# Control of Integral Processes with Dead Time

# Part II: Quantitative Analysis*

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**Abstract**

Several different control schemes for integral processes with dead time resulted in the same disturbance response. Moreover, it has already been shown that such a response is sub-ideal. Hence, it is quite necessary to quantitatively analyze the achievable specifications and the robust stability regions. This paper is devoted to do so. As a result, the control parameter can be quantitatively determined with compromise between the disturbance response and the robustness. Four specifications — (normalized) maximal dynamic error, maximal decay rate, (normalized) control action bound and approximate recovery time — are given to characterize the step-disturbance response. It shows that any attempt to obtain a (normalized) dynamic error less than $\tau_m$ is impossible and a sufficient condition on the (relative) gain-uncertainty bound is $\frac{\sqrt{3}}{2}$.

**Index Terms**: Dead-time compensator, robustness, integral processes with dead time, disturbance observer, Smith predictor

## 1 Introduction

In recent years, the control of processes with an integrator and long dead time has received much attention because of the instability of the processes. Watanabe and Ito [1] presented *process model control* for general unstable processes with dead time. Åström *et al.* [2] introduced a novel structure to decouple the disturbance response from the setpoint response and hence they can be designed separately. Zhang and Sun [3] presented a much easier tuning formula for it. Matausek and Micic presented a modified SP by introducing a minor loop to stabilize the processes with a proportional gain [4] and a high-pass filter [5]. Normey-Rico and Camacho [6] analyzed the robustness of a modified structure of Watanabe-Ito's. Majhi and Atherton [7] presented another scheme for such a system as well as stable/unstable systems. In Part I [8] of this series of papers, a disturbance

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observer-based control scheme was shown to be an effective way to control such processes. The controller can be designed to meet many requirements.

It is very interesting that, although the structures proposed in [3, 6] are different from the control scheme presented in Part I [8], the resulted disturbance responses are the same in essence when the controllers are properly tuned. Further research has shown that such a response is sub-ideal [9]. Hence, it is quite necessary to quantitatively analyze the system performance. The relationship between the control parameter and the achievable specifications and the robust stability regions will be clearly shown in this paper, based on the disturbance observer-based control scheme studied in Part I. The control parameter can hence be decided by a given allowable maximal dynamic error, in terms of the *Lambert W* function [10]. For a given control parameter, a control action bound is required. The maximal decay rate, based on which the recovery time is approximately obtained, is given to measure the responding speed to the disturbance. For a given gain uncertainty, dead-time uncertainty or both of them, the corresponding stability region is derived. Hence, the control parameter can be analytically determined with compromise between robustness and the disturbance response. A sufficient condition on the relative gain-uncertainty bound is $\frac{\sqrt{3}}{2}$.

The rest of the paper is organized as follows. The achievable specifications of the disturbance response are quantitatively analyzed in Section 2, in addition to a short review of the disturbance observer-based control scheme. The robust stability regions for gain uncertainties, dead-time uncertainties and both of them are analyzed in Section 3. The stability of the controller itself is analyzed in Section 4. Conclusions are made in Section 5.

## 2 Achievable Specifications of the Sub-ideal Disturbance Response

The integral processes with dead time (IPDT) can be described as,

$$G(s) = G_p(s)e^{-\tau s} = \frac{k}{s}e^{-\tau s}, \tag{1}$$

where the pure dead time $\tau$ and the static gain $k$ are all positive.

The disturbance observer-based control scheme is revisited in Figure 1(a) for the readers' convenience. $\tau_m$ is the estimated dead time and $G_m(s)$ is the nominal delay-free part. $C(s)$ is designed as a proportional controller $C(s) = \frac{1}{kT}$ and $Q(s)$ is a low-pass filter

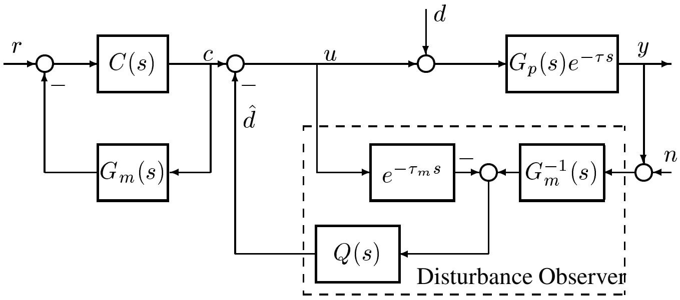
$$Q(s) = \frac{(2\lambda + \tau_m)s + 1}{(\lambda s + 1)^2} \tag{2}$$

designed to reject step disturbances without a static error, where $\lambda$ is a design parameter to trade-off the disturbance rejection, robustness and, possibly, the control action bound. A casual and internally stable, equivalent structure is shown in Figure 1(b).

