Energy based stabilization of the angular velocity of a rigid body operating in failure configuration

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Abstract

The stabilization problem for the angular velocity of a rigid body, modeling a simple satellite, is discussed and solved from a new perspective. New state feedback control laws, asymptotically stabilizing the zero equilibrium of the underactuated Euler's equations, are proposed. The control laws are such that the closed loop systems possess the *structure* of Hamiltonian systems.

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1 Introduction

The problem of stabilization of the angular velocity of a rigid body, modeling a simple satellite, has been addressed by several authors. From a control theoretic point of view the most interesting and studied case is the one of a body operating in failure configuration, *i.e.* with only one or two independent actuators acting on the system. More precisely, in the papers [1, 2, 3, 4] it was shown that the zero solution of Euler's angular velocity equations can be made asymptotically stable by means of two control torques, whereas in the works [5, 6, 7, 8] the same problem has been addressed and solved in the case of only one control torque. Robust stabilization has been studied in [9] and [10], and stabilization using partial state information has been addressed in [11]. Finally, the problem of stabilization of non-zero (relative) equilibria has been studied in [12, 11, 13]. In almost all the aforementioned papers (the papers [12, 3, 13] are notable exception) the stabilization problem has been addressed and solved without making use of the special structure of the system to be controlled, *i.e.* stabilizing control laws have been derived using various techniques, such as back-stepping, center manifold theory, homogeneity considerations and control Lyapunov functions methods, but without exploiting the physical and mathematical properties of the system. On the contrary, in the present paper the *Hamiltonian* structure of the system is exploited in the design of stabilizing control laws. It is worth noting that the design procedures proposed in this paper not only make use of the Hamiltonian structure of the system to be controlled, but yield closed loop systems possessing a Hamiltonian structure.

There are various advantages in preserving (in the closed loop) the physical (Hamiltonian) structure of the system, see [14] for further detail. Firstly, the control action has a clear physical interpretation as interconnection of the system with the controller. Hence, stabilization can be understood in terms of *energy balance* between system and controller. Secondly, since Hamiltonian systems are passive with respect to physically meaningful outputs, a margin of robustness *vis a vis* uncertain parameters and unmodeled dynamics is ensured. Finally, the control parameters can be interpreted as playing the role of dampers and springs.

2 The model

Consider a rigid body in an inertial reference frame and let x_1, x_2 and x_3 denote the components of the angular momentum vector along a body fixed reference frame having the origin at the center of gravity and consisting of the three principal axes. The Euler's equations for the rigid body subject to m external control torques, generated by pairs of gas jet actuators[1], are

$$
\begin{aligned}
\dot{x}_1 &= \left(\frac{1}{I_3} - \frac{1}{I_2}\right) x_2 x_3 + B_1 u, \\
\dot{x}_2 &= \left(\frac{1}{I_1} - \frac{1}{I_3}\right) x_3 x_1 + B_2 u, \\
\dot{x}_3 &= \left(\frac{1}{I_2} - \frac{1}{I_1}\right) x_1 x_2 + B_3 u
\end{aligned}
\tag{1}
$$

where $I_1 > 0$, $I_2 > 0$ and $I_3 > 0$ denote the principal moments of inertia, $u \in I\!R^m$ the control torques and $B_i^T \in I\!R^m$ are coefficients depending on the locations of the actuators. System (1) can be re-written as

$$
\dot{x} = [J(x) - R(x)]\frac{\partial H^T}{\partial x} + Bu,
$$

with $B^T = \left[B_1^T, B_2^T, B_3^T\right]$, *i.e.* as a port-controlled Hamiltonian system with internal Hamiltonian (energy) $H(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$, interconnection (Poisson structure)

$$
J(x) = -J^T(x) = \begin{bmatrix} 0 & \frac{x_3}{I_3} & -\frac{x_2}{I_2} \\ -\frac{x_3}{I_3} & 0 & \frac{x_1}{I_1} \\ \frac{x_2}{I_2} & -\frac{x_1}{I_1} & 0 \end{bmatrix},
$$

and internal dissipation $R(x) = R^T(x) = 0$. It is worth noting that, along the trajectories of the uncontrolled system one has $\dot{H} = 0$, *i.e.* the function $H(x)$ is a Lyapunov function for the zero equilibrium and it is conserved along the motion

[1]Note that gas jet actuators are operated in an on-off mode, hence generate a discontinuous control action. In what follows we disregard this important issue and concentrate on the design of continuous control laws.

of the system with $u = 0$.

