that all modes associated with the asymptotically stable part of the system are detectable. On the other hand, the rest of the system is unobservable if and only if $K_\nu = 0$ for some $\nu = 1, \cdots i$ or $k_{1\nu}^2 + k_{2\nu}^2 = 0$ for some $\nu = 1, \cdots j$. (A block diagram representation of the (linear approximation of the) system (30) for an empty matrix J_o is given in Figure 2.) Because the systems with transfer functions $\frac{1}{s}$ and $\frac{1}{(s^2+\omega^2)}$ are not exponentially stable, the closed loop of Figure 2 could never be GAS if $K_\nu = 0$ for any $\nu = 1, \cdots i$ or $k_{1\nu}^2 + k_{2\nu}^2 = 0$ for any $\nu = 1, \cdots j$, which is a contradiction. Therefore the pair $\{K, J\}$ is detectable.

This completes the proof of Proposition 2. $\triangleleft$

Remark 8 Note that, as proved in [25], if the subsystem $\dot{\xi} = f(\xi)$ is GAS-LES, assumption (A1) is without loss of generality because the control can always be rescaled appropriately. However, if this rescaling is undesired, the minimal assumption is that the system is *ISS with some restriction* [14]. For example the system $\dot{x} = -x^3 + (1+x^3)u$ is not ISS, but it is ISS with restriction one.

Remark 9 The result of Proposition 2 can also be proven employing the idea of *absolute stability* instead of the small gain theorem. According to that, global asymptotic stability is established, if the 'virtual' closed loop system described by the matrices (A_{cl}, B, KC) is absolutely stable for all nonlinearities $\psi(\eta)$ belonging to the sector $[-1, 0]$. For that to happen it is sufficient (see for example [31] or [10, Theorem 10.2]) for the Nyquist diagram of the transfer function $KC(sI - A_{cl})^{-1}B$ to lie on the left of a vertical line that goes through the point $(1, 0)$. While this condition is satisfied when the $\mathcal{H}_\infty$-norm ($\mathcal{L}_2$-gain) is (not larger than) one, it can also hold if the $\mathcal{H}_\infty$-norm is larger than one, but the Nyquist diagram is all on the left of the vertical line through $(1, 0)$. Note that this is a less restrictive condition.

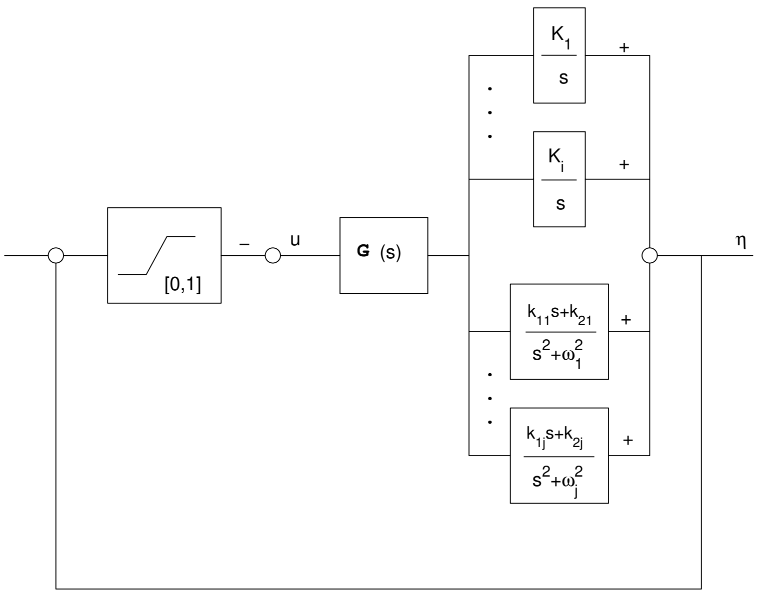


Figure 2: Block diagram of system (35) with J as in (41).

Remark 10 System (30) with output η is not, in general, minimum phase, nor with a relative degree one. This fact distinguishes the present stabilization method from a family of other nonlinear control results that rely on some passivity property of the system, see for example [7, Proposition 14.1.5], or even the results in [28] and [27].

Remark 11 An extension of Proposition 2 in the spirit of Corollary 1, *i.e.* in the presence of delays in the loop, can be derived.

It is easy to see that cascades with a simple integrator for the upper system (see also Proposition 1) belong to the class of systems described by equations (30) with $J = 0$ and $r(\xi) = 0$. In this case we can name the 'desired output' mentioned in the stabilization problem to be $\eta = \kappa z$, where κ is as described in Proposition 1. In general, when integrators are present, special attention has to be given in the choice of the nonlinearity $\sigma(\cdot)$. Note for example, that using the non-negative nonlinearity $\sigma_+(\cdot)$ for the system (2) we cannot achieve GAS, since there are no isolated equilibria (the system can converge to any point $[z_-,\ 0]$, where $z_- \leq 0$). However, when J is a full rank matrix, the equilibrium is always uniquely defined, hence GAS can be achieved. On the other hand, if the aim is not to globally asymptotically stabilize system (30) but to *practically stabilize* it, *i.e.* to achieve convergence to a small enough neighbourhood of the origin, then the saturation function could be like $\sigma_q(\cdot)$ of Table 1. This discussion can be formally summarized as follows.

Corollary 2 *Consider system (30). Suppose conditions (A1) and (A2) hold. Suppose moreover that J is a full rank matrix. Then there exists $\epsilon^\star > 0$ and a matrix $K \in \mathbb{R}^{1\times p}$ such that if $\epsilon \in (0, \epsilon^\star)$, the static partial state feedback control law*

$$u = -\epsilon\sigma_+(\frac{1}{\epsilon}Kz) \quad \left(or \quad u = \epsilon\sigma_+(-\frac{1}{\epsilon}Kz)\right) \tag{42}$$

globally asymptotically stabilizes *system (30), Moreover,* $u(t) \leq 0$ *(or* $u(t) \geq 0$[5]*) for all* $t \geq 0$.