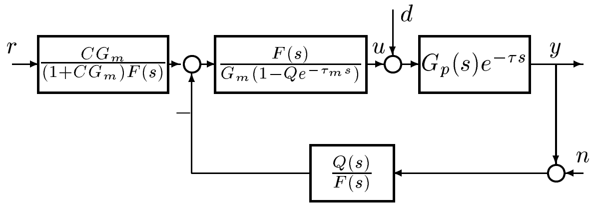
Under the nominal condition, $G_m(s) = G_p(s) = \frac{k}{s}$ and $\tau_m = \tau$, the setpoint response and the disturbance response are respectively:

$$G_{yr}(s) = \frac{1}{Ts+1}e^{-\tau_m s},$$

$$G_{yd}(s) = \frac{k}{s}\left(1 - Q(s)e^{-\tau_m s}\right)e^{-\tau_m s}. \tag{3}$$




(a) Original structure for design




(b) Equivalent structure for implementation

Figure 1: Disturbance observer-based control scheme

It is quite easy to evaluate the achievable specifications of the set-point response. However, it is not that easy to evaluate the achievable specifications of the disturbance response. The disturbance response obtained using (2) is sub-ideal [9]. A typical step disturbance response is shown in Figure 2. Four specifications — maximal dynamic error, maximal decay rate, control action bound and approximate recovery time — will be given to characterize the disturbance response.

## 2.1 Maximal dynamic error

**Theorem 1** For the process (1) controlled as in Figure 1, the (normalized) maximal dynamic error $e_m$ [1] under a step disturbance is not less than $\tau_m$, *i.e.*, $e_m \geq \tau_m$ .

**Proof.** The normalized response, $y_{d0}(t) = \frac{y_d(t)}{d_0 k}$, of a step disturbance $d_0$, disturbing at $t_0 = -\tau$ without loss of generality, is

$$\begin{aligned} y_{d0}(t) &= \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(1 - Q(s)e^{-\tau_m s}\right)\right] \\ &= t - \mathcal{L}^{-1}\left[\frac{1}{s^2}Q(s)\right] \cdot 1(t-\tau_m) \end{aligned} \tag{4}$$

$y_{d0}(t)$ can be divided into the following three stages as shown in Figure 2:

- Stage I: the output is not affected due to the effect of the dead time in the process;
- Stage II: after dead time $\tau$ the step disturbance acts, the output responds proportionally to time $t$, during which the control action is delayed by the dead time;
- Stage III: after delay $\tau + \tau_m$, the control action starts to affect the output.

[1] $e$ without a subscript stands for, as usual, the exponential constant.

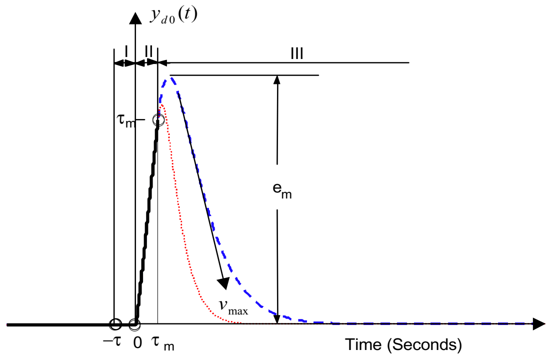


Figure 2: The normalized response of a step disturbance

The first two stages are not controllable. At the end of Stage II, the dynamic error reaches $\tau_m$. After that, the dynamic error maybe increase or decrease. Whatever it varies, the normalized maximal dynamic error is not less than $\tau_m$. This completes the proof.

**Remark:**

The attempt to obtain a dynamic error less than $\tau_m$ for such a system is impossible.

Substitute equation (2) into (4), the normalized response of a step disturbance can be obtained as

$$\begin{aligned} y_{d0}(t) &= t - \mathcal{L}^{-1}\left[\frac{(2\lambda+\tau_m)s+1}{s^2(\lambda s+1)^2}\right]\cdot 1(t-\tau_m) \\ &= \begin{cases} t & (0 \le t \le \tau_m) \\ (\frac{\lambda+\tau_m}{\lambda}t - \frac{\tau_m^2}{\lambda})e^{-\frac{t-\tau_m}{\lambda}} & (t \ge \tau_m) \end{cases}. \end{aligned} \tag{5}$$

**Theorem** 2 (Allowable Control Parameter $\lambda$) If process (1) is controlled as in Figure 1 and the low-pass filter is chosen in the form of equation (2), then for a given normalized maximal dynamic error $e_m > \tau_m$ the control parameter $\lambda$ should satisfy the condition

$$0 < \lambda \le \frac{\tau_m}{-W(-\frac{\tau_m}{e_m e})} - \tau_m, \tag{6}$$

where $W(\cdot)$ is the *Lambert W* function [10].

