Spectral properties of infinite-dimensional closed-loop systems

George WEISS
Dept. of Electrical and Electronic Engineering
Imperial College of Science, Technology and Medicine
Exhibition Road, London SW7 2BT, UK
e-mail: G.Weiss@ic.ac.uk

Cheng-Zhong XU
Lagep, Bât. 308G, Université de Lyon
43, Boulevard du 11/11/1918
69622, Villeurbanne Cedex, France
e-mail: xu@lagep.univ-lyon1.fr

Abstract: We consider feedback systems obtained from infinite-dimensional well-posed linear systems by output feedback. Thus, our framework allows for unbounded control and observation operators. Our aim is to investigate the relationship between the open-loop system, the feedback operator K, and the spectrum (in particular, the eigenvalues and eigenvectors) of the closed-loop generator A^K. We give a useful characterization of that part of the spectrum $\sigma(A^K)$ which lies in the resolvent set of A, the open-loop generator, via the "characteristic equation" involving the open-loop transfer function. We show that certain points of $\sigma(A)$ cannot be eigenvalues of A^K if the input and output are scalar (so that K is a number) and $K \neq 0$. We devote special attention to the case when the output feedback operator K is compact. It is relatively easy to prove that in this case, $\sigma_e(A)$, the essential spectrum of A, is invariant, i.e., it is equal to $\sigma_e(A^K)$. A related but much harder problem is to determine the largest subset of $\sigma(A)$ which remains invariant under compact output feedback. This set, which we call the unmovable spectrum of A, strictly contains $\sigma_e(A)$. We give an explicit characterization of the unmovable spectrum and we investigate the consequences of our results for a certain class of well-posed systems obtained by interconnecting an infinite chain of identical systems.

Keywords: strongly continuous semigroup, eigenvalues and eigenvectors, well-posed linear system, static output feedback, characteristic equation, essential spectrum, unmovable spectrum.

1. Introduction and main results about locating the spectrum of the closed-loop generator

In this paper we consider a well-posed linear system Σ with input u and output y connected in a feedback loop, as shown in Figure 1. If the bounded feedback operator K is admissible for Σ, then the feedback connection $u = Ky + v$ gives rise to a new well-posed linear system with input v, called the *closed-loop system* and denoted by Σ^K. (We refer to Section 3 for some background on the concepts used here.) We denote by $\mathbf{G}$ and $\mathbf{G}^K$ the transfer functions of Σ and Σ^K, so that

$$\mathbf{G}^K = (I - \mathbf{G}K)^{-1}\mathbf{G} = \mathbf{G}(I - K\mathbf{G})^{-1}. \tag{1.1}$$

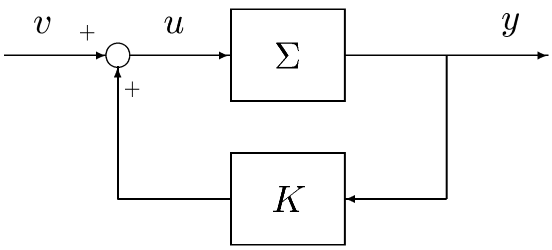

Figure 1: A well-posed linear system Σ with output feedback via K. If K is admissible, then this is a new well-posed linear system Σ^K, called the closed-loop system.

We denote by A and A^K the semigroup generators of Σ and Σ^K. The aim of this paper is to investigate the relationship between the spectra of A and A^K. The general framework in which we look at the feedback system is as in Weiss [13]. We assume that the reader has some understanding of the material in [13], in order to avoid reintroducing the concepts in this paper. (The formulas needed from [13] will be written down for easy reference, in Sections 1 and 3.)

We denote by U the input space of Σ, by X its state space and by Y its output space; all three are Hilbert spaces. We denote by B and B^K the control operators of Σ and Σ^K, and by C and C^K their observation operators. Thus, the evolution of the state of the feedback system, denoted by $x(t)$, satisfies both of the equations

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad \dot{x}(t) = A^K x(t) + B^K v(t),$$

where $u = v + Ky$. We have $x(t) \in X$ for all $t \geq 0$, $u, v \in L^2_{loc}([0,\infty), U)$, $y \in L^2_{loc}([0,\infty), Y)$ and $K \in \mathcal{L}(Y, U)$. If we denote $X_{-1} = \mathcal{D}(A^*)'$ and $X^K_{-1} = \mathcal{D}((A^K)^*)'$, both with respect to the pivot space X, then $B \in \mathcal{L}(U, X_{-1})$ and $B^K \in \mathcal{L}(U, X^K_{-1})$ (see also Section 3 for other details on Σ^K).

If $u = 0$ and $x(0) \in \mathcal{D}(A)$, then $y(t) = Cx(t)$ (for all $t \geq 0$), while if $v = 0$ and $x(0) \in \mathcal{D}(A^K)$, then $y(t) = C^K x(t)$ (for all $t \geq 0$). If we denote $X_1 = \mathcal{D}(A)$ and $X^K_1 = \mathcal{D}(A^K)$, both with the graph norm, then $C \in \mathcal{L}(X_1, Y)$ and $C^K \in \mathcal{L}(X^K_1, Y)$.

For general u (or v) and $x(0)$, the formula for y is more complicated (see [9]), but in terms of Laplace transforms we have the simple relationships

$$\hat{y}(s) = C(sI - A)^{-1}x(0) + \mathbf{G}(s)\hat{u}(s),$$
$$\hat{y}(s) = C^K(sI - A^K)^{-1}x(0) + \mathbf{G}^K(s)\hat{v}(s).$$

In the sequel, we state several results about $\sigma(A^K)$ (the spectrum of A^K) in the general framework introduced above. These results concern the general problem of locating the points of $\sigma(A^K)$ (in particular, its eigenvalues), especially those in the resolvent set of A. The results in this section will be proved in Section 3.

Our first result (Theorem 1.1 below) gives a method to compute eigenvalues of A^K which are not in the spectrum of A, and also the corresponding eigenvectors. The method amounts to solving a characteristic equation, and it is not new: it has been used often in the literature, for various particular systems. Salamon [8] and Rebarber and Townley [7] have formulated this method as fairly general abstract results. We include this theorem because we think that our formulation is somewhat more general, powerful and precise than the corresponding results in [8] and [7].

We need the notation $\sigma_p(A)$ for the point spectrum (the set of eigenvalues) of A. Recall that for any $\lambda \in \sigma_p(A)$, $\dim\{\mathrm{Ker}\,(\lambda I - A)\}$ is called the *geometric multiplicity* of λ, while $\sup_{k\in\mathbb{N}} \dim\{\mathrm{Ker}\,(\lambda I - A)^k\}$ is called the *algebraic multiplicity* of λ (see Kato [5]). The transfer function $\mathbf{G}$ is extended to $\rho(A)$, the resolvent set of A, as in [9, 10], and similarly $\mathbf{G}^K$ is extended to $\rho(A^K)$ (see also Section 3).

Theorem 1.1. *We use the notation* Σ, U, X, Y, A, B, C, $\mathbf{G}$ *for an open-loop well-posed system, and* K, Σ^K, A^K, B^K, C^K, $\mathbf{G}^K$ *for an admissible feedback operator and the corresponding closed-loop system, as introduced earlier in this section. Assume that* $\mathrm{Ker}\, B = \{0\}$ *and* $\lambda \in \rho(A)$. *Then the following statements are equivalent:*

(1) $\lambda \in \sigma_p(A^K)$, *i.e.,* $X_\lambda = \mathrm{Ker}\,(\lambda I - A^K) \subset \mathcal{D}(A^K)$ *is non-trivial.*

(2) $U_\lambda = \mathrm{Ker}\,[I - K\mathbf{G}(\lambda)] \subset U$ *is non-trivial.*

If the above statements are true, then X_λ *and* U_λ *are isomorphic, via an invertible operator* $T_\lambda : U_\lambda \to X_\lambda$. *The operator* T_λ *and its inverse* T_λ^{-1} *are given by*

$$T_\lambda \mathrm{v} = (\lambda I - A)^{-1}B\mathrm{v}, \qquad T_\lambda^{-1}x = KC^K x. \tag{1.2}$$

This theorem enables us to find eigenvalues of A^K which are in $\rho(A)$, by solving the "characteristic equation" (with unknowns λ and v)

$$[I - K\mathbf{G}(\lambda)]\,\mathrm{v} = 0, \quad \mathrm{v} \neq 0.$$

According to the theorem, for each $\lambda \in \rho(A)$ for which solutions v exist, the dimension of the corresponding solution space U_λ is the geometric multiplicity of the eigenvalue λ. If U is finite-dimensional, then we may start by looking for the zeros of $\det[I - K\mathbf{G}(\lambda)]$. Once we have λ and v, we can easily obtain the corresponding eigenvector of A^K via $x = (\lambda I - A)^{-1}B\mathrm{v}$, which is a consequence of (1.2).

The following theorem does not concern only the eigenvalues of A^K, but its spectrum in general. It is closely related to the last theorem.

Theorem 1.2. *With the assumptions and the notation of Theorem* 1.1*, for any* $\lambda \in \rho(A)$*, the following statements are equivalent:*

(1) $\lambda \in \sigma(A^K)$.