Proof. Note that if J is a full rank matrix, then the linear approximation of system (30) is controllable as long as the ξ- subsystem is controllable, and the matrices H and R are not both zero. Also, the half space defined by

$$\mathbb{S} \triangleq \{z \in \mathbb{R}^p : \sigma_+(\frac{1}{\epsilon}Kz) = 0\} = \{z \in \mathbb{R}^p : Kz \leq 0\}$$

$$\left(\text{or} \quad \mathbb{S} \triangleq \{z \in \mathbb{R}^p : \sigma_+(-\frac{1}{\epsilon}Kz) = 0\} = \{z \in \mathbb{R}^p : Kz \geq 0\}\right)$$

contains the point $z = 0$ but does not contain any neighbourhood of $z = 0$. Therefore the only trajectory of $\dot{z} = Jz$ contained in $\mathbb{S}$ is $z = 0$[6]. As a result, conditions (C1) and (C2) are satisfied, and the result follows from Proposition 2. $\triangleleft$

Corollary 3 *Consider system (2). Suppose that assumptions (H1) and (H2) of Proposition 1 hold. Then there exist* $\kappa^\star > 0$, $\epsilon^\star > 0$ *and* $t_o \in \mathbb{R}_+$ *such that for any* $\kappa \in (0, \kappa^\star)$ *and* $\epsilon \in (0, \epsilon^\star)$ *all trajectories* $(z(t), \xi(t))$ *of the closed loop system*

$$\begin{array}{rcl} \dot{z} &=& h(\xi) \\ \dot{\xi} &=& f(\xi) - g(\xi)\epsilon\sigma_q(\frac{\kappa}{\epsilon}z) \end{array} \tag{43}$$

are such that

$$\lim_{t\to\infty} \xi(t) = 0, \quad and \quad |z(t)| \leq \frac{\epsilon}{\kappa}, \forall t \geq t_o.$$

Proof. As in the proof of Proposition 2, we focus on the linearized system. For such a system consider the Lyapunov function

$$V = [z,\, \xi'] \left[\begin{array}{cc} \chi & Y \\ Y' & P_\xi \end{array} \right] \left[\begin{array}{c} z \\ \xi \end{array} \right] \tag{44}$$

with $P_\xi = P_\xi' > 0$ such that $F'P_\xi + P_\xi F = -Q_\xi$, $Q_\xi = Q_\xi' > 0$ and $Y = -\chi H F^{-1}$. Along the trajectories of (43) one has:

$$\dot{V} = -\xi'(Q_\xi + \chi(H'HF^{-1} + F^{-T}H'H))\xi - 2\kappa z\epsilon\sigma_q(\frac{\kappa}{\epsilon}z) - 2\xi' P_\xi G\epsilon\sigma_q(\frac{\kappa}{\epsilon}z), \tag{45}$$

with $\kappa = -\chi H F^{-1} G > 0$. Consider now the following two exclusive cases.

- $|z| \geq \frac{\epsilon}{\kappa}$. In this case $\sigma_q(\frac{\kappa}{\epsilon}z) \neq 0$, hence, using the fact that $-y\sigma(y) \leq -[\sigma(y)]^2$,

$$\begin{aligned} \dot{V} &\leq -\xi'(Q_\xi + \chi(H'HF^{-1} + F^{-T}H'H))\xi - 2\epsilon^2[\sigma_q(\tfrac{\kappa}{\epsilon}z)]^2 - 2\xi' P_\xi G\epsilon\sigma_q(\tfrac{\kappa}{\epsilon}z) \\ &= -[\sigma_q(\tfrac{\kappa}{\epsilon}z),\, \xi'] \left[\begin{array}{cc} 2\epsilon^2 & \epsilon G' P_\xi \\ \epsilon P_\xi G & Q_\xi + \chi(H'HF^{-1} + F^{-T}H'H) \end{array} \right] \left[\begin{array}{c} \sigma_q(\tfrac{\kappa}{\epsilon}z) \\ \xi \end{array} \right]. \end{aligned}$$

  It is easy to see (calculating the Schur complement of the above matrix) that, for a small enough χ, $\dot{V} < 0$. Therefore, V and thus $|z|$ and $\|\xi\|$, decrease everywhere in the area $\{z \in \mathbb{R} : |z| \geq \frac{\epsilon}{\kappa}\} \times \mathbb{R}^n$.

[5] Note that $\sigma_+(-\eta) = -\sigma_-(\eta)$, where $\sigma_-(\cdot)$ is defined in similar way to $\sigma_+(\cdot)$, but is equal to zero for all $\eta > 0$. Like the nonlinearities of Table 1, $\sigma_-(\cdot)$ also belongs to the sector $[0, 1]$.

[6] This is due to the fact that because $\det J \neq 0$, the system $\dot{z} = Jz$ has no constant trajectory.

- $|z| < \frac{\epsilon}{\kappa}$. In this case $\sigma_q(\frac{\kappa}{\epsilon}z) = 0$ and $\dot{V} = -\xi'(Q_\xi + \chi(H'HF^{-1} + F^{-T}H'H))\xi$. Note that the matrix $Q_\xi + \chi(H'HF^{-1} + F^{-T}H'H)$ can be made positive definite with an appropriate choice of Q_ξ and a small enough $\chi > 0$. Thus ξ is asymptotically converging to zero.

The claim is then proven. $\triangleleft$

The extension of Corollary 3 for system (30) is straightforward and is omitted here for the sake of brevity. Another extension of Proposition 2 is obtained if the 'lower' subsystem is the "trivial system". In this case, the following facts can be proven.