**Proof.** As stated in Theorem 1, the output can be affected only when $t > \tau_m$. Hence, we only consider the output in Stage III. It can be represented as

$$y_{d03}(t) = (\frac{\lambda+\tau_m}{\lambda}t - \frac{\tau_m^2}{\lambda})e^{-\frac{t-\tau_m}{\lambda}}. \tag{7}$$

$y_{d03}(t)$ reaches the peak value, *i.e.* the maximal dynamic error $e_m$:

$$e_m = y_{d03}(t_m) = (\beta+1)\tau_m e^{-\frac{\beta}{\beta+1}} \tag{8}$$

(where $\beta = \lambda/\tau_m$) at the moment

$$t_m = \frac{\lambda^2 + \lambda\tau_m + \tau_m^2}{\lambda+\tau_m} = \tau_m + \frac{\beta^2}{\beta+1}\tau_m.$$

Solve $\beta$ from (8), then

$$\beta = \frac{1}{-W(-\frac{\tau_m}{e_m e})} - 1 > 0.$$

Differentiating equation (8) with respect to $\beta$ or $\lambda$ shows that the maximal dynamic error $e_m$ is a monotonic increasing function with respect to $\lambda$. The larger the $\lambda$ the larger the $e_m$. This completes the proof.

## 2.2 Maximal decay rate

Differentiate equation (7) with respect to $t$ twice and let it be 0, then the maximal decay rate $v_{max}$ can be obtained as

$$v_{max} = -\frac{\lambda + \tau_m}{\lambda e} e^{-\frac{\lambda}{\lambda+\tau_m}} \tag{9}$$

at the moment $t_{vm} = \lambda + \tau_m + \frac{\lambda^2}{\lambda+\tau_m}$. $v_{max}$ is a monotonic increasing function with respect to $\lambda$ and can be served as a specification of the recovery time. The larger the $|v_{max}|$ is, the shorter the recovery time and the faster the disturbance response. There exists a minimum value $|v_{max}|_{min} = e^{-2}$ when $\lambda \to \infty$. This is the slowest one.

## 2.3 Control action bound

**Theorem** 3 (Control Action Bound) If process (1) is controlled by the above proposed method and the low-pass filter is chosen in the form of equation (2), then a (normalized) control action bound $1 + \frac{\lambda+\tau_m}{\lambda e} e^{-\frac{\lambda}{\lambda+\tau_m}}$ is required. In other words, for a given normalized control action bound $u_m$, the control parameter should satisfy the following condition:

$$\lambda \geq \frac{W(\frac{1/e}{u_m-1})}{1 - W(\frac{1/e}{u_m-1})}\tau_m. \tag{10}$$

**Proof.** The transfer function from disturbance $d$ to control action $u$ under the nominal case is

$$G_{ud}(s) = Q(s)e^{-\tau s}.$$

Assume that a step disturbance $d_0$, disturbing at $t_0 = -\tau$ without loss of generality, then the normalized control action, $u_0(t) = \frac{u(t)}{d_0}$, for $t \geq 0$ is

$$\begin{aligned} u_0(t) &= \mathcal{L}^{-1}\left[\frac{1}{s}Q(s)\right] \\ &= 1 + (\frac{\tau_m t}{\lambda^2} + \frac{t}{\lambda} - 1)e^{-\frac{t}{\lambda}}. \end{aligned}$$

$u_0(t)$ reaches the peak value, *i.e.* the normalized control action bound $u_m$:

$$u_m = 1 + \frac{\lambda + \tau_m}{\lambda e} e^{-\frac{\lambda}{\lambda+\tau_m}} \tag{11}$$

at the moment $t_{um} = \lambda + \frac{\lambda^2}{\lambda+\tau_m}$. For a given normalized control action bound $u_m$, the control parameter $\lambda$ can be solved as

$$\lambda = \frac{W(\frac{1/e}{u_m-1})}{1 - W(\frac{1/e}{u_m-1})}\tau_m.$$

It can be shown that $u_m$ is a monotonic decreasing function with respect to $\lambda$. Hence, for a given control bound $u_m > 0$, the control parameter should satisfy condition (10). This completes the proof.

**Remark:**

1. For the minimal maximal dynamic error $e_m \to \tau_m$, there should be $\lambda \to 0$ and $u_m \to \infty$. Hence, we cannot obtain so less dynamic error even under the nominal case because of the infinite control action. This means that only a sub-ideal disturbance response can be obtained.

2. There exists a low bound for $u_m$: $u_m \to 1 + e^{-2} \approx 1.14$ when $\lambda \to \infty$.

Combine Theorem 2 with Theorem 3, the following results are obtained:

**Corollary 1 :** For a given normalized maximal dynamic error $e_m > \tau_m$, a control action bound $u_m = 1 + \frac{e_m}{\lambda e}$ is required, where $\lambda$ is limited by Theorem 2.