3 Structure preserving stabilization

In this section we briefly discuss two design methods, namely the methods of structure preserving stabilization and of asymptotic structure preserving stabilization, for general port-controlled Hamiltonian systems.

The problem of structure preserving stabilization, or equivalently of interconnection and damping assignment, for a general port-controlled Hamiltonian system can be posed as follows.

Structure preserving stabilization problem

Given the (port-controlled) Hamiltonian system

$$\dot{x} = [J(x) - R(x)]\frac{\partial H^T}{\partial x} + B(x)u \tag{2}$$

with state x, control u, internal Hamiltonian $H(x)$, interconnection $J(x) = -J^T(x)$, dissipation $R(x) = R^T(x)$, input matrix $B(x)$, and a point x_e in the state space, find (if possible) a modified Hamiltonian function $H_a(x)$, a modified interconnection $J_a(x) = -J_a^T(x)$, a modified dissipation $R_a(x) = R_a^T(x)$, and a (state feedback) control law $u = u(x)$, such that the closed loop system is a

Hamiltonian system with *modified* energy, interconnection and damping, *i.e.*

$$\dot{x} = [J(x) - R(x)]\frac{\partial H^T}{\partial x} + B(x)u(x) = [J(x) + J_a(x) - (R(x) + R_a(x))]\frac{\partial (H + H_a)^T}{\partial x};$$

and the point x_e is a locally (globally) asymptotically stable equilibrium of the closed loop system.

The first requirement can be re-cast in terms of a system of pde's in the (unknown) functions $H_a(x)$ and $u(x)$, once candidate modifications of the interconnection $J_a(x)$ and of the dissipation $R_a(x)$ have been selected. The solvability of this system of pde's has been studied in detail in [15], where a set of sufficient conditions for solvability has been proposed. It is worth noting that to design a stabilizing control law $u = u(x)$ it is not necessary to solve explicitly the system of pde's, but it is enough to guarantee the existence of a solution with some special boundary conditions.

The second requirement is achieved if $\left.\frac{\partial (H + H_a)}{\partial x}\right|_{x=x_e} = 0$, *i.e.* x_e is an equilibrium of the closed loop system; if x_e is a strict local (global) minimum of $H(x) + H_a(x)$ and $R(x) + R_a(x) \geq 0$, *i.e.* the closed loop system is Lyapunov stable; and if the trajectories of the closed loop system converge asymptotically to x_e.

Note that a basic ingredient in the previous problem formulation is the fact that the state spaces of the system to be controlled and of the target closed loop system have the same dimension. This constraint can be removed at the expenses of a weaker design goal, as explained hereafter.

Asymptotic structure preserving stabilization problem

Given the port-controlled Hamiltonian system (2) with state x and control u, and the *target* autonomous port-controlled Hamiltonian system with dissipation

$$\dot{\xi} = [j(\xi) - r(\xi)]\frac{\partial V^T}{\partial \xi} \tag{3}$$

with state ξ, internal Hamiltonian $V(\xi)$, interconnection $j(\xi) = -j^T(\xi)$ and dissipation $r(\xi) = r^T(\xi) \geq 0$, find (if possible) a *mapping* $z = \phi(x)$ and a control law $u = u(x)$ such that the resulting closed loop system has the following properties:

- the set $z = 0$ is invariant and attractive;
- the (well-defined) restriction of the closed loop system to the set $z = 0$ is a diffeomorphic copy of the target port-controlled Hamiltonian system (3).

It is obvious that the asymptotic structure preserving stabilization problem is simpler to solve than the structure preserving stabilization one. Nevertheless, in both cases the solvability of the problem relies upon the solvability of a system of pde's with special boundary conditions.

Remark 1 From the above definition it is apparent that the structure preserving stabilization problem is a *special* case of the asymptotic structure preserving stabilization problem. Moreover, the latter takes its name from the fact that the preservation of the structure is an asymptotic property, *i.e.* the trajectories of the closed loop system *converge* toward the trajectories of the target Hamiltonian system as the time goes to infinity.