(2) $I - K\mathbf{G}(\lambda)$ *is not invertible.*

Example 1.3. The above results say nothing about eigenvalues of A^K which are also in the spectrum of A. Such eigenvalues may occur, even if the system Σ is finite-dimensional and minimal (i.e., controllable and observable) and $K = I$. Indeed, consider Σ to be any minimal realization (A, B, C) of

$$\mathbf{G}(s) = \begin{bmatrix} \frac{-1}{s+1} & \frac{1}{s} \\ 0 & \frac{-1}{s+1} \end{bmatrix},$$

i.e., $\mathbf{G}(s) = C(sI - A)^{-1}B$ holds for all $s \in \mathbb{C}$ except the poles of $\mathbf{G}$. Note that such a realization must be of order 3, i.e., $A \in \mathbb{R}^{3\times 3}$. Then, with $K = I$, $\lambda = 0$ is an eigenvalue of both A and A^K, because $s = 0$ is a pole of both $\mathbf{G}$ and $\mathbf{G}^K$ (this remains true for any diagonal K). Note that any minimal realization of the above $\mathbf{G}$ will also have the properties $\operatorname{Ker} B = \operatorname{Ker} C^* = \{0\}$ (because $\mathbf{G}(s)$ has rank 2). In the above example the algebraic multiplicity of $0 \in \sigma(A)$ (which is 1) is preserved under feedback. In general, this must not be the case. Indeed, if instead of $K = I$ we take $K = -I$, then A^K will have an eigenvalue of algebraic multiplicity 3 at $\lambda = 0$, because $\mathbf{G}^K$ has a pole of order 3 at $s = 0$. The geometric multiplicity remains unchanged at 1, but we can construct examples where this changes too.

Under several additional assumptions, we can guarantee that the eigenvalues of A^K and A are disjoint. Indeed, in many applications, $\sigma(A)$ consists of points λ such that $\bar{\lambda} \in \sigma_p(A^*)$ (where $\overline{\lambda}$ denotes the complex conjugate of λ). This is the case, for example, if $(sI - A)^{-1}$ is compact for some (hence, for every) $s \in \rho(A)$. If, moreover, $U = Y = \mathbb{C}$, Σ is approximately controllable and approximately observable and $K \neq 0$, then $\sigma(A) \cap \sigma(A^K) = \emptyset$. This follows from the following theorem. The controllability and observability concepts used here are recalled in Section 3.

Theorem 1.4. *With the notation of Theorem* 1.1*, assume that* $U = Y = \mathbb{C}$*,* (A, B) *is approximately controllable,* (A, C) *is approximately observable and* $K \neq 0$*.*

If $\overline{\lambda} \in \sigma_p(A^*)$*, then* $\lambda \notin \sigma_p(A^K)$*.*

For finite-dimensional SISO (single-input single-output) systems, the above result expresses the well-known fact that the root-locus "never comes back" to any of the open-loop poles. Indeed, the poles of $\mathbf{G}^K$ are the eigenvalues of A^K if the realization of Σ (and hence, of Σ^K) is minimal. Now we see that this remains true for well-posed systems with scalar input and output. From here, a short argument involving repeated feedbacks shows that the root-locus curves of SISO systems never intersect themselves (another well-known fact in finite dimensions).

Remark 1.5. With the assumptions and the notation of Theorem 1.1, let $\mathcal{D} \subset \mathbb{C}$ be an open set such that $\sigma(A) \subset \mathcal{D}$, and denote

$$r = \frac{1}{\sup_{s \notin \mathcal{D}} \|\mathbf{G}(s)\|} . \tag{1.3}$$

(This r could be 0 or $+\infty$, with the convention $1/0 = +\infty$ and $1/+\infty = 0$.)

If $\|K\| < r$, then $\sigma(A^K) \subset \mathcal{D}$.

This statement follows easily from Theorem 1.2, and it is contained in Rebarber and Townley [7, Proposition 2.2]. (In [7] there are additional assumptions about the system, which are not needed for this result.) The above statement looks like an upper semicontinuity result for $\sigma(A^K)$ as a function of K, in the spirit of Kato [5] (see Theorem 3.1 and Remark 3.3 in Chapter IV of [5]). However, it is not true that $\sigma(A^K)$ is upper semicontinuous as a function of K, because r defined in (1.3) could be zero. This is illustrated in Example 4.1 in [14], where $\sigma(A^K)$ is not upper semicontinuous at $K = -I$, where $\sigma(A^K)$ becomes empty. However, $\sigma(A^K)$ is upper semicontinuous if we allow only open sets $\mathcal{D}$ whose complement is compact (these too form a topology on $\mathbb{C}$), because then $r > 0$ for each $\mathcal{D}$.

2. Fredholm eigenvalues and the unmovable spectrum

In this section we state our main results about points in $\sigma(A)$ which must be also in $\sigma(A^K)$. We work in the general framework introduced in Section 1, and often (but not always) we assume that the output feedback operator K is compact. The notation X will always stand for a Hilbert space (often, the state space of a system). The proofs of all the results in this section will be given in Section 4.

We introduce the following notation used in Kato [5, Chapter IV]. If A is a closed operator on X (not necessarily a semigroup generator) and $\lambda \in \mathbb{C}$, we put

$$\nu(\lambda) = \dim \{\operatorname{Ker}(\lambda I - A)\} , \qquad \mu(\lambda) = \dim \{\operatorname{Ran}(\lambda I - A)\}^{\perp} .$$

Thus, $\nu(\lambda) > 0$ means that $\lambda \in \sigma_p(A)$, while $\mu(\lambda) > 0$ means that $\overline{\lambda} \in \sigma_p(A^*)$. Clearly, if we replace λ with $\overline{\lambda}$ and A with A^*, then $\nu(\lambda)$ and $\mu(\lambda)$ switch places.

Definition 2.1. If A is a closed operator on X, a point $\lambda \in \sigma(A)$ is called a *Fredholm eigenvalue* of A if $\operatorname{Ran}(\lambda I - A)$ is closed and

$$0 < \nu(\lambda) = \mu(\lambda) < \infty .$$

In other words, using some of the terminology of [5], λ is a Fredholm eigenvalue of A if and only if it is an eigenvalue of A and moreover, $\lambda I - A$ is a Fredholm operator with index zero. We denote the set of Fredholm eigenvalues of A by $\sigma_F(A)$. Obviously we have $\sigma_F(A) \subset \sigma_p(A)$. It is useful to note that if $\lambda \in \sigma_p(A)$ is an isolated point of $\sigma(A)$ and $\nu(\lambda) < \infty$, then $\lambda \in \sigma_F(A)$. This follows from Remark 6.23 in [5, Chapter III]. In particular, it follows that if $(\beta I - A)^{-1}$ is compact for some (hence, for every) $\beta \in \rho(A)$, then $\sigma_F(A) = \sigma(A)$.

To avoid misunderstandings: the concept of a Fredholm eigenvalue does not appear in [5], we introduced it here to suit our needs. Since it is a simple and natural concept, we believe that it exists in the literature, perhaps under another name.

Definition 2.2. We denote by $\mathcal{K}(X)$ the space of all compact operators in $\mathcal{L}(X)$. Let $A : \mathcal{D}(A) \to X$ be a closed operator. The *unmovable spectrum* of A, denoted $\sigma_u(A)$, consists of those complex numbers λ which satisfy

$$\lambda \in \sigma(A + P) \qquad \forall\, P \in \mathcal{K}(X).$$

The following proposition gives an explicit characterization of $\sigma_u(A)$ as the complement of $\sigma_F(A)$ in $\sigma(A)$.

Proposition 2.3. *Let A be a closed operator on X. Then*

$$\sigma(A) = \sigma_F(A) \cup \sigma_u(A), \qquad \sigma_F(A) \cap \sigma_u(A) = \emptyset.$$

Note that the above proposition implies, in particular, that $\sigma_u(A)$ contains the continuous spectrum of A as well as its residual spectrum.

Recall from Kato [5] that the *essential spectrum* of A, denoted by $\sigma_e(A)$, consists of those complex numbers λ for which one of the following conditions holds: either $\mathrm{Ran}\,(\lambda I - A)$ is not closed, or it is closed and $\nu(\lambda) = \mu(\lambda) = \infty$. We have

$$\sigma_e(A) \subset \sigma_u(A) \subset \sigma(A), \tag{2.1}$$

and each of these inclusions may be strict. Indeed, it follows from the last proposition that $\sigma_u(A)$ consists of those complex numbers λ for which either $\mathrm{Ran}\,(\lambda I - A)$ is not closed, or it is closed and $\nu(\lambda) = \mu(\lambda) = \infty$, or it is closed and $\nu(\lambda) \neq \mu(\lambda)$.

We remark that the definition of $\sigma_e(A)$ given in Nikolskii [6, p. 313] (and some other books) is different: instead of "$\nu(\lambda) = \mu(\lambda) = \infty$" he requires "$\nu(\lambda) = \infty$ or $\mu(\lambda) = \infty$". With this modified definition $\sigma_e(A)$ becomes larger, but (2.1) still holds and the inclusions may be strict. The definition of the essential spectrum in Dunford and Schwartz [4, p. 1393] is again different, they only require that $\mathrm{Ran}\,(\lambda I - A)$ is not closed, so that $\sigma_e(A)$ is smaller than in [5], and obviously (2.1) holds.

Example 2.4. Let $X = l^2$ and let $\mathbf{S}^*$ be the left-shift operator on X (the adjoint of the right-shift $\mathbf{S}$). Then it is easy to see that $\nu(0) = 1$ and $\mu(0) = 0$, so that $0 \in \sigma_u(\mathbf{S}^*)$, but clearly $0 \notin \sigma_e(\mathbf{S}^*)$. In fact, $\sigma_e(\mathbf{S}^*)$ is the unit circle, while $\sigma_u(\mathbf{S}^*) = \sigma(\mathbf{S}^*)$ is the unit disk. We remark that in this case $\sigma_e(\mathbf{S}^*)$ is exactly the continuous spectrum of $\mathbf{S}^*$. This example will be used in Section 5.