**Corollary 2 :** For a given normalized maximal dynamic error $e_m > \tau_m$ and a normalized control action bound $u_m$, the control parameter should meet the following condition:

$$\frac{W(\frac{1/e}{u_m-1})}{1 - W(\frac{1/e}{u_m-1})}\tau_m \leq \lambda \leq \frac{\tau_m}{-W(-\frac{\tau_m}{e_m e})} - \tau_m. \tag{12}$$

## 2.4 Approximate recovery time

Combining equations (8) and (9), the following relation between $e_m$ and $v_{max}$ holds:

$$e_m = -\lambda e v_{max}.$$

This means if the response decays as a rate $v_{max}$, then it decays to 0 after $\lambda e$ from the peak time $t_m$. Assume that the average decay rate is half of the maximal decay rate, which is reasonable, then the recovery time $t_r$ under the nominal case can be approximated, to some extent, as the following formula:

$$\begin{aligned} t_r &\approx \tau + t_m + 2\lambda e \\ &= \left(1 + \beta + 2\beta e + \frac{1}{\beta+1}\right)\tau_m. \end{aligned}$$

Many simulations show that a more accurate and simpler formula for the $5\%$ error band is:

$$t_r \approx (7\beta + 2)\tau_m.$$

Hence, from the recovery-time point of view, it is better to choose $\beta = 0 \sim 1$, *i.e.* $\lambda = (0 \sim 1)\tau_m$.

For a given control parameter $\beta = \lambda/\tau_m$, the achievable specifications of the step disturbance response are shown in Figure 3.

# 3 Robust Stability Regions

Three cases of different uncertainties (gain uncertainties, dead-time uncertainties and both) will be considered below.

The complementary sensitivity function of the nominal system is

$$T(s) = Q(s)e^{-\tau_m s}.$$

According to the well-known small gain theorem, the closed-loop system is robustly stable if $\|Q(s)\|_\infty < \frac{1}{\|\Delta(s)\|_\infty}$ for a multiplicative uncertainty $\Delta(s) \in H_\infty$. Since the steady-state gain of $Q(s)$ is 1, which means there exists an integral effect in the controller, a multiplicative uncertainty does not affect the steady-state performance of the system.

## 3.1 Case 1: Gain uncertainties

The magnitude response of $Q(s)$ is

$$|Q(j\omega)| = \frac{\sqrt{1 + (2\lambda + \tau_m)^2\omega^2}}{1 + \lambda^2\omega^2}$$

and the peak value is

$$|Q(j\omega)|_{max} = \frac{(2\beta + 1)^2}{2\beta\sqrt{(\beta + 1)(3\beta + 1)}}.$$

Hence, the following theorem holds:

**Theorem 4** For a gain uncertainty $\Delta(s) = \frac{\Delta k}{k}$, the closed-loop system is robustly stable if $\frac{|\Delta k|}{k} < \frac{2\beta\sqrt{(\beta+1)(3\beta+1)}}{(2\beta+1)^2}$.

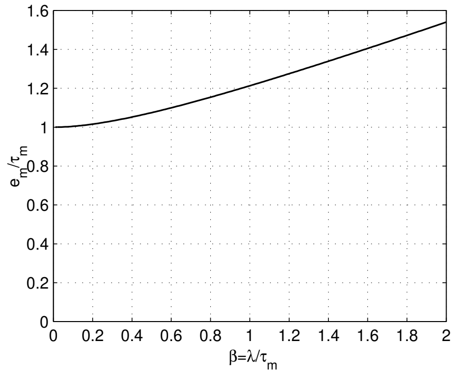
**Remark**:

When $\beta \to \infty$, $\frac{2\beta\sqrt{(\beta+1)(3\beta+1)}}{(2\beta+1)^2} \to \frac{\sqrt{3}}{2}$. This provides a bound, $\frac{\sqrt{3}}{2}$, for the gain uncertainty to guarantee the robust stability. This bound is about $0.63$ when $\beta = 1$. The relationship between the control parameter and the allowable gain uncertainty is shown in Figure 4(a).