4 Stabilization of the zero equilibrium

We now discuss the structure preserving stabilization problem for the zero equilibrium. It must be noted that the solvability of the problem and the complexity of the solution depend upon the structure of the input matrix B. In particular, the problem is trivial if the input matrix is constant and has a nonzero element in any row. In fact, if this is the case, the control law (*i.e.* a simple damping injection) $u = -B^T x$ achieves the goal with $H_a(x) = 0$, $J_a(x) = 0$ and $R(x) = R = BB^T \geq 0$. However, if the above conditions are not satisfied the problem is much more involved, as discussed in the following statement.

Proposition 1 *Consider system (1) with $B_1 = [1, 0]$, $B_2 = [0, 1]$, $B_3 = [0, 0]$, and the point $x_0 = (0, 0, 0)$. Assume $I_1 \neq I_2$. Then the structure preserving stabilization problem is (globally) solvable and a solution is*[2]

$$H_a(x) = -A_3 x_1 x_3 + \frac{1}{2} A_3^2 x_3^2 + x_3^2 x_2 + \frac{1}{2} x_3^4$$

[2] *The matrix $J_a(x)$ is skew-symmetric, hence the upper diagonal terms have not been specified.*

$$R_a(x) = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 + 4A_3^2 x_3^4 & -2A_3^2 x_3^3 \\ 0 & -2A_3^2 x_3^3 & A_3^2 x_3^2 \end{bmatrix}$$

$$J_a(x) = \begin{bmatrix} 0 & \star & \star \\ \frac{x_3}{I_3} - (A_3 x_1 - 2A_3^2 x_3^3 - 2x_3 x_2 - 2x_3^3)A_3 & 0 & \star \\ -A_3 x_3^2 (A_3^2 + 1) - \frac{x_2}{I_2} & -\frac{x_1}{I_2} & 0 \end{bmatrix}$$

$$\begin{aligned} u_1(x) &= A_3 x_3^3 - 2A_3 x_3 x_2^2 - 2A_3 x_2 x_3^3 + A_3^2 x_2 x_1 + 2A_3^3 x_3^3 - A_3^4 x_3^2 x_1 + A_3^5 x_3^3 \\ &\quad -(\frac{1}{I_3} - \frac{1}{I_2}) x_2 x_3 - R_1(x_1 - A_3 x_3) \\ u_2(x) &= -2A_3^2 x_3^2 x_2 - (\frac{1}{I_1} - \frac{1}{I_3}) x_1 x_3 - A_3 x_1 x_3 - R_2(x_2 + x_3^2), \end{aligned} \tag{4}$$

with $A_3 = \frac{1}{I_2} - \frac{1}{I_1}$ *and* R_i *positive constants.*

Proof. To begin with note that the function

$$H(x) + H_a(x) = \frac{1}{2}(x_1 - A_3 x_3)^2 + \frac{1}{2}(x_2 + x_3^2)^2 + \frac{1}{2} x_3^2, \tag{5}$$

is positive definite and proper and that the matrix R_a is positive definite for any nonzero x_3 and positive semidefinite for $x_3 = 0$. Moreover, along the trajectories of the closed loop system, one has

$$\dot{H} + \dot{H}_a = -\frac{\partial(H(x) + H_a(x))}{\partial x} R_a(x) \left[\frac{\partial(H(x) + H_a(x))}{\partial x}\right]^T \leq 0,$$

which proves Lyapunov stability of the closed loop system. Asymptotic stability is then proved using a standard La Salle argument and the properties of the matrix $R(x)$ discussed above. $\triangleleft$

Remark 2 It is interesting to observe that a substantial change in the interconnection has been introduced and that the damping matrix $R_a(x)$ has nonzero *cross-terms*, *i.e.* it is not a diagonal matrix.

Remark 3 The energy function (5), which is a control Lyapunov function for the system, can be also derived using back-stepping arguments, see [16] for further detail. However, the control laws resulting from a straightforward application of a back-stepping procedure do not possess, in general, the property discussed above, *i.e.* they do not preserve the structure of the system, or what is the same, the closed loop system can not be written as a port controlled Hamiltonian system with dissipation.

Simulations have been carried out and sample state and control histories are displayed in Figure 1. Note the non-exponential convergence rate.

5 Asymptotic structure preserving stabilization

In the previous section we have seen that the problem of structure preserving stabilization for the zero equilibrium of system (1) is solvable. However, the solution is very complex: the closed loop system is a port-controlled Hamiltonian system with an interconnection which is completely different from the interconnection of the uncontrolled system and the control law is composed of several terms. Simpler control laws can be obtained if the less ambitious problem of asymptotic structure preserving stabilization is posed.