Proposition 2.5. *With the assumptions and the notation of Theorem* 1.1*, suppose that K is compact. Then $\sigma_e(A) = \sigma_e(A^K)$.*

This proposition is reminiscent of the fact that $\sigma_e(A)$ cannot be moved by relatively compact perturbations, see Theorem 5.35 in [5, Chapter IV]. However, our result cannot be reduced to the one just mentioned from [5], because the perturbations considered here are not relatively compact, they are not even relatively bounded (since the domain of the operator may change). The last proposition is used in the proof of the following theorem:

Theorem 2.6. *With the assumptions and the notation of Theorem* 1.1*, suppose that K is compact (this is the case, for example, if U or Y is finite-dimensional).*

Then for every $\lambda \in \rho(A) \cap \sigma(A^K)$, we have $\lambda \in \sigma_F(A^K)$.

Remark 2.7. In the last theorem, the assumption that K is compact can be replaced with the less restrictive but more cumbersome assumption that $K\mathbf{G}(\lambda)$ is compact for the values of λ under consideration. The proof remains the same.

The last theorem is used in the proof of the following one, which tells us that $\sigma_u(A)$ cannot be moved by compact output feedback. This result looks similar to Proposition 2.5, but (counting the preliminary results) it is much harder to prove.

Theorem 2.8. *With the assumptions and the notation of Theorem* 1.1*, suppose that K is compact. Then $\sigma_u(A) = \sigma_u(A^K)$.*

We mention that Theorem 2.8 is optimal in the following sense: $\sigma_u(A)$ is the largest subset of $\sigma(A)$ which is contained in $\sigma(A^K)$ for *any* feedback perturbation obtained by embedding A into a well-posed system and then applying a compact admissible feedback operator K. This fact follows from the last theorem and the definition of $\sigma_u(A)$. Indeed, any bounded perturbation of A (such as P in Definition 2.2) can be regarded as a feedback perturbation corresponding to the system $\dot{x} = Ax + u$, $y = x$, with input u, state x and output y (so that $B = C = I$).

3. Proof of the results from Section 1

Throughout this section, X is a Hilbert space and $A : \mathcal{D}(A) \to X$ is the generator of a strongly continuous semigroup $\mathbb{T}$ on X. The Hilbert space X_1 is $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, where $\beta \in \rho(A)$ is fixed (this norm is equivalent to the graph norm). The Hilbert space X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1}z\|$. This space is isomorphic to $\mathcal{D}(A^*)'$, the dual space of $\mathcal{D}(A^*)$ and

$$X_1 \subset X \subset X_{-1},$$

densely and with continuous embeddings. $\mathbb{T}$ extends to a semigroup on X_{-1}, denoted by the same symbol. The generator of this extended semigroup is an extension of A, whose domain is X, so that $A : X \to X_{-1}$. We consider a well-posed linear system Σ with input space U, state space X, output space Y, semigroup generator A, control operator B, observation operator C and transfer function $\mathbf{G}$. Thus, $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$ and $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$. For the various concepts mentioned above we refer to [10, 11, 12, 13] and the references therein. B is called *bounded* if $B \in \mathcal{L}(U, X)$, and C is called *bounded* if it can be extended such that $C \in \mathcal{L}(X, Y)$.

Denote the growth bound of $\mathbb{T}$ by $\omega_0(\mathbb{T})$. Originally, $\mathbf{G}$ is defined on the right half-plane where $\operatorname{Re} s > \omega_0(\mathbb{T})$. For all s, β in this half-plane, we have

$$\mathbf{G}(s) - \mathbf{G}(\beta) = C \left[(sI - A)^{-1} - (\beta I - A)^{-1} \right] B. \tag{3.1}$$

Using the above formula with $\operatorname{Re} \beta > \omega_0(\mathbb{T})$ but allowing $s \in \rho(A)$, we extend $\mathbf{G}$ to an analytic function defined on $\rho(A)$. Then, this function $\mathbf{G}$ satisfies (3.1) for all $s, \beta \in \rho(A)$. This extension of $\mathbf{G}$ has been used in [8, 9, 10] and it may be different from the extension by analytic continuation, adopted in [12, 13] (see [12, Remark 4.8]). However, if $\rho(A)$ is connected, then this extension of $\mathbf{G}$ is the same as its analytic continuation. The function $\mathbf{G}$ is *well-posed*, which means that it is a bounded and analytic $\mathcal{L}(U, Y)$-valued function on some right half-plane (in particular, on any half-plane which is strictly to the right of $\omega_0(\mathbb{T})$).

An operator $K \in \mathcal{L}(Y, U)$ is called an *admissible feedback operator* for Σ (or for $\mathbf{G}$) if $I - \mathbf{G}K$ is invertible on some right half-plane and its inverse is well-posed (here, $I - \mathbf{G}K$ may be replaced equivalently by $I - K\mathbf{G}$). If this is the case, then the feedback system from Figure 1 is a new well-posed linear system Σ^K. We use the notation A^K, B^K, C^K and $\mathbf{G}^K$ as introduced in Section 1, so that in particular (1.1) holds on some right half-plane. What is less obvious is that (1.1) *remains valid for all* $s \in \rho(A) \cap \rho(A^K)$ (in particular, $I - K\mathbf{G}(s)$ is invertible for all such s). Indeed, this follows from Corollary 5.3 and Remark 5.5 in [10]. Let $\mathbb{T}^K$ denote the strongly continuous semigroup generated by A^K. The following identities, proved in [13], will be useful:

$$[I - \mathbf{G}(s)K] C^K (sI - A^K)^{-1} = C(sI - A)^{-1}, \tag{3.2}$$

$$(sI - A^K)^{-1} B^K [I - K\mathbf{G}(s)] = (sI - A)^{-1} B, \tag{3.3}$$

$$\begin{aligned} (sI - A^K)^{-1} - (sI - A)^{-1} &= (sI - A)^{-1} B K C^K (sI - A^K)^{-1} \\ &= (sI - A^K)^{-1} B^K K C (sI - A)^{-1}. \end{aligned} \tag{3.4}$$

According to [13], these identities are valid on the right half-plane where $\operatorname{Re} s > \max\{\omega_0(\mathbb{T}), \omega_0(\mathbb{T}^K)\}$. We remark that, using again [10, Section 5], it follows that (3.2)–(3.4) remain valid for all $s \in \rho(A) \cap \rho(A^K)$. Note that (3.3) implies that $\operatorname{Ker} B = \{0\}$ if and only if $\operatorname{Ker} B^K = \{0\}$. It is worth noting that (3.4) implies (via an easy computation) that for all $x \in \mathcal{D}(A^K)$ and for all $z \in \mathcal{D}(A)$,

$$A^K x = \left(A + BKC^K \right) x, \qquad Az = \left(A^K - B^K K C \right) z. \tag{3.5}$$

It is not difficult to prove, using (3.2), that C^K is bounded if and only if C is bounded. Similarly, (3.3) implies that B^K is bounded if and only if B is bounded.

An important subclass of the well-posed linear systems are the regular linear systems, see for example [12, 13]. The system Σ is called *regular* if for each $\mathrm{v} \in U$, the limit

$$D\mathrm{v} = \lim_{\lambda \to +\infty} \mathbf{G}(\lambda)\mathrm{v} \tag{3.6}$$

exists (in Y). The operator $D \in \mathcal{L}(U, Y)$ is then called the *feedthrough operator* of Σ. We define the Λ-extension of C by

$$C_\Lambda x = \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1}x \qquad \forall\, x \in \mathcal{D}(C_\Lambda), \tag{3.7}$$

where $\mathcal{D}(C_\Lambda)$ is the space of those $x \in X$ for which the above limit exists. The system Σ is regular if and only if $(sI - A)^{-1}BU \subset \mathcal{D}(C_\Lambda)$ for some (hence, for every) $s \in \rho(A)$ and, if this is the case, then $\mathbf{G}(s) = C_\Lambda(sI - A)^{-1}B + D$ for all $s \in \rho(A)$. Moreover, for a regular system, $y(t) = C_\Lambda x(t) + Du(t)$ holds for almost every $t \geq 0$ (for every initial state $x(0) \in X$ and every input signal $u \in L^2_{loc}([0, \infty), U)$).

If Σ is regular, K is an admissible feedthrough operator for Σ and $I - DK$ is invertible, then Σ^K is also regular and the relationship between the open and closed-loop systems becomes more transparent. Indeed, in this case A^K, B^K, C^K and D^K can be expressed in terms of A, B, C and D, see [13], in particular

$$A^K x = \left[A + BK(I - DK)^{-1}C_\Lambda\right] x, \qquad C^K x = (I - DK)^{-1}C_\Lambda x, \tag{3.8}$$

for all $x \in \mathcal{D}(A^K)$. Thus, regularity simplifies matters, and it will be useful in the examples, however, in general we will only assume the well-posedness of Σ.