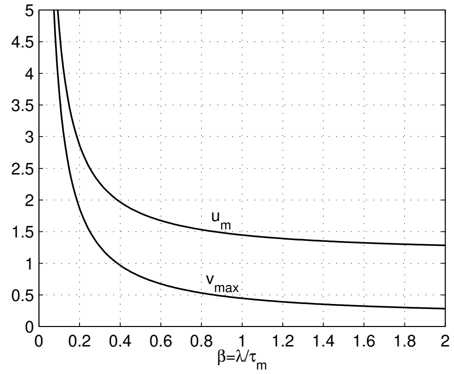
## 3.2 Case 2: Dead-time uncertainties

Assume there exists a dead-time uncertainty, *i.e.*

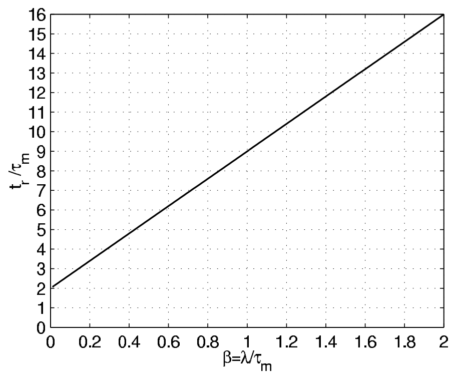
$$G_p(s) = \frac{k}{s} \text{ and } \tau = \tau_m + \tau_\Delta,\ (-\tau_m \leq \tau_\Delta).$$




(a) maximal dynamic error

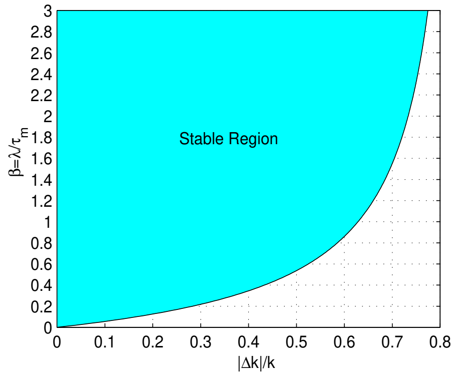



(b) control action bound and maximal decay rate

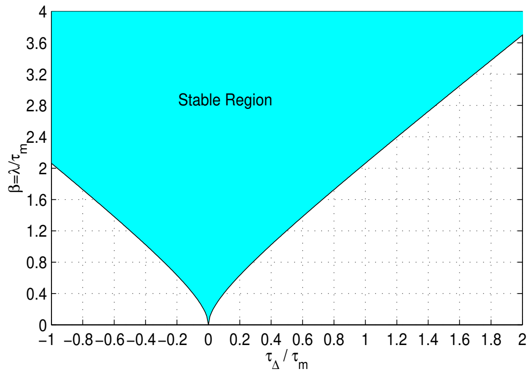



(c) approximate recovery time

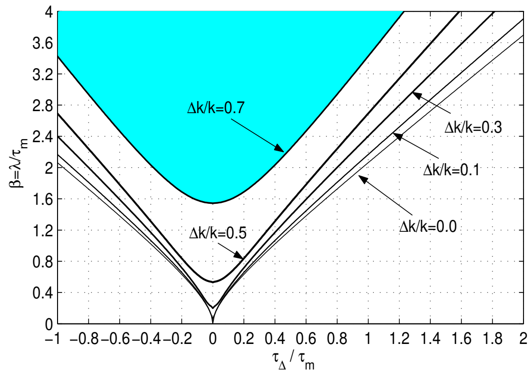
Figure 3: Achievable specifications of the disturbance response



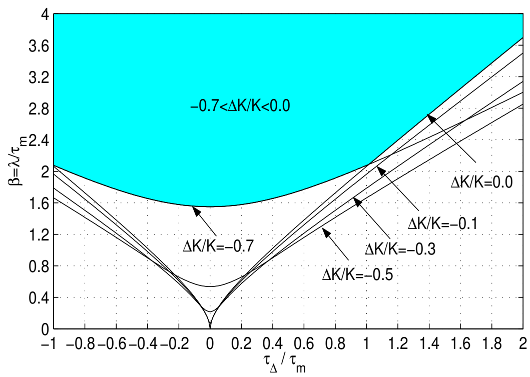

(a) Gain uncertainty




(b) Dead-time uncertainty




(c) Positive gain uncertainty with dead-time uncertainty




(d) Negative gain uncertainty with dead-time uncertainty

Figure 4: Control parameter *v.s.* uncertainty

This uncertainty can be converted to a multiplicative uncertainty $\Delta(s) = e^{-\tau_\Delta s} - 1$. Hence, the robust stability is guaranteed if the following condition is met:

$$\begin{aligned} |Q(j\omega)| &= \frac{\sqrt{1 + (2\lambda + \tau_m)^2\omega^2}}{1 + \lambda^2\omega^2} \\ &< \frac{1}{|e^{-j\tau_\Delta \omega} - 1|} = \frac{1}{\sqrt{2 - 2\cos(\tau_\Delta \omega)}} \ (\forall \omega > 0), \end{aligned}$$