Proposition 2 *Consider system (1) with* $B_1 = [1,0]$*,* $B_2 = [0,1]$*,* $B_3 = [0,0]$*,* $I_1 \neq I_2$*, and the target port-controlled Hamiltonian system* $\dot{\xi} = -\alpha\xi^3$*, with* $\xi \in I\!R$ *and* $\alpha > 0$, i.e. $j(\xi) = 0$*,* $r(\xi) = \alpha\xi^2$ *and* $V(\xi) = \frac{1}{2}\xi^2$.

Then the asymptotic structure preserving stabilization problem is (globally) solvable and a solution is

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + \lambda_1 x_3^2 \\ x_2 + \lambda_2 x_3 \end{bmatrix} \tag{6}$$

$$\begin{aligned} u_1(x) &= -A_1 x_2 x_3 - 2\lambda_1 A_3 x_1 x_2 x_3 - k_1(x_1 + \lambda_1 x_3^2) \\ u_2(x) &= -A_2 x_1 x_3 - \lambda_2 A_3 x_1 x_2 - k_2(x_2 + \lambda_2 x_3) \end{aligned} \tag{7}$$

with $\lambda_1\lambda_2 A_3 = -\alpha$*,* $k_1 > 0$*,* $k_2 > 0$*,* $A_1 = \dfrac{1}{I_3} - \dfrac{1}{I_2}$*,* $A_2 = \dfrac{1}{I_1} - \dfrac{1}{I_3}$*, and* $A_3 = \dfrac{1}{I_2} - \dfrac{1}{I_1}$.

Proof. Consider the closed loop system (1)-(7). Simple calculations show that

$$\dot{z}_1 = -k_1 z_1 \qquad \dot{z}_2 = -k_2 z_2,$$

i.e. the set $z = 0$ is invariant and attractive. Moreover, the well defined restriction of the closed loop system (1)-(7) to the set $z = 0$ is

$$\dot{x}_3 = A_3 \left(x_1 x_2 \right)_{z=0} = A_3 \lambda_1 \lambda_2 x_3^3 = -\alpha x_3^3,$$

hence the claim. ◁

Remark 4 The selection of $\dot{\xi} = -\alpha \xi^3$ as target Hamiltonian system is related to the fact that the system (1) can not be exponentially stabilized by any continuous control law because, under the assumption of Proposition 2, the linearized system has one uncontrollable eigenvalue on the imaginary axis. However, if non-differentiable control laws are considered, it is possible to prescribed *faster* target dynamics, *e.g.* $\dot{\xi} = -|\xi|^\beta \xi$ with $\beta > 0$.

Remark 5 It is worth noting that the control law (7) is much simpler than the control law (4), yet it provides asymptotic stability and it is such that the closed loop system behaves like the target port-controlled Hamiltonian system asymptotically.

Simulations have been carried out and state and control histories, with the same initial conditions used for the simulations in Figure 1, are displayed in Figure 2.

Note the non-exponential convergence rate.

6 Conclusions

The stabilization problem for the angular velocity of a rigid body around the zero equilibrium has been studied from a new point of view. In particular, it is shown that it is possible to achieve asymptotic stability using control laws which admit an interpretation in terms of energy flow and dissipation. Further work is in progress to obtain stabilizing control laws for the complete model (*i.e.* kinematic and dynamic) of a simple satellite.