Lemma 3.1. *If $\lambda \in \rho(A)$, $\mathrm{v} \in U$, K is an admissible feedback operator for $\mathbf{G}$, $B\mathrm{v} \neq 0$ and $B^K[I - K\mathbf{G}(\lambda)]\mathrm{v} = 0$, then $\lambda \in \sigma_p(A^K)$ and $x = (\lambda I - A)^{-1}B\mathrm{v}$ is a corresponding eigenvector of A^K (i.e., $x \in \mathcal{D}(A^K)$, $x \neq 0$ and $(\lambda I - A^K)x = 0$).*

Proof: We apply $sI - A^K$ (as an operator from X to X^K_{-1}) to both sides of (3.3), obtaining

$$\left(sI - A^K\right)(sI - A)^{-1}B = B^K[I - K\mathbf{G}(s)], \tag{3.9}$$

valid for all s in some right half-plane. We want to show that (3.9) remains valid for all $s \in \rho(A)$. Denote $Q(s) = (sI - A^K)(sI - A)^{-1}$, for all $s \in \rho(A)$. Rewrite the second formula in (3.5) in the form $(sI - A^K)z = (sI - A)z - B^K KCz$, which is valid for all $z \in \mathcal{D}(A)$ and for all $s \in \mathbb{C}$. This implies, taking $z = (sI - A)^{-1}x$, that for all $s \in \rho(A)$,

$$Q(s) = I - B^K KC(sI - A)^{-1}. \tag{3.10}$$

Hence, for all $s, \beta \in \rho(A)$,

$$Q(s) - Q(\beta) = -B^K KC\left[(sI - A)^{-1} - (\beta I - A)^{-1}\right].$$

If we compose with B from the right and use (3.1), we obtain that for all $s, \beta \in \rho(A)$,

$$[Q(s) - Q(\beta)]\, B = \; - B^K K\, [\mathbf{G}(s) - \mathbf{G}(\beta)]\, .$$

If we add formula (3.9), with β in place of s (where β is a number for which (3.9) holds) to the last formula above, we obtain that (3.9) holds for $s \in \rho(A)$.

Now, if λ and v are as in the lemma, we see from (3.9) that

$$\left(\lambda I - A^K\right)(\lambda I - A)^{-1} B\mathrm{v} = 0.$$

Since $x = (\lambda I - A)^{-1} B\mathrm{v} \neq 0$, x is an eigenvector of A^K. ∎

Now we state a converse of Lemma 3.1.

Lemma 3.2. *If $\lambda \in \rho(A)$, K is an admissible feedback operator for $\mathbf{G}$ and $x \in \mathcal{D}(A^K)$, $x \neq 0$ and $(\lambda I - A^K)x = 0$, then denoting $\mathrm{v} = KC^K x$ we have*

$$B^K[I - K\mathbf{G}(\lambda)]\mathrm{v} = 0 \quad \textit{and} \quad x = (\lambda I - A)^{-1} B\mathrm{v}\, .$$

Proof: We rewrite the first formula in (3.5) in the form $(sI - A^K)\xi = (sI - A)\xi - BKC^K\xi$, which is valid for all $\xi \in \mathcal{D}(A^K)$ and for all $s \in \mathbb{C}$. This implies, applying $(sI - A)^{-1}$ to both sides, that for all $s \in \rho(A)$,

$$\xi - (sI - A)^{-1}(sI - A^K)\xi \, = \, (sI - A)^{-1} BKC^K \xi\, . \tag{3.11}$$

Taking $\xi = x$, $\mathrm{v} = KC^K x$ and $s = \lambda$ in (3.11), we obtain $x = (\lambda I - A)^{-1} B\mathrm{v}$. Now we apply both sides of (3.9) to v, with $s = \lambda$, obtaining

$$(\lambda I - A^K)x \, = \, B^K[I - K\mathbf{G}(\lambda)]\mathrm{v}.$$

Since the left-hand side is zero by definition, we obtain the desired identity. ∎

Proof of Theorem 1.1: Take $\lambda \in \rho(A)$ and let $T_\lambda = (\lambda I - A)^{-1} B$, which is a bounded operator from U to X. Like in the theorem, we define $X_\lambda = \mathrm{Ker}\,(\lambda I - A^K)$ and $U_\lambda = \mathrm{Ker}\,[I - K\mathbf{G}(\lambda)]$, which are closed subspaces of X and U, possibly of dimension zero. To prove the equivalence of (1) and (2) in the theorem, it is sufficient to prove that T_λ is a continuous and invertible operator from U_λ onto X_λ. Let $\mathrm{v} \in U_\lambda$. By Lemma 3.1, $x = T_\lambda \mathrm{v}$ satisfies $(\lambda I - A^K)x = 0$, so that $x \in X_\lambda$. Thus, after restriction to U_λ, we have $T_\lambda \in \mathcal{L}(U_\lambda, X_\lambda)$. It is also clear that T_λ is one-to-one. To show that T_λ (restricted to U_λ) is onto, take $x \in X_\lambda$. By Lemma 3.2, if we denote $\mathrm{v} = KC^K x$ then $T_\lambda \mathrm{v} = x$. Recall that $\mathrm{Ker}\, B = \{0\}$ implies $\mathrm{Ker}\, B^K = \{0\}$. Hence, using again Lemma 3.2, we see that $\mathrm{v} \in U_\lambda$. Thus, T_λ is an isomorphism and we have also proved that its inverse is as claimed in (1.2). ∎

Proof of Theorem 1.2: As a first step, we rewrite (3.10) in the form

$$I = (sI - A^K)(sI - A)^{-1} + B^K KC(sI - A)^{-1} \qquad \forall\, s \in \rho(A). \tag{3.12}$$

To prove that (1) implies (2), we prove that if (2) is false then also (1) is false. Thus, assume that $I - K\mathbf{G}(\lambda)$ is invertible. Then from (3.9) (which we have shown to hold for all $s \in \rho(A)$) we obtain

$$B^K = (\lambda I - A^K)(\lambda I - A)^{-1} B [I - K\mathbf{G}(\lambda)]^{-1}.$$

If we substitute this into (3.12), we obtain

$$I = (\lambda I - A^K)(\lambda I - A)^{-1} \left[I + B[I - K\mathbf{G}(\lambda)]^{-1} KC(sI - A)^{-1} \right].$$

This shows that $\lambda I - A^K$ is onto. Since by our assumption $U_\lambda = \operatorname{Ker}[I - K\mathbf{G}(\lambda)] = \{0\}$, by Theorem 1.1 λ is not an eigenvalue of A^K, so that $\lambda I - A^K$ is one-to-one. Hence, $\lambda I - A^K$ is invertible, so that (1) is false.

Conversely, suppose that (1) false, i.e., $\lambda \in \rho(A^K)$ (and recall that $\lambda \in \rho(A)$ was assumed). It follows from (1.1) that $I - K\mathbf{G}(\lambda)$ is invertible, so that (2) is false. ∎

Remark 3.3. Our first two theorems are closely related to Lemma 4.4 in Salamon [8]. By Salamon's lemma we get the equivalence between (1) and (2) in our Theorem 1.1. However, our theorem asserts additionally the isomorphism between U_λ and X_λ. By Salamon's lemma the implication (1) $\Rightarrow$ (2) in our Theorem 1.2 is true. Our theorem asserts additionally the implication (2) $\Rightarrow$ (1).

The system Σ (or the pair (A, C)) is called *approximately observable* if the following holds: if $x_0 \in \mathcal{D}(A)$ and $C\mathbb{T}_t x_0 = 0$ for all $t \geq 0$, then $x_0 = 0$. (Sometimes, this property is called approximate observability in infinite time.) The system Σ (or the pair (A, B)) is called *approximately controllable* if (A^*, B^*) is approximately observable. (Approximate controllability can be characterized also directly, without using duality, but this is not needed here.)

Proof of Theorem 1.4: Suppose that $x \in \mathcal{D}(A^K)$, $x \neq 0$ and $(\lambda I - A^K)x = 0$. We show that this assumption leads to a contradiction, so that $\lambda \notin \sigma_p(A^K)$.

The first formula in (3.5) implies that

$$(\lambda I - A)x = BKC^K x.$$

Since (A, C) is approximately observable, (A^K, C^K) is also approximately observable (see [13, Section 6]) and hence $C^K x \neq 0$. Thus, $\mathrm{v} = KC^K x \neq 0$ and

$$(\lambda I - A)x = B\mathrm{v}. \tag{3.13}$$

Since $\overline{\lambda} \in \sigma_p(A^*)$, there exists $z \in \mathcal{D}(A^*)$ such that $z \neq 0$ and $(\overline{\lambda} I - A^*)z = 0$. Using the duality pairing between $\mathcal{D}(A^*)$ and X_{-1}, we have from (3.13) that

$$\langle (\lambda I - A)x, z \rangle = \langle B\mathrm{v}, z \rangle ,$$

which becomes $\langle \mathrm{v}, B^* z \rangle = 0$. But $U = \mathbb{C}$ and $\mathrm{v} \neq 0$, and hence $B^* z = 0$. This shows that (A^*, B^*) is not approximately observable. Hence, (A, B) is not approximately controllable, in contradiction with our assumptions. ∎

4. Proof of the results from Section 2

We continue to use the notation introduced at the beginning of Section 3.

Proof of Proposition 2.3: Let us show the first equality in the proposition. For this, we observe that $\sigma_e(A) \subset \sigma_u(A)$. Indeed, this inclusion can be obtained directly from Theorem 5.35 in Kato [5, Chapter IV], which says that $\sigma_e(A)$ is invariant under relatively compact perturbations. Thus, we only have to show that any point $\lambda \in \sigma(A) \setminus \sigma_e(A)$ is in the union of $\sigma_F(A)$ and $\sigma_u(A)$. Let λ be such a point. Then $\operatorname{Ran}(\lambda I - A)$ is closed and at least one of $\nu(\lambda)$ and $\mu(\lambda)$ is finite. We cannot have $\nu(\lambda) = \mu(\lambda) = 0$, because that would imply $\lambda \in \rho(A)$. If $\nu(\lambda) = \mu(\lambda)$, then $\lambda \in \sigma_F(A)$. If $\nu(\lambda) \neq \mu(\lambda)$, then $\lambda I - A$ has a non-zero index. This index is invariant under compact perturbations, see Theorem 5.26 in [5, Chapter IV]. This implies that $\lambda \in \sigma(A + P)$ for any compact P, so that $\lambda \in \sigma_u(A)$.