That is,

$$\begin{aligned} 2 - 2\cos(\tau_\Delta \omega) &< \frac{(1 + \lambda^2\omega^2)^2}{1 + (2\lambda + \tau_m)^2\omega^2} \\ &= \frac{\left(1 + (\frac{\lambda}{\tau_\Delta})^2(\tau_\Delta \omega)^2\right)^2}{1 + (\frac{2\lambda + \tau_m}{\tau_\Delta})^2(\tau_\Delta \omega)^2} \\ &= \frac{\left(1 + (\frac{\beta}{\tau_\Delta/\tau_m})^2(\tau_\Delta \omega)^2\right)^2}{1 + (\frac{2\beta + 1}{\tau_\Delta/\tau_m})^2(\tau_\Delta \omega)^2} \ (\forall \omega > 0). \end{aligned}$$

There does not exist an analytical solution for $\beta$. Denote $\omega_\Delta = \tau_\Delta \omega$, $f_1(\omega_\Delta) = 2 - 2\cos(\omega_\Delta)$ and $f_2(\omega_\Delta) = \frac{\left(1 + (\frac{\beta}{\tau_\Delta/\tau_m})^2\omega_\Delta^2\right)^2}{1 + (\frac{2\beta+1}{\tau_\Delta/\tau_m})^2\omega_\Delta^2}$. Since $f_1(\omega_\Delta)$ and $f_2(\omega_\Delta)$ are all even functions with respect to $\omega_\Delta$, we only consider $\omega_\Delta \geq 0$ hereafter. The only intersection of $f_1(\omega_\Delta)$ and $f_2(\omega_\Delta)$ can be obtained when

$$f_1(\omega_\Delta) = f_2(\omega_\Delta) \quad \text{and} \quad \dot{f}_1(\omega_\Delta) = \dot{f}_2(\omega_\Delta).$$

This can be solved numerically. The solutions of the lower bound of $\beta$ to guarantee the robust stability for $\frac{\tau_\Delta}{\tau_m} \in [-1, 2]$ are shown in Figure 4(b). For a large dead-time uncertainty $\frac{\tau_\Delta}{\tau_m} > 1$, the lower bound of the control parameter is about $1.625\frac{\tau_\Delta}{\tau_m} + 0.45$.

### 3.3 Case 3: Dead-time and gain uncertainties

Assume that there exist uncertainties in both dead time $\tau$ and gain $k$, *i.e.*

$$G_p(s) = \frac{k}{s}(1 + \frac{\Delta k}{k}) \text{ and } \tau = \tau_m + \tau_\Delta, \ (-\tau_m \leq \tau_\Delta),$$

then the corresponding multiplicative uncertainty is

$$\Delta(s) = (1 + \frac{\Delta k}{k})e^{-\tau_\Delta s} - 1.$$

The robust stability is guaranteed if the following condition is met:

$$\frac{\sqrt{1 + (2\lambda + \tau_m)^2\omega^2}}{1 + \lambda^2\omega^2} < \frac{1}{\sqrt{1 + (1 + \frac{\Delta k}{k})^2 - 2(1 + \frac{\Delta k}{k})\cos(\tau_\Delta \omega)}} \ (\forall \omega > 0).$$

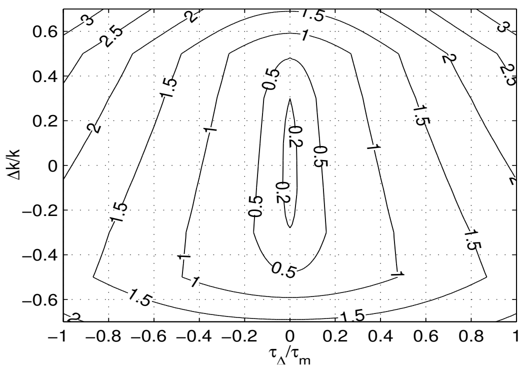



Figure 5: Rough contour of $\beta$ v.s. uncertainties

This inequality cannot be analytically solved, either. For some given gain uncertainties $\frac{\Delta k}{k}$, the relationship between the control parameter $\beta$ and the dead-time uncertainty can be obtained numerically, as shown in Figure 4(c, d). For a positive gain uncertainty together with a dead-time uncertainty, the stability region is sharp. The larger the gain uncertainty, the sharper the stability region. For a negative gain uncertainty together with a dead-time uncertainty, the stability region is blunt. The larger the (absolute) gain uncertainty, the blunter the stability region. For a gain uncertainty $q \leq \frac{\Delta k}{k} \leq p$ ($p, q \in [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$), the stability region is the intersection of the corresponding stability regions for $\frac{\Delta k}{k} = p$ and $\frac{\Delta k}{k} = q$. For example, the shading area in Figure 4(d) is the stability region for a gain uncertainty $-0.7 \leq \frac{\Delta k}{k} \leq 0.0$ while the shading area in Figure 4(c) is the stability region for a gain uncertainty $-0.7 \leq \frac{\Delta k}{k} \leq 0.7$ because the stability region for $\frac{\Delta k}{k} = 0.7$ is included by that of $\frac{\Delta k}{k} = -0.7$. In order to obtain a better view of the relationship, a rough contour of $\beta$ with respect to $\frac{\Delta k}{k}$ and $\frac{\tau_\Delta}{\tau_m}$ is shown in Figure 5. This is very useful for determining $\beta$ to meet the uncertainties. Figure 5 indicates that the nominal gain should be chosen near the maximal possible gain rather than the center of the gain range.