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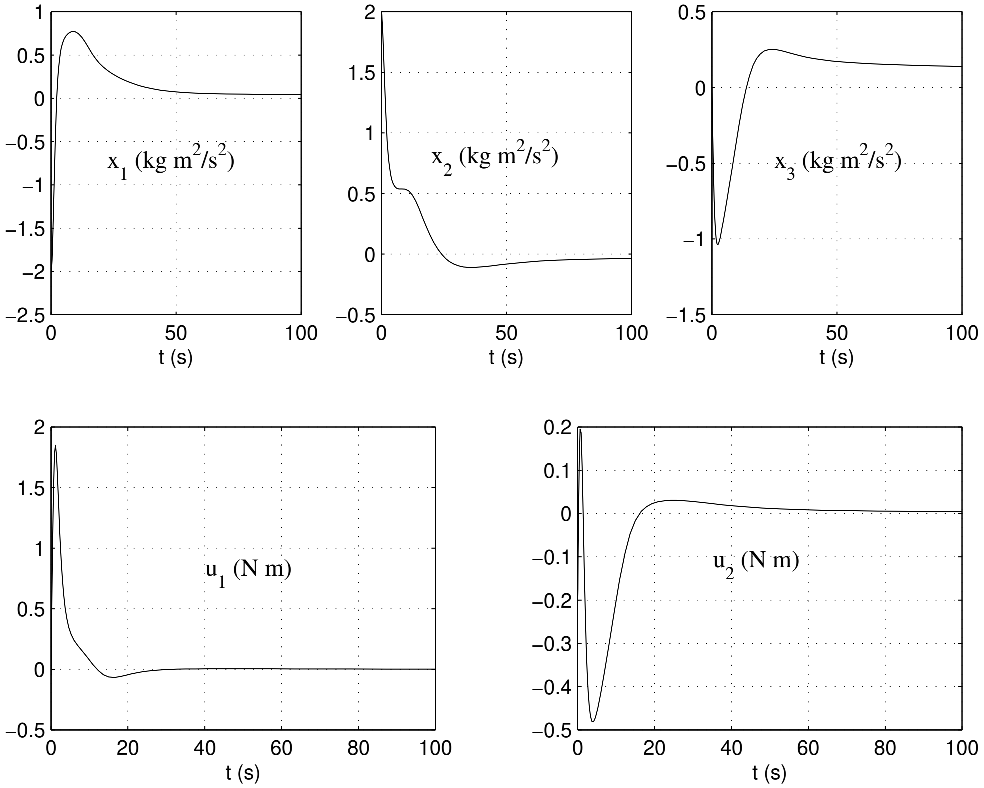

Figure 1: State and control histories for the control law achieving structure preserving stabilization of the zero equilibrium.

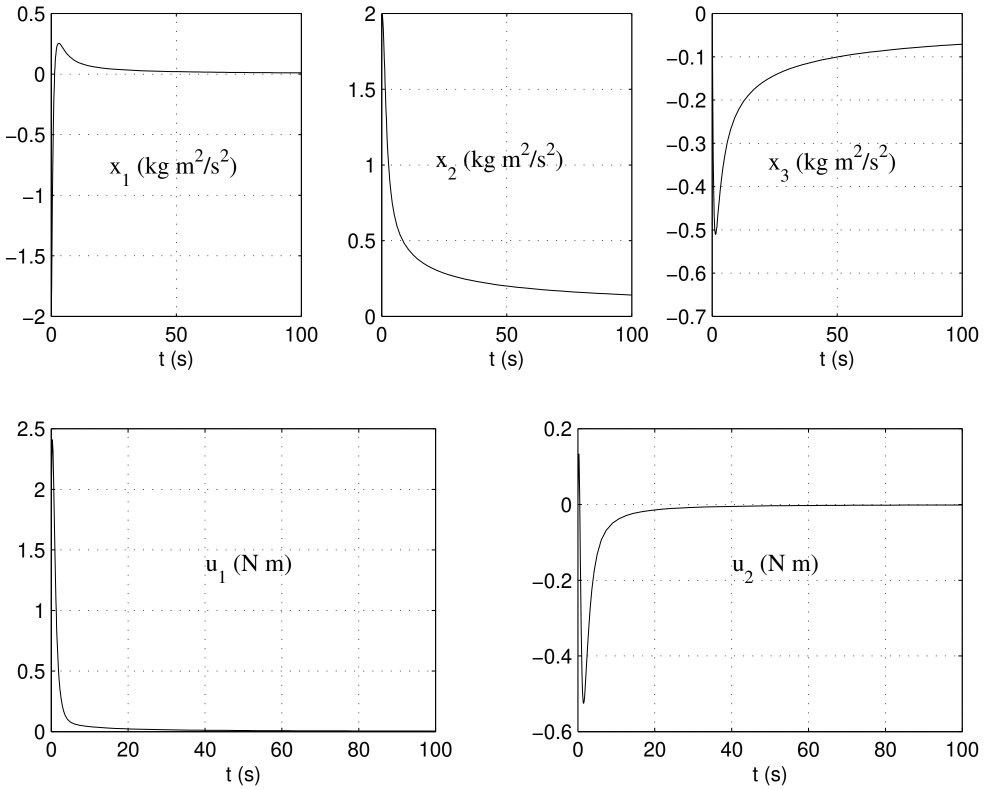

Figure 2: State and control histories for the control law achieving asymptotic structure preserving stabilization of the zero equilibrium.