It remains to prove the second equality in the proposition. Suppose that $\lambda \in \sigma_F(A)$. Denote $N = \operatorname{Ker}(\lambda I - A)$ and $R = \operatorname{Ran}(\lambda I - A)$. By the definition of $\sigma_F(A)$, R is closed. We introduce two decompositions of X:

$$X = N \oplus N^\perp, \qquad X = R^\perp \oplus R. \tag{4.1}$$

Let T be the restriction of $\lambda I - A$ to $N^\perp \cap \mathcal{D}(A)$, regarded as a closed operator from $N^\perp$ to R. Then T is one-to-one and onto, so that it is boundedly invertible. Still by the definition of $\sigma_F(A)$, the dimensions of N and $R^\perp$ are equal and finite. Let P_0 be an invertible operator from N to $R^\perp$. We extend P_0 to an operator $P \in \mathcal{F}(X)$ by putting $Px = 0$ for all $x \in N^\perp$. Using the first decomposition from (4.1) in the domain space X and the second in the range space X, we have the matrix representations

$$\lambda I - A = \begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix}, \qquad P = \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is now easy to see that $\lambda I - A - P$ is boundedly invertible, so that $\lambda \notin \sigma(A + P)$. According to Lemma 4.2, we have $\lambda \notin \sigma_u(A)$, so that $\sigma_F(A) \cap \sigma_u(A) = \emptyset$. ∎

Proof of Proposition 2.5: Take $\lambda \in \rho(A) \cap \rho(A^K)$. Since $(sI - A)^{-1}B \in \mathcal{L}(U, X)$ and $C^K(sI - A^K)^{-1} \in \mathcal{L}(X, Y)$, (3.4) shows that $(sI - A^K)^{-1} - (sI - A)^{-1}$ is compact. Now the statement follows from Problem 5.38 in Kato [5, Chapter IV]. ∎

Lemma 4.1. *If $K \in \mathcal{L}(Y, U)$ and $G \in \mathcal{L}(U, Y)$, then*

$$\dim\{\mathrm{Ker}\,(I - KG)\} = \dim\{\mathrm{Ker}\,(I - GK)\}.$$

The easy proof of this fact is left to the reader.

Proof of Theorem 2.6: Take $\lambda \in \rho(A) \cap \sigma(A^K)$. First we show that $\lambda \in \sigma_p(A^K)$. According to Theorem 1.2, $I - K\mathbf{G}(\lambda)$ is not invertible. Since K is compact, this implies that $I - K\mathbf{G}(\lambda)$ has a non-trivial kernel and

$$\nu = \dim\{\mathrm{Ker}\,[I - K\mathbf{G}(\lambda)]\} = \dim\{\mathrm{Ran}\,[I - K\mathbf{G}(\lambda)]\}^{\perp} < \infty. \tag{4.2}$$

According to Theorem 1.1, the first equality above implies that $\lambda \in \sigma_p(A^K)$ and the number $\nu(\lambda)$ (now defined with A^K in place of A) is given by $\nu(\lambda) = \nu$.

Now we introduce the dual system Σ^d, which has input space Y, state space X and output space U, see [8, 10]. The semigroup generator of Σ^d is A^*, its control operator is C^*, its observation operator is B^* and its transfer function $\mathbf{G}^d$ is given by $\mathbf{G}^d(s) = \mathbf{G}(\overline{s})^*$. Then K^* is an admissible feedback operator for Σ^d and the corresponding closed-loop system is the dual of Σ^K:

$$\left(\Sigma^d\right)^{K^*} = \left(\Sigma^K\right)^d.$$

Since $\overline{\lambda} \in \rho(A^*) \cap \sigma((A^K)^*)$, we can argue like in the first step of this proof, but with Σ^d in place of Σ. This leads to

$$\dim\left\{\mathrm{Ker}\,\left(\lambda I - A^K\right)^*\right\} = \dim\left\{\mathrm{Ker}\,[I - K^*\mathbf{G}(\lambda)^*]\right\}. \tag{4.3}$$

Using Lemma 4.1, we can rewrite this in the form

$$\dim\left\{\mathrm{Ran}\,\left(\lambda I - A^K\right)\right\}^{\perp} = \dim\left\{\mathrm{Ker}\,[I - \mathbf{G}(\lambda)^*K^*]\right\}.$$

Note that the left-hand side above is the $\mu(\lambda)$ (now defined with A^K in place of A). Using the compactness of K^*, we obtain that

$$\mu(\lambda) = \dim\{\mathrm{Ran}\,[I - K\mathbf{G}(\lambda)]\}^{\perp}.$$

Now using (4.2), we conclude that $0 < \mu(\lambda) = \nu(\lambda) < \infty$.

What we have obtained is almost the definition of a Fredholm eigenvalue, but we still have to show that $\mathrm{Ran}\,(\lambda I - A^K)$ is closed. We argue by contradiction, assuming that this is not the case. Then, by definition, $\lambda \in \sigma_e(A^K)$ and we can use Proposition 2.5 to conclude that $\lambda \in \sigma_e(A)$. This is in contradiction with our assumption that $\lambda \in \rho(A)$, so that $\mathrm{Ran}\,(\lambda I - A^K)$ must be closed. ∎

In order to prove Theorem 2.8, we need some preliminary material, mostly concerning the unmovable spectrum of an operator A, denoted $\sigma_u(A)$, defined in Section 2. The following lemma gives an alternative definition of $\sigma_u(A)$.

Lemma 4.2. *We denote by $\mathcal{F}(X)$ the space of all finite-rank operators in $\mathcal{L}(X)$. If $A:\mathcal{D}(A)\rightarrow X$ is closed and $\lambda\in\mathbb{C}$, then $\lambda\in\sigma_u(A)$ if and only if*

$$\lambda\in\sigma(A+P) \qquad \forall\, P\in\mathcal{F}(X).$$

Proof: We denote by $\sigma_f(A)$ the set of all λ which satisfy the condition in the lemma. Since $\mathcal{F}(X)\subset\mathcal{K}(X)$, it is clear that $\sigma_u(A)\subset\sigma_f(A)$. To prove the opposite inclusion, take $\lambda\in\sigma_f(A)$ and $P\in\mathcal{K}(X)$. We have to show that $\lambda\in\sigma(A+P)$. Let (F_n) be a sequence of finite-rank operators such that $\|P-F_n\|\rightarrow 0$. By definition, we have $\lambda\in\sigma(A+F_n)$. Since the spectrum of an operator is an upper semicontinuous function of the operator (see Kato [5, p. 208]), it follows that $\lambda\in\sigma(A+P)$. ∎

Lemma 4.3. *With the assumptions and the notation of Theorem* 1.1*, suppose that K is compact. Then $\sigma_u(A)\subset\sigma(A^K)$.*

Proof: Recall from [13, Remark 6.4] that M is an admissible feedback operator for Σ^K if and only if $K+M$ is an admissible feedback operator for Σ. Moreover, in this case, $(\Sigma^K)^M=\Sigma^{K+M}$. In particular, $-K$ is an admissible feedback operator for the system Σ^K, and the closed-loop system corresponding to this feedback is the original open-loop system Σ.

Take $\mu\in\sigma_u(A)$. Arguing by contradiction, assume that $\mu\in\rho(A^K)$. Then according to Theorem 2.6 (with the roles of Σ and Σ^K reversed), we have $\mu\in\sigma_F(A)$. But this contradicts Proposition 2.3, so that $\mu\in\sigma(A^K)$ must hold. ∎

Proof of Theorem 2.8: We show that $\sigma_u(A)\subset\sigma_u(A^K)$. Suppose that $\mu\in\sigma_u(A)$. We argue by contradiction. Thus, suppose that μ is not in $\sigma_u(A^K)$. Then, by Lemma 4.2, there exists a finite-rank operator $P\in\mathcal{L}(X)$ such that $\mu\in\rho(A^K+P)$. We factor $P=B_1C_1$, where $B_1\in\mathcal{L}(W,X)$, $C_1\in\mathcal{L}(X,W)$ and W is a finite-dimensional vector space. We consider the system $\widetilde{\Sigma}$ obtained from Σ by adding an additional input and an additional output, so that its control operator is $[B\ B_1]$, its observation operator is $[C\ C_1]^\top$ and there are no feedthrough terms introduced by the new input and output. We make this more clear: the input space of $\widetilde{\Sigma}$ is $U\times W$ and its output space is $Y\times W$. The state trajectories of $\widetilde{\Sigma}$ are described by

$$\dot{x}(t)\,=\,Ax(t)+Bu(t)+B_1w(t),$$

where w is the new input signal. We denote the components of the output of $\widetilde{\Sigma}$ by y and ξ. The first is given in terms of Laplace transforms by

$$\hat{y}(s)\,=\,C(sI-A)^{-1}x(0)+\mathbf{G}(s)\hat{u}(s)+C(sI-A)^{-1}B_1\hat{w}(s).$$

In particular, if Σ is regular, with feedthrough operator D, then for almost every $t\geq 0$, $y(t)=C_\Lambda x(t)+Du(t)$. The second component of the output of $\widetilde{\Sigma}$ is given by

$$\xi(t)\,=\,C_1x(t) \qquad \forall\, t\geq 0.$$

Some reasoning shows that the feedback operator $\widetilde{K} = \mathrm{diag}\,(K, I)$ is admissible for $\widetilde{\Sigma}$ and the semigroup of the closed-loop system $\widetilde{\Sigma}^{\widetilde{K}}$ is $A^K + P$. Since $\widetilde{K}$ is compact, by Lemma 4.3, $\mu \in \sigma(A^K + P)$. This contradicts our earlier conclusion that $\mu \in \rho(A^K + P)$. This shows that $\sigma_u(A) \subset \sigma_u(A^K)$. The converse inclusion is proved similarly, with the roles of Σ and Σ^K reversed. ∎

We denote the continuous spectrum of a closed operator T by $\sigma_c(T)$. The following proposition will be useful in the next section.