It should be pointed out that the stability regions obtained above are suffiecient conditions but not necessary conditions.

# 4 Stability of the Controller

One always attempts to obtain a stable controller, in addition to a stable closed-loop control system. The stability of the controller itself will be analyzed in this section.

The characteristic equation of the controller can be equivalent to

$$\frac{(\lambda s+1)^2}{(2\lambda+\tau)s+1} = e^{-\tau s}. \tag{13}$$

The stability can be determined using the dual-locus diagram method [11, 12]. The loci of the both sides in (13) are shown in Figure 6. If the two loci do not intersect, then the controller is stable, otherwise it is unstable. In other words, if the locus of $\frac{(\lambda s+1)^2}{(2\lambda+\tau)s+1}$ crosses the unit circle at which before $e^{-\tau s}$ locus arrives, then the system is stable; if $e^{-\tau s}$ arrives before $\frac{(\lambda s+1)^2}{(2\lambda+\tau)s+1}$ then the controller is unstable.

The locus of $\frac{(\lambda s+1)^2}{(2\lambda+\tau)s+1}$ intersects with the unit circle at frequency $\omega_c = \frac{\sqrt{2\beta^2+4\beta+1}}{\beta^2\tau}$ ($\beta = \frac{\lambda}{\tau}$). The phase angle of $\frac{(\lambda s+1)^2}{(2\lambda+\tau)s+1}$

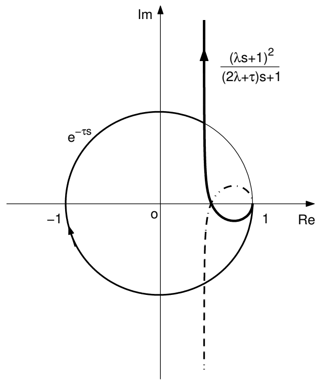



Figure 6: The dual-locus diagram of the controller

at $\omega_c$ is

$$\phi_\lambda = 2\arctan\frac{\sqrt{2\beta^2+4\beta+1}}{\beta} - \arctan\frac{(2\beta+1)\sqrt{2\beta^2+4\beta+1}}{\beta^2}$$

If the locus of $e^{-\tau s}$ also arrives this point at $\omega_c$, then the following equation should be met:

$$\phi_\lambda = 2\pi - \tau\omega_c$$

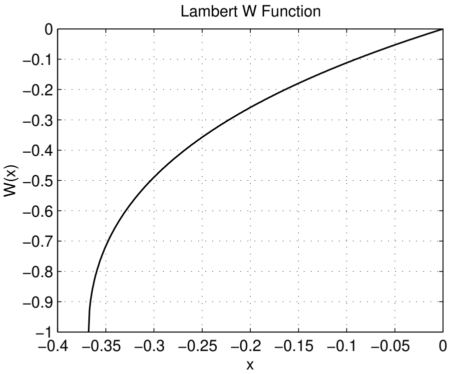
i.e.,

$$2\arctan\frac{\sqrt{2\beta^2+4\beta+1}}{\beta} - \arctan\frac{(2\beta+1)\sqrt{2\beta^2+4\beta+1}}{\beta^2} = 2\pi - \frac{\sqrt{2\beta^2+4\beta+1}}{\beta^2}.$$

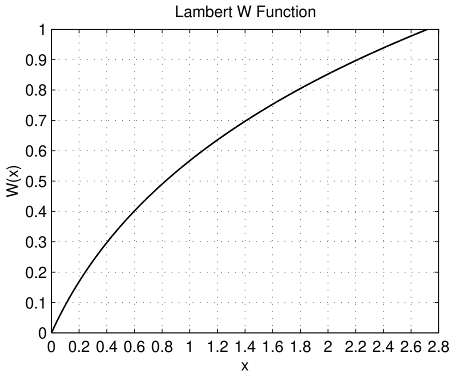
This equation has a unique solution at about $0.63$. For $\beta > 0.63$, the locus of $\frac{(\lambda s+1)^2}{(2\lambda+\tau)s+1}$ crosses the unit circle at which before $e^{-\tau s}$ locus arrives. Hence, the controller is stable for $\beta > 0.63$.