Corollary 4 *Consider the system*

$$\dot{z} = Jz + Ru, \tag{46}$$

with $z \in \mathbb{R}^p$, $u \in \mathbb{R}$, and $J + J' \leq 0$. Assume that the pair $\{J, R\}$ is controllable and $\det(J) \neq 0$. *Then there exists a matrix $K \in \mathbb{R}^{1\times p}$ such that*

1. *the state feedback control law*

$$u = -\epsilon\sigma_s(\frac{1}{\epsilon}Kz) \quad or \quad u = -\epsilon\sigma_+(\frac{1}{\epsilon}Kz) \quad or \quad u = \epsilon\sigma_+(-\frac{1}{\epsilon}Kz) \tag{47}$$

   globally asymptotically stabilizes *system (46) for all $\epsilon > 0$.*

2. *there exists $t_o \geq 0$ such that all trajectories of system (46) in closed loop with the quantized state feedback control law*

$$u = -\epsilon\sigma_q(\frac{1}{\epsilon}Kz) \tag{48}$$

   are such that

$$\|Kz(t)\| \leq \epsilon, \ \forall t \geq t_o.$$

Proof. Because of controllability of the pair $\{J, R\}$, there will exist a matrix K such that

$$J_{cl} = J - RK$$

is Hurwitz. Setting $\eta = Kz$, and denoting all types of saturations with $\sigma(\cdot)$[7], the closed loop system (46)-(47) (or (46)-(48)) can be written as

$$\dot{z} = Jz - RKz + RKz - R\epsilon\sigma(\frac{1}{\epsilon}\eta) = J_{cl}z - R(\epsilon\sigma(\frac{1}{\epsilon}\eta) - \eta) = J_{cl}z - R\psi(\eta),$$

where $\psi(\eta) = \epsilon\sigma(\frac{1}{\epsilon}\eta) - \eta$ is a nonlinear function belonging to the sector $[-1, 0]$. As explained before, the $\mathcal{L}_2$-gain of $\psi(\eta)$ is less than or equal to one. As for the $\mathcal{L}_2$-gain of system $\dot{z} = J_{cl}z + Rv$ with output η, we know that it is less than or equal to one if and only if there exists a positive definite matrix P, such that

$$J_{cl}'P + PJ_{cl} + PRR'P + K'K \leq 0,$$

or, what is the same,

$$(K' - PR)(K' - PR)' + PJ + J'P \leq 0. \tag{49}$$

[7] See Footnote 5.

Choosing $P = I$, it is obvious that, selecting $K = R'$, (49) holds. Next, note that for the nonlinearities used in either of the feedback laws (47) the $\mathcal{L}_2$-gain of $\psi(\eta)$ is either strictly less than one for all z, or is one in an area that cannot contain any trajectories of $\dot{z} = Jz$ either than $z = 0$[8] and strictly less than one everywhere else. Thus, according to the result in [22], the closed loop system is globally asymptotically stable.

Finally note that choosing a nonlinearity of the type $\sigma_q(\cdot)$ the $\mathcal{L}_2$-gain of $\psi(\eta)$ is strictly less than one for $\mathbb{S}_1 \stackrel{\triangle}{=} \{z \in \mathbb{R}^p : \|Kz(t)\| \geq \epsilon\}$ and exactly one in $\mathbb{S}_2 \stackrel{\triangle}{=} \{z \in \mathbb{R}^p : \|Kz(t)\| < \epsilon\}$. As a result, the area $\mathbb{S}_2$ is attractive and stable. This proves the claim. $\triangleleft$

Remark 12 Note that the solution $K = R'$ resulting from the proof of Corollary 4 can be retrieved from the procedure followed in the proof of Lemma 1 if the matroces F and H are taken to be empty.

In Figures 3 and 4 we illustrate the conclusion of Corollary 4 with some simulation results for a fourth order system with open loop eigenvalues at $\pm 2j$, $\pm j$.

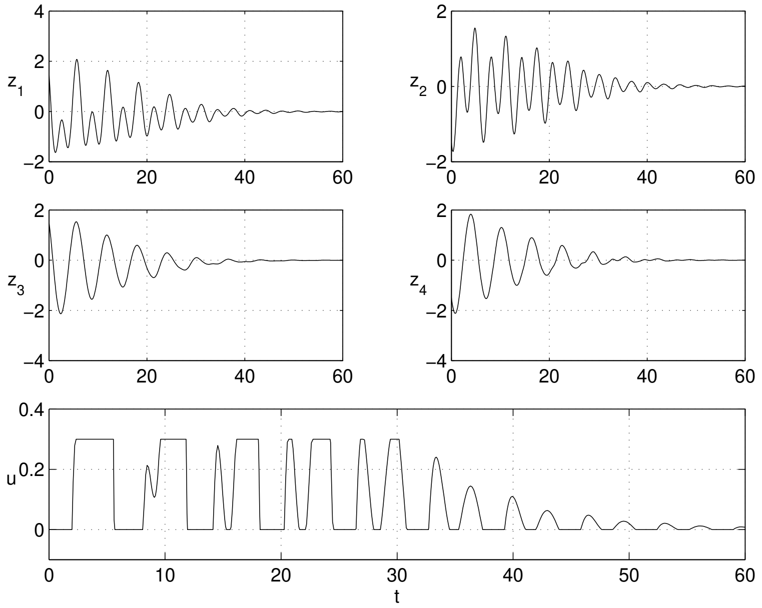


Figure 3: A 4^{th} order linear system with 2 pairs of open loop imaginary eigenvalues, in closed loop with a positive control law of the form (47). State histories on the top and control action on the bottom. Note that $u(t) \geq 0$, for all $t \geq 0$.

5 Stabilization with sensors saturations

In this section we consider the asymptotic stabilization of linear stable systems for which the measured output is subject to a constraint, for example the case where the measurement device has some range limitations.

Consider a SISO linear system with saturated output, namely

$$\begin{aligned} \dot{z} &= Jz + \bar{H}w \\ \eta &= \sigma(Kz), \end{aligned} \tag{50}$$

[8]See Footnote 6.