Proposition 4.4. *With the assumptions and the notation of Theorem* 1.1*, we assume additionally that B is bounded from below. If for some $\lambda \in \rho(A)$ we have $1 \in \sigma_c(K\mathbf{G}(\lambda))$, then $\lambda \in \sigma_c(A^K)$.*

The fact that $B \in \mathcal{L}(U, X_{-1})$ is bounded from below can be formulated equivalently as follows: for some $\beta \in \rho(A)$ and some $m > 0$, we have

$$\|(\beta I - A)^{-1} B u\|_X \geq m \cdot \|u\|_U \qquad \forall\, u \in U\,.$$

This is the case, for example, if U is finite-dimensional and $\mathrm{Ker}\, B = \{0\}$.

Proof: Suppose that $\lambda \in \rho(A)$ and $1 \in \sigma_c(K\mathbf{G}(\lambda))$. Then $\mathrm{Ker}\,[I - K\mathbf{G}(\lambda)] = \{0\}$ and $\mathrm{Ran}\,[I - K\mathbf{G}(\lambda)]$ is dense in U (it is not closed). By Theorem 1.1, $\mathrm{Ker}\,(\lambda I - A^K) = \{0\}$. The range of $\lambda I - A^K$ cannot be X, because that would imply (using the closed graph theorem) that $\lambda \in \rho(A^K)$, in contradiction with Theorem 1.2. To finish the proof, we only have to show that $\mathrm{Ran}\,(\lambda I - A^K)$ is dense in X. We will argue by contradiction, assuming that $\mathrm{Ran}\,(\lambda I - A^K)$ is not dense, so that $(\lambda I - A^K)^*$ has a non-trivial kernel. Recall that, using the dual of the closed-loop system in the proof of Theorem 2.6, we have obtained the formula (4.3). This formula implies that $I - K^*\mathbf{G}(\lambda)^*$ has a non-trivial kernel. By Lemma 4.1, it follows that $I - \mathbf{G}(\lambda)^* K^*$ has a non-trivial kernel. This is equivalent to the fact that the range of $I - K\mathbf{G}(\lambda)$ is not dense in U, in contradiction with our conclusion at the beginning of this proof. This contradiction shows that the range of $\lambda I - A^K$ must be dense. ∎

The following two results (the last two of this section) are not used in this paper. However, we think that they are of independent interest in this area.

Theorem 4.5. *We use the assumptions of Proposition* 4.4*. If for some $\lambda \in \rho(A)$,* $\mathrm{Ran}\,[I - K\mathbf{G}(\lambda)]$ *is closed, then also* $\mathrm{Ran}\,(\lambda I - A^K)$ *is closed.*

Proof: First note that if B is bounded from below (as an operator from U to X_{-1}) then also B^K is bounded from below (as an operator from U to X_{-1}^K). Indeed, this follows from the formula (3.3), valid for s with sufficiently large real part, since for such s, $I - K\mathbf{G}(s)$ is invertible. (We have used the characterization of boundedness from below of B given immediately after Proposition 4.4.)

Let $\lambda \in \rho(A)$ be such that $\operatorname{Ran}[I - K\mathbf{G}(\lambda)]$ is closed. We introduce Π_U, the orthogonal projector from U to $\{\operatorname{Ker}[I - K\mathbf{G}(\lambda)]\}^\perp$, and Π_X, the orthogonal projector from X to $\{\operatorname{Ker}(\lambda I - A^K)\}^\perp$. Then Theorem 1.1 implies that

$$\Pi_X(\lambda I - A)^{-1}B[I - \Pi_U] = 0.$$

We apply Π_X to both sides of (3.11) and we use the above formula, obtaining

$$\Pi_X\left[\xi - (\lambda I - A)^{-1}(\lambda I - A^K)\xi\right] = \Pi_X(\lambda I - A)^{-1}B\Pi_U K C^K \xi, \tag{4.4}$$

for all $\xi \in \mathcal{D}(A^K)$. Another useful formula is obtained by rewriting (3.9), using the fact that $(\lambda I - A^K) = (\lambda I - A^K)\Pi_X$:

$$\left(\lambda I - A^K\right)\Pi_X(\lambda I - A)^{-1}B = B^K[I - K\mathbf{G}(\lambda)]. \tag{4.5}$$

By the closed graph theorem, the closedness of $\operatorname{Ran}[I - K\mathbf{G}(\lambda)]$ (which has been assumed) is equivalent to the fact that $I - K\mathbf{G}(\lambda)$ restricted to $\{\operatorname{Ker}[I - K\mathbf{G}(\lambda)]\}^\perp$ is bounded from below. Similarly, the closedness of $\operatorname{Ran}(\lambda I - A^K)$ (which has to be proved) is equivalent to the fact that $(\lambda I - A^K)$ restricted to $\{\operatorname{Ker}(\lambda I - A^K)\}^\perp \cap \mathcal{D}(A^K) = \Pi_X \mathcal{D}(A^K)$ is bounded from below. Arguing by contradiction, we assume that $\operatorname{Ran}(\lambda I - A^K)$ is not closed. Then there exists a sequence (ξ_n) with terms in $\Pi_X \mathcal{D}(A^K)$ such that $\|\xi_n\|_X = 1$ and $(\lambda I - A^K)\xi_n \to 0$ (in X). Denote $v_n = \Pi_U K C^K \xi_n$, then clearly $v_n \in \{\operatorname{Ker}[I - K\mathbf{G}(\lambda)]\}^\perp$ and from (4.4) we see that

$$\lim_{n\to\infty} \|\xi_n - \Pi_X(\lambda I - A)^{-1}Bv_n\|_X = 0, \tag{4.6}$$

which implies

$$\lim_{n\to\infty} \left\|\left(\lambda I - A^K\right)\xi_n - \left(\lambda I - A^K\right)\Pi_X(\lambda I - A)^{-1}Bv_n\right\|_{X_{-1}^K} = 0.$$

Combining this with (4.5), we obtain

$$\lim_{n\to\infty} \left\|\left(\lambda I - A^K\right)\xi_n - B^K[I - K\mathbf{G}(\lambda)]v_n\right\|_{X_{-1}^K} = 0.$$

Since the first term above tends to zero in X, it also tends to zero in X_{-1}^K. This implies that also the second term tends to zero in X_{-1}^K. Since B^K is bounded from below as an operator from U to X_{-1}^K, it follows that

$$\lim_{n\to\infty} \|[I - K\mathbf{G}(\lambda)]v_n\|_U = 0.$$

The formula (4.6) together with $\|\xi_n\| = 1$ also shows that the sequence $(\|v_n\|)$ is eventually bounded from below (i.e., $\|v_n\| \geq \varepsilon > 0$ for all n sufficiently large). This, together with the last limit shows that $I - K\mathbf{G}(\lambda)$ is not bounded from below on $\{\operatorname{Ker}[I - K\mathbf{G}(\lambda)]\}^\perp$. Hence, $\operatorname{Ran}[I - K\mathbf{G}(\lambda)]$ is not closed, which contradicts our assumption. This contradiction shows that $\operatorname{Ran}(\lambda I - A^K)$ is closed. ∎

Proposition 4.6. *We use the assumptions of Proposition* 4.4 *(in particular, B is bounded from below). If for some $\lambda \in \rho(A)$, $I - K\mathbf{G}(\lambda)$ is not bounded from below, then also $\lambda I - A^K$ is not bounded from below.*

Proof: Suppose that $\lambda \in \rho(A)$ is such that $I - K\mathbf{G}(\lambda)$ is not bounded from below. Thus, there exists a sequence (v_n) with terms in U such that $\|\mathrm{v}_n\| = 1$ and $[I - K\mathbf{G}(\lambda)]\mathrm{v}_n \to 0$ (in U). It follows from (3.9) that

$$\left(\lambda I - A^K\right)(\lambda I - A)^{-1}B\mathrm{v}_n = B^K[I - K\mathbf{G}(\lambda)]\mathrm{v}_n \to 0 \quad (\text{in } X^K_{-1}).$$

Denote $z_n = (\lambda I - A)^{-1}B\mathrm{v}_n$. Since B is bounded from below (as an operator from U to X_{-1}) and $(\lambda I - A)^{-1}$ is bounded from below (as an operator from X_{-1} to X), we have that $\|z_n\| \geq \varepsilon > 0$ for all $n \in \mathbb{N}$, while $(\lambda I - A^K)z_n \to 0$ (in X^K_{-1}).