# 5 Conclusions

This paper quantitatively analyzes the step-disturbance response and stability regions for integral processes with dead time. The response can be divided into three stages. The first two stages are not controllable and as a result the minimum of the normalized maximal dynamic error is not less than $\tau_m$. Four specifications: the (normalized) maximal dynamic error, the (normalized) control action bound, the maximal decay rate and an approximate recovery time are used to characterize the sub-ideal disturbance response. For a given (reasonable) maximal dynamic error, the control parameter is determined in terms of the *Lambert W* function. For a chosen control parameter, there exists a requirement of the control action bound. Moreover, the (sufficient) stability regions for gain uncertainties, dead-time uncertainties and both of them are shown in figures. A sufficient condition on the (relative) gain-uncertainty bound is $\frac{\sqrt{3}}{2}$. The controller itself is stable provided that $\beta > 0.63$. Hence, the disturbance observer-based control scheme can be quantitatively designed according to the requirements of the disturbance response, the robustness and the stability of the controller itself.




(a) $-e^{-1} \leq x < 0$




(b) $0 \leq x \leq e$

Figure 7: The principle branch of Lambert W function

# Appendix: *Lambert W* Function

*Lambert W* function is named after a great mathematician, *Johann Heinrich Lambert* in the 18-th century, and also in memory of the pioneering work on this function by E.M. Wright. In some literature, it is also called $\Omega-$function.

It is defined to be the multi-valued inverse of a very simple function $x = we^w$. However, it is a very interesting function not only in the eyes of a mathematician. It has been widely used in, for example, the enumeration of trees, the calculation of water-wave heights, the solution of linear delay equations with constant coefficients and so on. For the history of this function and applications, see [10] and the references therein.

Here, only a graphic description of this function is to be shown. Since $e_m \geq \tau_m$, then $-e^{-1} \leq -\frac{\tau_m}{e_m e} < 0$. The principle branch of the *Lambert W* function in (6) is shown in Figure 7(a). Another case, as shown in Figure 7(b), corresponds to the control parameter with respect to a given control action bound (10), where $0 \leq \frac{1/e}{u_m - 1} \leq e$.

# References


[1] K. Watanabe and M. Ito, "A process-model control for linear systems with delay," *IEEE Trans. Automat. Control*, vol. 26, no. 6, pp. 1261–1269, 1981.

[2] K.J. Åström, C.C. Hang, and B.C. Lim, "A new Smith predictor for controlling a process with an integrator and long dead-time," *IEEE Trans. Automat. Control*, vol. 39, pp. 343–345, 1994.

[3] W.D. Zhang and Y.X. Sun, "Modified Smith predictor for controlling integrator/time delay process," *Ind. Eng. Chem. Res.*, vol. 35, pp. 2769–2772, 1996.

[4] M.R. Matasek and A.D. Micic, "A modified Smith predictor for controlling a process with an integrator and long dead-time," *IEEE Trans. Automat. Control*, vol. 41, no. 8, pp. 1199–1202, 1996.

[5] M.R. Matausek and A.D. Micic, "On the modified Smith predictor for controlling a process with an integrator and long dead-time," *IEEE Trans. Automat. Control*, vol. 44, no. 8, pp. 1603–1606, 1999.

[6] J.E. Normey-Rico and E.F. Camacho, "Robust tuning of dead-time compensators for process with an integrator and long dead-time," *IEEE Trans. Automat. Control*, vol. 44, no. 8, pp. 1597–1603, 1999.

[7] S. Majhi and D.P. Atherton, "Modified Smith predictor and controller for processes with time delay," *IEE Proc.-Control Theory Appl.*, vol. 146, no. 5, pp. 359–366, 1999.

[8] Q.-C. Zhong and J.E. Normey-Rico, "Control of integral processes with dead-time (Part I: Disturbance-observer based 2DOF control scheme)," *IEE Proc.Control Theory Appl.*, 2002, In press.

[9] L. Mirkin and Q.-C. Zhong, "Coprime parametrization of 2DOF controller to obtain sub-ideal disturbance response for processes with dead time," in *Proc. of the 40th IEEE Conf. on Decision & Control*, Orlando, USA, December 2001, pp. 2253–2258.

[10] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, and D.E. Knuth, "On the Lambert W function," *Advances in Computational Mathematics*, vol. 5, pp. 329–359, 1996, Also available at http://www.apmaths.uwo.ca/ rcorless/frames/PAPERS/LambertW/, accessed on 7/4/2002.

[11] O.J.M. Smith, *Feedback Control Systems*, McGraw-Hill Book Company, Inc., 1958.

[12] Q.-C. Zhong, "Robust stability analysis of simple control algorithms in communication networks using dual-locus diagram," submitted to Automatica, 2002.