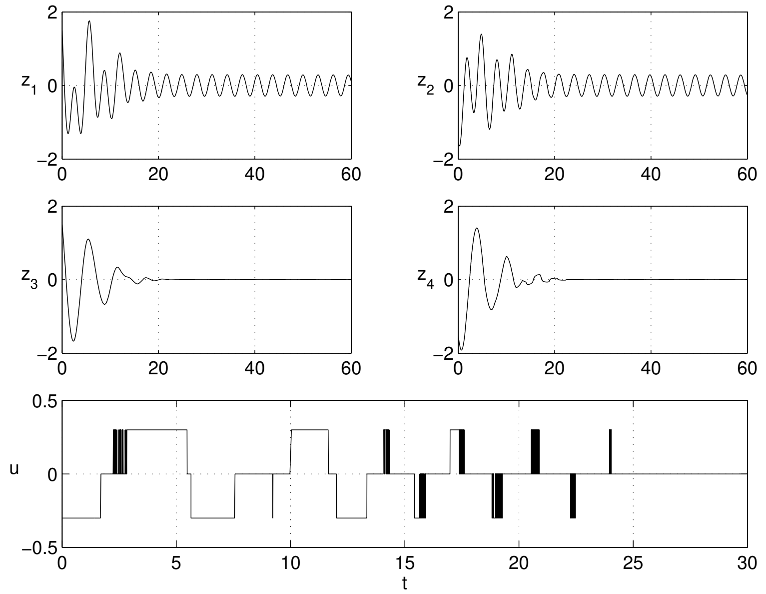


Figure 4: A 4^{th} order linear system with 2 pairs of open loop imaginary eigenvalues, in closed loop with a quantized control law of the form (48). State histories on the top and the control action on the bottom. Note that $u \in \{-0.3,\, 0,\, 0.3\}$ and $u = 0$, for all $t \geq 25s$.

with J such that (A2) is satisfied (*i.e.* $J + J' \leq 0$). Goal of this section is to show that system (50) is globally asymptotically stabilizable by dynamic output feedback, as illustrated in the following Proposition.

Proposition 3 *Consider system (50) with $z \in \mathbb{R}^p$, $w \in \mathbb{R}$, $\eta \in \mathbb{R}$, and J and $\sigma(\cdot)$ such that (A2) and (C2) hold. Assume that the pair $\{J', \bar{H}\}$ is controllable and the pair $\{K, J\}$ is observable. Then there exist matrices $\Gamma \in \mathbb{R}^{1\times p}$, $G \in \mathbb{R}^{p\times 1}$ and a Hurwitz matrix $F \in \mathbb{R}^{p\times p}$ such that (50) in closed loop with the dynamic controller*

$$\begin{array}{rcl} \dot{\xi} & = & F\xi - G\eta \\ w & = & \Gamma\xi \end{array} \tag{51}$$

is globally asymptotically (locally exponentially) stable.

Proof. It is trivial to verify that the closed loop system (50)-(51) is described by equations of the form

$$\begin{array}{rcl} \dot{z} & = & Jz + \bar{H}\Gamma\xi \\ \dot{\xi} & = & F\xi + Gu \\ u & = & -\sigma(Kz), \end{array} \tag{52}$$

i.e. it is the feedback interconnection of a system of the form (13) with $H = \bar{H}\Gamma$ and $R = 0$, and the nonlinear feedback $u = -\sigma(Kz)$. Hence, selecting Γ, G and a Hurwitz matrix F as in the proof of Lemma 2 and using arguments similar to those in the proof of Proposition 2, it follows that the interconnection is globally asymptotically stable.

$\lhd$

Remark 13 Proposition 3 can be easily extended, using the same arguments as in the proof of Corollary 2 and Corollary 3, to the case $\sigma(\cdot) = \sigma_+(\cdot)$, provided that $\det(J) \neq 0$, or to the case $\sigma(\cdot) = \sigma_q(\cdot)$, if one is interested in practical, rather than asymptotic, stability.

6 Applications

In this section we consider three applications of the main results of Section 4, namely the global stabilization of a chain of integrators with bounded input, the global asymptotic stabilization of linear null controllable systems by positive (negative) control and the global stabilization of the benchmark TORA system.

6.1 Stabilization of a chain of integrators with bounded control revisited

The problem of global asymptotic stabilization of a chain of integrators with bounded control has been extensively studied by several researchers. In this section we revisit it, and in the light of the results of Propositions 1 and 2, we present a novel stabilizing bounded control law, complete with some remarks on its robustness.

Proposition 4 *Let λ_1, λ_2, ..., λ_{n-1}, λ_n be positive numbers such that the polynomials*

$$s^{n-i} + \lambda_n s^{n-i-1} + \cdots + \lambda_{i+2} s + \lambda_{i+1}$$

for $i = n-1, \cdots, 0$ are Hurwitz and have all real roots. Then the system

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \vdots & & \\ \dot{x}_{n-1} & = & x_n \\ \dot{x}_n & = & u \end{array} \tag{53}$$

in closed loop with

$$u = -\frac{\epsilon}{2}\sigma(\frac{2\lambda_n}{\epsilon}x_n) - \frac{\epsilon}{4}\sigma(\frac{4\lambda_{n-1}}{\epsilon}x_{n-1}) - \frac{\epsilon}{8}\sigma(\frac{8\lambda_{n-2}}{\epsilon}x_{n-2}) - \ldots - \frac{\epsilon}{2^n}\sigma(\frac{2^n\lambda_1}{\epsilon}x_1) + w \tag{54}$$

is LES and ISS with the restriction $|w| < \frac{\epsilon}{2^{n+1}}$. Moreover, if $w = 0$, $|u| < \epsilon$.

Proof. The proof can be carried out iteratively. For, set

$$u = -\frac{\epsilon}{2}\sigma(\frac{2\lambda_n}{\epsilon}x_n) + v_{n-1}$$

and note that the system

$$\begin{array}{rcl} \dot{x}_{n-1} & = & x_n \\ \dot{x}_n & = & -\frac{\epsilon}{2}\sigma(\frac{2\lambda_n}{\epsilon}x_n) + v_{n-1} \end{array} \tag{55}$$

satisfies the assumptions of Proposition 1 for every $\lambda_n > 0$. It is also obvious that the closed loop system will be ISS with restriction $|v_{n-1}| < \frac{\epsilon}{2}$. As a result

$$v_{n-1} = -\frac{\epsilon}{4}\sigma(\frac{4\lambda_{n-1}}{\epsilon}x_{n-1}) + v_{n-2},$$

achieves ISS of system (55), with the restriction $|v_{n-2}| < \frac{\epsilon}{4}$, and LES for $v_{n-2} = 0$.