Choose $\beta \in \rho(A^K)$ and denote $x_n = (\beta I - A^K)^{-1}z_n$, so that $x_n \in \mathcal{D}(A^K)$. Then the previous conclusions about the sequence (z_n) translate into

$$\left\|\left(\beta I - A^K\right)x_n\right\| \geq \varepsilon > 0 \quad \text{and} \quad \left(\lambda I - A^K\right)x_n \to 0 \quad (\text{in } X). \tag{4.7}$$

We rewrite the first part of (4.7) in the form $\|(\beta - \lambda)x_n + (\lambda I - A^K)x_n\| \geq \varepsilon > 0$. Now the second part of (4.7) implies that for sufficiently large $n \in \mathbb{N}$ we have $\|(\beta - \lambda)x_n\| \geq \varepsilon/2 > 0$, so that the sequence (x_n) is eventually bounded from below (in X). This fact, together with the second part of (4.7) shows that $\lambda I - A^K$ is not bounded from below. ∎

5. Infinite chains of interconnected identical systems

In this section we apply the results of the previous sections to study the well-posedness and spectrum of certain infinite chains of interconnected linear systems. We believe that the study of these simple interconnected systems could be a first step towards understanding more complex infinite interconnections, which could be obtained by discretizing partial differential equations.

The first interconnected system that we consider, Σ^{casc} is a cascade connection of infinitely many copies of a well-posed system Σ_0, as shown in Figure 2 below. For simplicity, we assume that each of the identical systems Σ_0 in the chain is single-input single-output, so that all the signals in the figure are scalar. The input signal of Σ^{casc} is $v = (v_1, v_2, \ldots)$ with values in $U = l^2$ and the output signal is $y = (y_1, y_2, \ldots)$ with values in $Y = l^2$. If we denote the state space of Σ_0 by X_0, then the state space of Σ^{casc} is $X = l^2(X_0)$, i.e., square summable sequences with values in X_0. Information propagates only to the left, so that the infinite block matrix of the semigroup $\mathbb{T}^{casc}$ of this system must be upper triangular. It is not clear when this system is well-posed, and if it is, we would like to know something about the spectrum of A^{casc}, the infinitesimal generator of $\mathbb{T}^{casc}$.

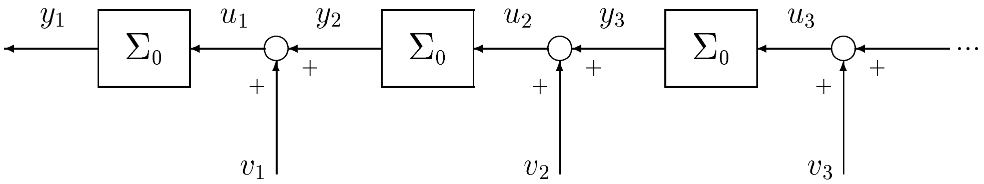

Figure 2: The linear system Σ^{casc} obtained by connecting an infinite chain of identical systems Σ_0 in cascade, with a well defined last system in the chain (the system on the left edge of the diagram). The input and output space is l^2.

Consider the system Σ obtained as the collection of all the systems Σ_0 in the chain working independently (i.e., with no interconnection). The input, state and output spaces are the same as for Σ^{casc}. In this simpler system, with input signal $u = (u_1, u_2, \ldots)$, u_1 only influences y_1, u_2 only influences y_2 etc. If we denote the transfer function of Σ_0 by $\mathbf{G}_0$, then $\mathbf{G}_0(s)$ is scalar and the transfer function of Σ is $\mathbf{G} = \mathbf{G}_0 I$, where I is the identity on l^2. This system is clearly well-posed and its semigroup $\mathbb{T}$ is block diagonal. The spectrum of its generator A is the same as the spectrum of the semigroup generator A_0 corresponding to Σ_0.

The system Σ^{casc} can be thought of as being obtained from Σ via the infinite sequence of interconnections $u_k = y_{k+1} + v_k$, where v_k are the components of the new input signal. In other words, Σ^{casc} is the closed-loop system obtained from Σ via static output feedback through the feedback operator $K = \mathbf{S}^*$, where $\mathbf{S}^*$ is the left-shift operator on l^2 encountered in Example 2.4.

For Σ^{casc} to be a well-posed system, we need that $\mathbf{S}^*$ should be an admissible feedback operator for Σ. This condition and the spectral properties of Σ^{casc} can be expressed in terms of $\mathbf{G}_0$, as the following proposition shows.

Proposition 5.1. (1) $\mathbf{S}^*$ *is an admissible feedback operator for* Σ *if and only if there exists an* $r < 1$ *such that for all* s *in some right half-plane, we have* $|\mathbf{G}_0(s)| \leq r$.

(2) *Suppose that* $\mathbf{S}^*$ *is an admissible feedback operator for* Σ*, denote by* Σ^{casc} *the corresponding closed-loop system and by* A^{casc} *its semigroup generator. Let* $\lambda \in \rho(A_0)$*. Then* $\lambda \in \sigma(A^{casc})$ *if and only if* $|\mathbf{G}_0(\lambda)| \geq 1$.

(3) *With the assumptions and the notation of part* (2)*, if* $|\mathbf{G}_0(\lambda)| > 1$ *then* $\lambda \in \sigma_p(A^{casc})$*, with multiplicity one.*

(4) *With the assumptions and the notation of part* (2)*, suppose that* B_0*, the control operator of* Σ_0*, is bounded from below. If* $|\mathbf{G}_0(\lambda)| = 1$ *then* $\lambda \in \sigma_c(A^{casc})$.

Proof: First we prove part (1). According to the feedback theory recalled in Section 3, $\mathbf{S}^*$ is an admissible feedback operator for Σ if and only if $\Delta = I - \mathbf{G}K = I - \mathbf{G}_0\mathbf{S}^*$ has a well-posed inverse. Take $s \in \rho(A_0)$. For $\mathbf{G}_0(s) = 0$ we have

$\Delta(s)^{-1} = I$, while for $\mathbf{G}_0(s) \neq 0$ we can factor $\Delta(s) = \mathbf{G}_0(s)\left(\frac{1}{\mathbf{G}_0(s)} I - \mathbf{S}^*\right)$. The last factor is invertible if and only if $\frac{1}{\mathbf{G}_0(s)}$ is not in $\sigma(\mathbf{S}^*)$, which is the closed unit disk (see Example 2.4). Thus, for each $s \in \rho(A_0)$, $\Delta(s)$ is invertible if and only if $|\mathbf{G}_0(s)| < 1$. Since $\Delta(s)$ is an anti-causal Toeplitz operator (i.e., the adjoint of a shift invariant operator), its inverse (when it exists) is also an anticausal Toeplitz operator. The symbol of $\Delta(s)^{-1}$ is

$$\varphi(z) = \frac{1}{1 - \mathbf{G}_0(s) z}, \qquad |z| = 1.$$

As is well known, the norm of a Toeplitz operator is equal to the L^∞ norm of its symbol. This gives, after a short computation,

$$\left\| \Delta(s)^{-1} \right\| = \frac{1}{1 - |\mathbf{G}_0(s)|}.$$

This shows that Δ^{-1} is well-posed if and only if there exists an $r < 1$ such that for all s in some right half-plane, we have $|\mathbf{G}(s)| \leq r$.

To prove part (2), assume that Σ^{casc} is well-posed and take $\lambda \in \rho(A_0)$. According to Theorem 1.2, $\lambda \in \sigma(A^{casc})$ if and only if $\Delta(\lambda)$ is not invertible. As we have seen in the first part of this proof, this happens if and only if $|\mathbf{G}_0(\lambda)| \geq 1$.

To prove part (3), assume that Σ^{casc} is well-posed and take $\lambda \in \rho(A_0)$ such that $|\mathbf{G}_0(\lambda)| > 1$. Then $1/\mathbf{G}_0(\lambda) \in \sigma_p(\mathbf{S}^*)$ and the multiplicity of this eigenvalue is one, as it is easy to verify. Hence, $1 \in \sigma_p(\mathbf{S}^* \mathbf{G}(\lambda))$, with multiplicity one. According to Theorem 1.1, we have $\lambda \in \sigma_p(A^{casc})$, with multiplicity one.

Now we prove (4), so we assume that B_0 is bounded from below. This implies that B, the control operator of Σ, is also bounded from below. If $|\mathbf{G}_0(\lambda)| = 1$ then $1/\mathbf{G}_0(\lambda) \in \sigma_c(\mathbf{S}^*)$ (see Example 2.4), so that $1 \in \sigma_c(\mathbf{S}^* \mathbf{G}(\lambda))$. According to Proposition 4.4, we have $\lambda \in \sigma_c(A^{casc})$. ∎

Now we consider a more complicated interconnected system Σ^K, shown in Figure 3. The constituent blocks as well as the input, state and output spaces are the same as before, but now information can propagate also to the right, via the static gain blocks α, where $\alpha \in \mathbb{R}$. The interconnections are described by

$$u_k = y_{k+1} + \alpha y_{k-1} + v_k \qquad \forall\, k \in \mathbb{N} \tag{5.1}$$

($y_0 = 0$ by definition). For $\alpha = 0$ this system reduces to Σ^{casc} considered earlier.

Like Σ^{casc} before, the system Σ^K can be obtained from the open-loop system Σ by static output feedback. This time, the feedback operator is $K = \mathbf{S}^* + \alpha \mathbf{S}$, where $\mathbf{S}$ is the right-shift operator on l^2. Indeed, the formula (5.1) can be written as $u = \mathbf{S}^* y + \alpha \mathbf{S} y + v$. Note that K is a Toeplitz operator on l^2, with symbol

$$\varphi_K(z) = z + \alpha z^{-1}, \qquad |z| = 1.$$

It is easy to see that the range of φ_K is an ellipse, symmetric with respect to the real and the imaginary lines, which passes through the points $\pm(1+\alpha)$ and $\pm i(1-\alpha)$.

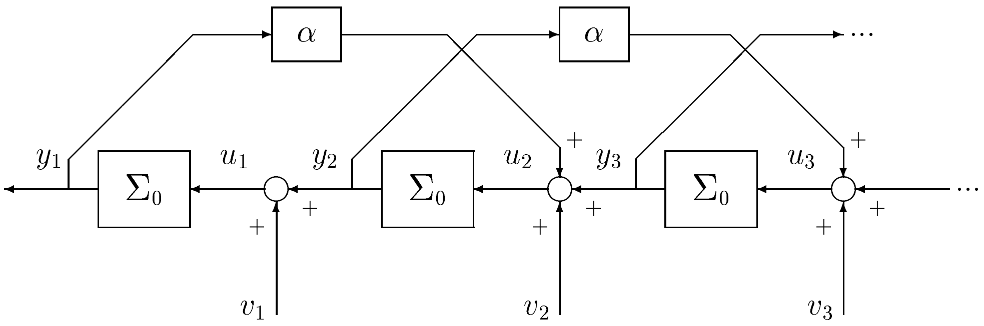

Figure 3: The linear system Σ^K obtained by interconnecting an infinite sequence of identical systems Σ_0 and static gains α. For $\alpha \neq 0$, information can flow in both directions. Note that for $\alpha = 0$, this system reduces to the one in Figure 2.

We denote by $\mathcal{E}$ the compact set containing 0 and whose boundary is this ellipse (equivalently, $\mathcal{E}$ is the convex hull of the ellipse). In particular, for $\alpha = 1$ we get a segment on the real line while for $\alpha = -1$ we get a segment on the imaginary line. We denote by $d(\zeta, \mathcal{E})$ the distance from a point $\zeta \in \mathbb{C}$ to the set $\mathcal{E}$.

Proposition 5.2. (1) *$K = \mathbf{S}^* + \alpha\mathbf{S}$ is an admissible feedback operator for Σ if and only if there exists a $\delta > 0$ such that for all s in some right half-plane, we have*

$$d\left(\frac{1}{\mathbf{G}_0(s)}, \mathcal{E}\right) \geq \delta. \tag{5.2}$$

(2) *Suppose that K is an admissible feedback operator for Σ and denote by A^K the semigroup generator of the corresponding closed-loop system. Let $\lambda \in \rho(A_0)$. Then $\lambda \in \sigma(A^K)$ if and only if $1/\mathbf{G}_0(\lambda) \in \mathcal{E}$.*

To prove this proposition, we need a certain corollary of a theorem by Krein, Widom and Devinatz (see, for example, Douglas [2, pp. 185-186] or Nikolskii [6, p. 319]), which we state here as a lemma:

Lemma 5.3. *Let T be a Toeplitz operator on l^2, with continuous symbol φ. Then $\sigma(T)$ consists of the curve $\{\varphi(z) \mid |z| = 1\}$ together with all the points not on the curve which are encircled by this curve (i.e., around which the winding number of the curve is not zero).*

Proof of Proposition 5.2: Recall that the admissibility of K is equivalent to the fact that $\Delta = I - \mathbf{G}_0 K$ has a well-posed inverse. Take $s \in \rho(A_0)$. For $\mathbf{G}_0(s) = 0$ we have $\Delta(s)^{-1} = I$, while for $\mathbf{G}_0(s) \neq 0$ we can factor $\Delta(s) = \mathbf{G}_0(s)\left(\frac{1}{\mathbf{G}_0(s)}I - K\right)$.

The last factor is invertible if and only if $1/\mathbf{G}_0(s)$ is not in $\sigma(K)$. By Lemma 5.3, $\sigma(K) = \mathcal{E}$. Thus, $\Delta(s)$ is invertible if and only if $1/\mathbf{G}_0(s) \notin \mathcal{E}$.

To prove the "if" part of (1), suppose that there exists $a \in \mathbb{R}$ and $\delta > 0$ such that for all $s \in \mathbb{C}$ with $\operatorname{Re} s > a$, (5.2) holds. In particular, this implies that $\mathbf{G}_0$ is bounded on this right half-plane. Let $\mathcal{G}$ be the closure of the image of this right half-plane through $\mathbf{G}_0$, so that $\mathcal{G}$ is compact. From (5.2) we see that $d(1/\zeta, \mathcal{E}) \geq \delta$ for all $\zeta \in \mathcal{G}$. The function $r(\zeta) = \|(I - \zeta K)^{-1}\|$ is continuous and hence bounded on $\mathcal{G}$. We have $\|\Delta(s)^{-1}\| = r(\mathbf{G}_0(s))$, so that the function $\|\Delta^{-1}\|$ is bounded on the half-plane where $\operatorname{Re} s > a$. This means that K is admissible for Σ.

To prove the "only if" part of (1), suppose that (5.2) is false on every right half-plane. Then there exists a sequence (s_n) in $\mathbb{C}$ with $\operatorname{Re} s_n \to \infty$ such that $\mathbf{G}_0(s_n) \neq 0$ and

$$\lim_{n\to\infty} d\left(\frac{1}{\mathbf{G}_0(s_n)}, \mathcal{E}\right) = 0.$$

For any bounded operator T and any $\lambda \in \rho(T)$ we have the estimate

$$\|(\lambda I - T)^{-1}\| \geq \frac{1}{d(\lambda, \sigma(T))},$$

see for example Butzer and Berens [1, p. 31]. Combining this with the previous formula, taking $\lambda = 1/\mathbf{G}_0(s_n)$ and $T = K$, we obtain

$$\|\Delta(s_n)^{-1}\| = \frac{1}{|\mathbf{G}_0(s_n)|} \cdot \left\| \left(\frac{1}{\mathbf{G}_0(s_n)} I - K\right)^{-1} \right\| \to \infty.$$

Hence, K is not an admissible feedback operator for Σ.

To prove (2), we assume that K is admissible, we denote by A^K the semigroup generator of the closed-loop system and we take $\lambda \in \rho(A_0)$. According to Theorem 1.2, $\lambda \in \sigma(A^K)$ if and only if $\Delta(\lambda)$ is not invertible. Thus, using a conclusion from the beginning of this proof, $\lambda \in \sigma(A^K)$ if and only if $1/\mathbf{G}_0(\lambda) \in \mathcal{E}$. ∎

Remark 5.4. Using the theorem of Krein, Widom and Devinatz as well as a proposition by Coburn (see [2, p. 185]), we can prove the following additional facts: Suppose that K is admissible and $\lambda \in \rho(A_0)$. If $|\alpha| < 1$ and $1/\mathbf{G}_0(\lambda)$ is in the interior of $\mathcal{E}$, then $\lambda \in \sigma_p(A^K)$, with multiplicity one, but $\overline{\lambda}$ is not an eigenvalue of $(A^K)^*$. If $|\alpha| > 1$ and $1/\mathbf{G}_0(\lambda)$ is in the interior of $\mathcal{E}$, then $\overline{\lambda} \in \sigma_p((A^K)^*)$, with multiplicity one, but λ is not an eigenvalue of A^K. (If $|\alpha| = 1$, then $\mathcal{E}$ has no interior.)

It would be interesting to understand more complex infinite interconnections. For example, it would be natural to consider chains of identical systems which have two inputs and two outputs each. One input and one output would be used to interconnect with the next system to the left, and the remaining input and output would be used for the exchange of information with the next system to the right. Understanding such interconnected systems would require the theory of block-Toeplitz operators, more precisely, Toeplitz operators that have a symbol which is a rational function whose values are 2×2 matrices. Such Toeplitz operators are discussed briefly in [2, p. 202] and in more detail in [3].

Acknowledgements. This work has been supported by a CNRS grant for scientific cooperation between France and the UK, 2001-2002.

References

[1] P.L. Butzer and H. Berens, *Semi-Groups of Operators and Approximation*, Die Grundl. der math. Wiss. in Einzeld. Vol. 145, Springer-Verlag, New York, 1967.

[2] R.G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York and London, 1972.

[3] R.G. Douglas, *Banach Algebra Techniques in the Theory of Toeplitz Operators*, American Math. Society, CBMS 15, Providence, 1973.

[4] N. Dunford and J.T. Schwartz, *Linear Operators, Part II, Spectral Theory*, Wiley-Interscience, New York, 1971.

[5] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1980.

[6] N.K. Nikol'skii, *Treatise on the Shift Operator*, Springer-Verlag, New York, 1986.

[7] R. Rebarber and S. Townley, Robustness and continuity of the spectrum for certain distributed parameter systems, *Automatica* **31**, pp.1533-1546, 1995.

[8] D. Salamon, Infinite-dimensional linear systems with unbounded control and observation: a functional analytic approach, *Trans. Am. Math. Soc.* **300**, pp.383-431, 1987.

[9] O. Staffans and G. Weiss, Transfer functions of regular linear systems. Part II: the system operator and the Lax-Phillips semigroup, *Trans. Amer. Math. Society*, to appear in 2002.

[10] O. Staffans and G. Weiss, Transfer functions of regular linear systems. Part III: inversions and duality, submitted.

[11] G. Weiss, Admissible observation operators for linear semigroups, *Israel J. Math.* **65**, pp.17-43, 1989.

[12] G. Weiss, Transfer functions of regular linear systems, Part I: Characterizations of regularity, *Trans. Amer. Math. Society* **342**, pp.827–854, 1994.

[13] G. Weiss, Regular linear systems with feedback, *Mathematics of Control, Signals and Systems* **7**, pp.23-57, 1994.

[14] C.Z. Xu and G. Weiss, Eigenvalues and eigenvectors of semigroup generators obtained from diagonal generators by output feedback, submitted in 2002.