The proof is then completed, noting that at each step $i = n, \cdots, 1$ it is possible to find $\lambda_i > 0$ such that the transfer function of the linear approximation of the closed loop sub-system of (53) with state $(x_i, \cdots, x_n)$, input v_{i-1} and output $y = x_i$ has no zeros, and all real negative poles. Hence, the observation in Remark 1 can be used. Note also, that by assumption the polynomial

$$s^n + \lambda_n s^{n-1} + \cdots + \lambda_2 s + \lambda_1$$

is stable, therefore the closed loop system is LES. Finally, by a trivial property of the geometric series, if $w = 0$,

$$|u| \leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \ldots + \frac{\epsilon}{2^n} + \frac{\epsilon}{2^{n+1}} < \epsilon,$$

and ϵ can be selected arbitrarily small. $\lhd$

Remark 14 The requirement that the polynomials

$$p_i(s) = s^{n-i} + \lambda_n s^{n-i-1} + \cdots + \lambda_{i+2} s + \lambda_{i+1}$$

for $i = n-1, \cdots, 0$ have all negative real eigenvalues can easily be satisfied. For, note that, if the polynomial $p_{i-1}(s) = s^{n-i-1} + \lambda_n s^{n-i-2} + \cdots + \lambda_{i+2}$ has real negative roots, all branches of the root locus of the transfer function

$$\frac{1}{sp_{i-1}(s)}$$

contain some parts on the real axis. Hence, a selection of a small enough (feedback gain) $\lambda_{i+1} > 0$ results in a polynomial $p_i(s)$ with all roots in the negative real axis.

Remark 15 System (53) is a special case of the class of systems studied in [15]. Therein, a similar construction has been performed. However, in the proposed design the *saturating gains*, namely $\frac{\epsilon}{2}$, $\frac{\epsilon}{4}$, ..., are constants, whereas in [15] the gains are functions of the state and have to satisfy some non-trivial conditions. Finally, for large values of $\|x\|$ the *saturating gains* in [15] tend to zero, and this is not the case for the control law (54).

The result in Proposition 4 can be easily extended to a larger class of systems, namely nonlinear chains of integrators described by equations of the form

$$\begin{array}{rcl} \dot{x}_1 & = & \phi_1(x_2) \\ \vdots & & \\ \dot{x}_{n-1} & = & \phi_{n-1}(x_n) \\ \dot{x}_n & = & \phi_n(u) \end{array} \tag{56}$$

with $d\phi_i(0) > 0$, for all $i = 1, \cdots, n$. For illustration purposes consider the system described by the equations

$$\begin{array}{rcl} \dot{x}_1 & = & \sin(x_2) \\ \dot{x}_2 & = & \sin(x_3) \\ \dot{x}_3 & = & \sin(x_4) \\ \dot{x}_n & = & \sin(u). \end{array} \tag{57}$$

In Figure 5 the response of system (57) in closed loop with

$$u = -\frac{\epsilon}{2}\sigma(\frac{2\lambda_4}{\epsilon}x_4) - \frac{\epsilon}{4}\sigma(\frac{4\lambda_3}{\epsilon}x_3) - \frac{\epsilon}{8}\sigma(\frac{8\lambda_2}{\epsilon}x_2) - \frac{\epsilon}{16}\sigma(\frac{16\lambda_1}{\epsilon}x_1) \tag{58}$$

is presented. For this particular case, global asymptotic stabilization can only be achieved if $\epsilon \in (0, \epsilon^\star]$ with $\epsilon^\star < \frac{\pi}{2}$. In the particular simulations $\epsilon = \frac{\pi}{4}$ is used.

Remark 16 Output feedback stabilization of system (53) with output $\eta = x_1$ can be addressed by a straightforward application of Proposition 4 and [27, Theorem 7.1]. It is noted that the system with output $\eta = x_1$ is observable and, for such a system, an exponentially converging observer can be implemented.

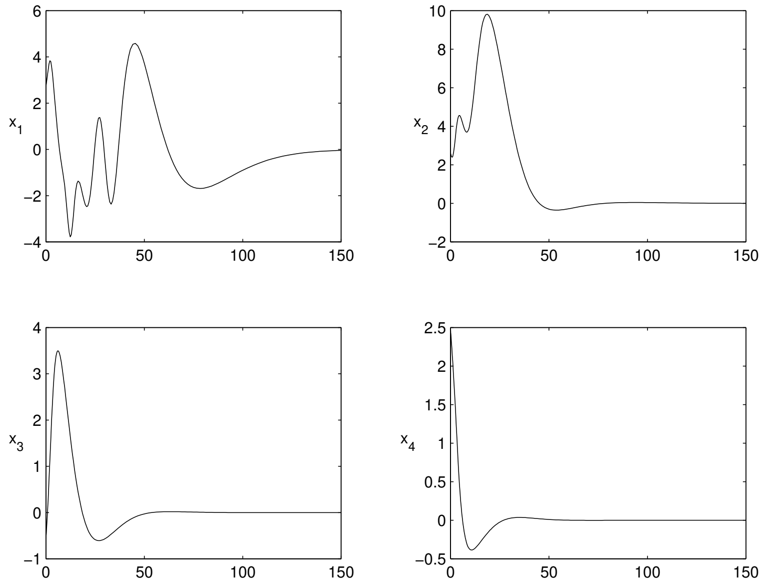


Figure 5: State histories of the closed loop system (57)–(58).

Finally, as a consequence of the result in Proposition 4 we state the following results.

Corollary 5 *Consider system (53) in closed loop with*

$$u = -\frac{\epsilon}{2}\sigma(\frac{2\lambda_n}{\epsilon}x_n(t-\tau_n)) - \frac{\epsilon}{4}\sigma(\frac{4\lambda_{n-1}}{\epsilon}x_{n-1}(t-\tau_{n-1})) - \cdots - \frac{\epsilon}{2^n}\sigma(\frac{2^n\lambda_1}{\epsilon}x_1(t-\tau_1)). \quad (59)$$

Then for every $\epsilon > 0$ and for every set of positive constants $\tau_1, \cdots, \tau_n$ there exist a set of positive constants $\lambda_1, \cdots, \lambda_n$ such that the closed loop system (53)-(59) is GAS - LES.

For illustration, we depict in Figure 6 the state histories of a chain of three integrators with output delays $\tau_1 = \tau_2 = \tau_3 = 1sec$, in closed loop with (59) with $\epsilon = 0.8$ and $[\lambda_1, \lambda_2, \lambda_3] = [0.015, 0.25, 1]$.

Corollary 6 *Consider the system described by the equations*

$$\begin{array}{rcl} \dot{x}_1 & = & \alpha_1 x_2 \\ \vdots & & \\ \dot{x}_{n-1} & = & \alpha_{n-1}x_n \\ \dot{x}_n & = & \alpha_n u, \end{array} \quad (60)$$

with $0 < \underline{\alpha}_j < \alpha_j < \overline{\alpha}_j$, $j = 1, \ldots, n$. Suppose that all $\underline{\alpha}_j$, $\overline{\alpha}_j$ are known and that there exist positive numbers λ_1, λ_2, $\ldots$, λ_{n-1}, λ_n such that the polynomials

$$s^{n-i} + \alpha_n\lambda_n s^{n-i-1} + \cdots + (\alpha_{i+2}\cdots\alpha_n)\lambda_{i+2}s + (\alpha_{i+1}\cdots\alpha_n)\lambda_{i+1}$$

for $i = n-1, \cdots, 0$ are Hurwitz and have all real roots for all $\alpha_j \in [\underline{\alpha}_j, \overline{\alpha}_j]$, $j = 1, \ldots, n$. Then the system (60) in closed loop with the control law (54) is LES and ISS with the restriction $|w| < \frac{\epsilon}{2^{n+1}}$. Moreover, if $w = 0$, $|u| < \epsilon$.

Robust stabilization of system (60) in the presence of uncertain system parameters has also been studied in [14]. The nested saturation scheme employed there also required some nontrivial algebraic conditions to be satisfied.

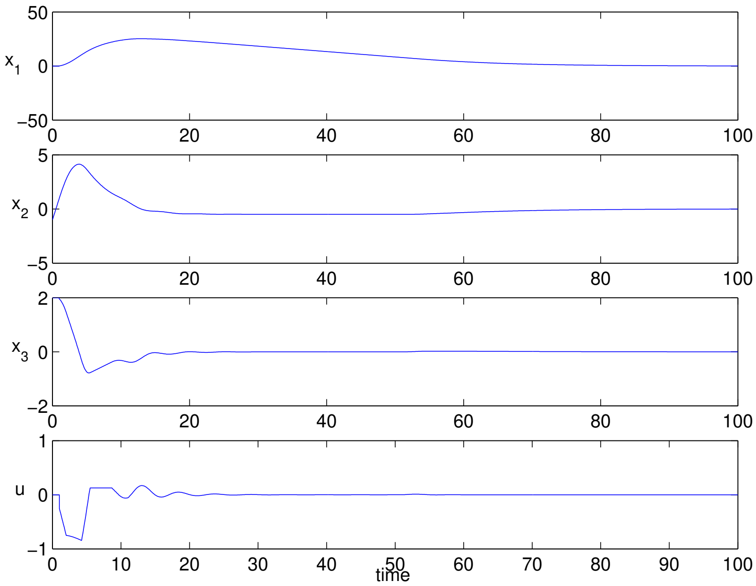


Figure 6: State histories and control action of the closed loop system (53)–(59).

6.2 Asymptotic stabilizability by control of constant sign

In this section we present a general result on the stabilizability of linear systems with bounded control of constant sign, that comes as a consequence of Proposition 2 and Corollary 4.

Proposition 5 *Any linear system*

$$\dot{x} = Ax + Bu \tag{61}$$

which is null controllable with bounded control and such that $\det(A) \neq 0$, *is* asymptotically stabilizable *by positive (or negative) control.*

Proof. Note first that because it is assumed that $\det(A) \neq 0$, the matrix A has no zero eigenvalue.

All null controllable with bounded control linear systems can be decomposed in the following structure

$$\begin{array}{rcl} \dot{z}_1 & = & J_1 z_1 + h_1(z_2, z_3, \cdots, \xi) + r_1 u \\ \dot{z}_2 & = & J_2 z_2 + h_2(z_3, \cdots, \xi) + r_2 u \\ & \vdots & \\ \dot{z}_k & = & J_k z_k + h_k(\xi) + r_k u \\ \dot{\xi} & = & F\xi + gu, \end{array} \tag{62}$$

where for $i = 1, \cdots, k$, $z_i \in \mathbb{R}^{p_i}$, $\xi \in \mathbb{R}^m$, $\sum_i p_i + m = n$, and $J_i + J_i' \leq 0$, $\det(J_i) \neq 0$. The last equation of the cascade (62) represents the asymptotically stable part of (61), if there is any, *i.e.* F is Hurwitz.

In the case that such an asymptotically stable part does not exist, note that according to Corollary 4 the following subsystem

$$\dot{z}_k = J_k z_k + r_k u \tag{63}$$

is *globally asymptotically stabilized* with the control law

$$u = -\frac{\epsilon}{2}\sigma_+(\frac{2}{\epsilon}K_k z_k) \quad \left(\text{or} \quad u = \frac{\epsilon}{2}\sigma_+(-\frac{2}{\epsilon}K_k z_k)\right) \tag{64}$$

for all $\epsilon > 0$ and for some appropriately chosen $K_k \in \mathbb{R}^{1\times p_k}$. According to Proposition 2, a similar control law[9] would also stabilize the cascade of the two last equations of (62) if the asymptotically stable part exists. Then, after recursive application of Proposition 2 we see that there exist $K_1,\ K_2, \cdots, K_k$ such that (62) is *globally asymptotically stabilized* by

$$\begin{array}{c} u = -\frac{\epsilon}{2}\sigma_+(\frac{2}{\epsilon}K_k z_k) - \cdots - \frac{\epsilon}{2^{k-1}}\sigma_+(\frac{2^{k-1}}{\epsilon}K_2 z_2) - \frac{\epsilon}{2^k}\sigma_+(\frac{2^k}{\epsilon}K_1 z_1) \\ \left(\text{or} \ \ u = \frac{\epsilon}{2}\sigma_+(-\frac{2}{\epsilon}K_k z_k) + \cdots + \frac{\epsilon}{2^{k-1}}\sigma_+(-\frac{2^{k-1}}{\epsilon}K_2 z_2) + \frac{\epsilon}{2^k}\sigma_+(-\frac{2^k}{\epsilon}K_1 z_1)\right). \end{array} \tag{65}$$

Moreover, it is trivial to see that $u(t) \leq 0$ (or $u(t) \geq 0$), for all $t \geq 0$. ◁

Remark 17 The results in Proposition 5 and Corollary 4 should be examined in the light of what established in [23], where it was proven that a linear system $\dot{x} = Ax + Bu$ is locally controllable at the origin with $u(t) \in [0, 1]$, for all t, if and only if the pair $\{A, B\}$ is controllable in the ordinary sense and all eigenvalues of A have nonzero imaginary parts. Using [23] and [3] it is easy to show that a linear system is asymptotically controllable with positive (or negative) bounded control if and only if $\mathbf{spec}(A) \subset \mathbf{C}^- \cup \{\mathbf{C^0} \setminus \{0\}\}$.

Remark 18 It is easy to verify that the control law described by equation (65) will also globally asymptotically (locally exponentially) stabilize the nonlinear version of system (62), *i.e.* when $h_i(z_{i+1}, \cdots, \xi)$ and $r_i = r_i(z_{i+1}, \cdots, \xi)$ are nonlinear functions and the ξ-subsystem is an ISS and LES system described by $\dot{\xi} = f(\xi) + g(\xi)u$.

6.3 Asymptotic stabilization of the TORA

In this section we apply the results of Section 4 to solve the asymptotic stabilization problem for the TORA [2]. After appropriate normalized transformations [30], the dynamics of the system are described by the equations

$$\begin{array}{rcl} \ddot{x}_d + x_d & = & \varepsilon(\dot{\phi}^2 \sin\phi - \ddot{\phi}\cos\phi) \\ \ddot{\phi} & = & u - \varepsilon\ddot{x}_d\cos\phi, \end{array} \tag{66}$$

where x_d is the translational position, $v_d = \dot{x}_d$ the translational velocity, ϕ the angular position, $\omega = \dot{\phi}$ the angular velocity and $0 < \varepsilon < 1$ a constant depending on the physical parameters of the device. The presence of the term in ω^2 in the model makes the stabilization of the system an intricate problem, especially considering that an ideal control law would utilize measurements of the translational and angular positions only. Consider (66) and the transformation [9],

$$\begin{array}{rcl} x_1 & = & x_d + \varepsilon\sin\phi \\ x_2 & = & v_d + \varepsilon\omega\sin\phi \\ x_3 & = & \phi \\ x_4 & = & \psi(x_3)\omega \end{array} \tag{67}$$

where

$$\psi(x_3) = -\sqrt{\frac{1-\varepsilon^2\cos^2 x_3}{1-\varepsilon^2}}.$$

[9] Not necessarily with the same K_k.

Then, in the new coordinates the system is described by the equations

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_1 + \varepsilon \sin x_3 \\ \dot{x}_3 & = & \frac{1}{\psi(x_3)} x_4 \\ \dot{x}_4 & = & \frac{\psi(x_3)}{1-\varepsilon^2 \cos^2 x_3} u + \frac{\varepsilon(x_1 - \varepsilon \sin x_3)\cos x_3}{1-\varepsilon^2 \cos^2 x_3} \psi(x_3). \end{array} \tag{68}$$

x_1 and x_3, which are functions of the translational and angular positions only, are the measured variables. In [9] global output feedback stabilization and tracking was achieved with a combination of a nonlinear observer and backstepping. Here, we propose a simpler output feedback scheme. First, consider the preliminary feedback transformation

$$u = u(v, x_1, x_3) = \frac{1 - \varepsilon^2 \cos^2 x_3}{\psi(x_3)} v - \varepsilon(x_1 - \varepsilon \sin x_3) \cos x_3, \tag{69}$$

and the subsystem

$$\begin{array}{rcl} \dot{x}_3 & = & \alpha(x_3) x_4 \\ \dot{x}_4 & = & v, \end{array} \tag{70}$$

with $\alpha(x_3) = \frac{1}{\psi(x_3)}$ and output $y = x_3$. Stabilization of (70) can be achieved using the dynamic output feedback

$$\begin{array}{rcl} \dot{\hat{\theta}} & = & -(\lambda(x_3)+1)\hat{\theta} - (\lambda(x_3)+1)\beta(x_3) - x_3\alpha(x_3) + (b-1)w \\ v & = & -x_3\alpha(x_3) - \hat{\theta} - \beta(x_3) + w, \end{array} \tag{71}$$

where w is a new control variable to be used in the next step, $\beta(x_3)$ and $\lambda(x_3)$ are functions satisfying

$$\frac{\partial \beta(x_3)}{\partial x_3} \alpha(x_3) = \lambda(x_3), \quad \lambda(x_3) > 0 \tag{72}$$

and b is a constant to be selected later. Asymptotic stabilization can be proven considering the new coordinate

$$z = \hat{\theta} + \beta(x_3) - x_4,$$

and noting that system (70)-(71) can be written as

$$\begin{array}{rcl} \dot{x}_3 & = & \alpha(x_3) x_4 \\ \dot{x}_4 & = & -x_3\alpha(x_3) - x_4 - z + w \\ \dot{z} & = & -\lambda(x_3) z + bw, \end{array} \tag{73}$$

which is LES-ISS. Condition (72) can be seen either as a differential equation for the definition of $\beta(x_3)$, if $\lambda(x_3)$ is selected by the designer, or as the definition of $\lambda(x_3)$, if $\beta(x_3)$ is selected. For example, $\beta(x_3) = -\beta x_3$ with $\beta > 0$ is a simple choice.
Consider now the cascade

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_1 + \varepsilon \sin x_3 \\ \dot{x}_3 & = & \alpha(x_3) x_4 \\ \dot{x}_4 & = & -x_3\alpha(x_3) - x_4 - z + w \\ \dot{z} & = & -\lambda(x_3) z + bw, \end{array} \tag{74}$$

that results from the first two of equations (68) and the system (73) and note that it satisfies assumptions (A1) and (A2), hence can be asymptotically stabilized with bounded control of the form

$$w = -\epsilon\sigma\left(\frac{1}{\epsilon} K \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right), \tag{75}$$

with K selected as in Lemma 1. However, to obtain an output feedback controller, K has to be of the form $K = [k_1 \;\; 0]$, for some k_1. We now show that such a K exists. For, consider the approximation of (74) for small $\|[x_3 \; x_4 \; z]'\|$, and define the matrices

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 0 \\ \varepsilon & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & -1 & -1 \\ 0 & 0 & -\lambda \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \\ b \end{bmatrix}, \tag{76}$$

where $\alpha = \alpha(0)$ and $\lambda = \lambda(0)$. ¿From Lemma 1

$$K = \chi G'\bar{Y} = \chi[\bar{Y}_{1,2} + b_3\bar{Y}_{1,3} \quad \bar{Y}_{2,2} + b_3\bar{Y}_{2,3}]$$

where χ is a positive constant, $\bar{Y}$ is the matrix that solves the Sylvester equation

$$H + \bar{Y}F + J'\bar{Y} = 0$$

and $\bar{Y}_{i,j}$ is the (i,j) entry of $\bar{Y}$. Selecting

$$b = -\frac{\bar{Y}_{2,2}}{\bar{Y}_{2,3}}$$

yields $K = [k_1 \;\; 0]$, hence $w = w(x_1)$.
The result is summarized in the following statement.

Proposition 6 *Consider system (68) and a nonlinearity $\sigma(\cdot) = \sigma_s(\cdot)$ or $\sigma(\cdot) = \sigma_+(\cdot)$, belonging to the sector $[0,1]$. There exist constants b, k_1, $\epsilon \in \mathbb{R}$, with $\epsilon > 0$ and a positive function $\lambda(x_3)$ such that system (68) in closed loop with the dynamic output feedback controller*

$$\begin{aligned} \dot{\hat{\theta}} &= -(\lambda+1)\hat{\theta} - (\lambda+1)\beta(x_3) - x_3\alpha(x_3) - (b-1)\epsilon\sigma\left(\frac{1}{\epsilon}k_1x_1\right) \\ v &= -x_3\alpha(x_3) - \hat{\theta} - \beta(x_3) - \epsilon\sigma\left(\frac{1}{\epsilon}k_1x_1\right) \\ u &= \frac{1-\varepsilon^2\cos^2 x_3}{1-\psi(x_3)}v - \varepsilon(x_1 - \varepsilon\sin x_3)\cos x_3 \end{aligned} \tag{77}$$

is GAS (LES).

The control law (77) is much simpler in structure and implementation than the output feedback designs proposed in [9] or [18], while in [24] only state feedback is considered. In Figure 7 some simulation results of the closed loop with the proposed controller are depicted. For the simulations we have used, as in [24], $\varepsilon = 0.1$, so that the results are directly comparable with the ones given in this reference. It can be concluded that full state feedback does not outperform the output feedback presented here.

7 Conclusions

The problem of stabilization of a class of cascaded systems with bounded control has been addressed and solved using the linear bounded real lemma and a generalized version of the small gain theorem. Globally asymptotically stabilizing control laws that require only partial state feedback have been designed. These control laws make use of typical saturating functions, constant sign saturations or quantizations. The general result is

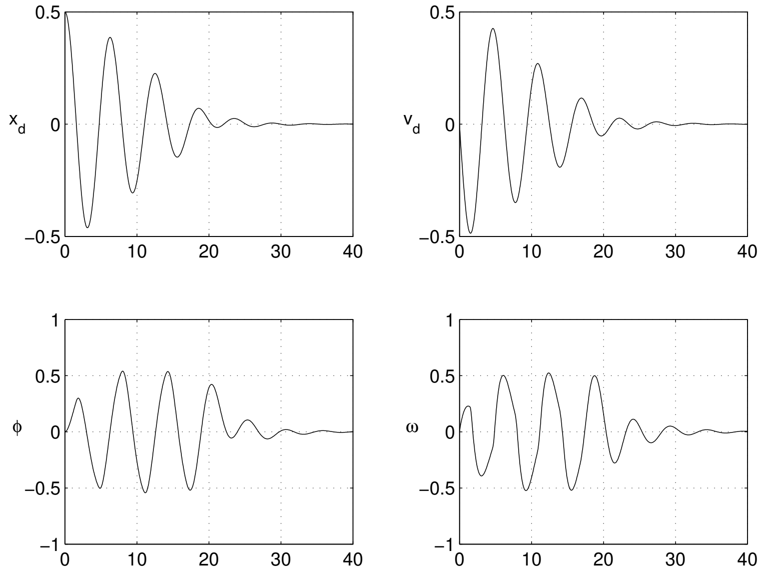


Figure 7: State histories of the closed loop system (68)-(77).

applied to the global stabilization problem for a chain of integrators subject to input saturation, yielding a control law that is significantly different from existing results and also to the global stabilization of the nonlinear benchmark system of TORA. At the same time, the new stabilization scheme provides motivation for a dynamic output feedback stabilization methodology, which can accommodate saturated outputs. This dynamic solution is clearly different from observation-based schemes available in the literature.

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