How to get a conservative well-posed linear system out of thin air

PART II: CONTROLLABILITY AND STABILITY

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Abstract: Let A_0 be a possibly unbounded positive operator on the Hilbert space H, which is boundedly invertible. Let C_0 be a bounded operator from $\mathcal{D}(A_0^{\frac{1}{2}})$ (with the norm $\|z\|_{\frac{1}{2}}^2 = \langle A_0 z, z\rangle$) to another Hilbert space U. In Part I of this work we have proved that the system of equations

$$\ddot{z}(t) + A_0 z(t) + \frac{1}{2} C_0^* C_0 \dot{z}(t) \,=\, C_0^* u(t),$$
$$y(t) \,=\, -\,C_0 \dot{z}(t) + u(t),$$

determines a well-posed linear system Σ with input u and output y, input and output space U and state space $X = \mathcal{D}(A_0^{\frac{1}{2}}) \times H$. Moreover, Σ is conservative, which means that a certain energy balance equation is satisfied both by the trajectories of Σ and by those of its dual system. In this paper we show that Σ is exactly controllable if and only if it is exactly observable, if and only if it is exponentially stable. Moreover, if we denote by A the generator of the contraction semigroup associated to Σ (which acts on X), then Σ is exponentially stable if and only if one of the entries in the second column of $(i\omega I - A)^{-1}$ is uniformly bounded as a function of $\omega \in \mathbb{R}$. We also show that, under a mild assumption, Σ is approximately controllable if and only if it is approximately observable, if and only if it is strongly stable, if and only if the dual system is strongly stable. We prove many related results and we give examples based on wave and beam equations.

Keywords: well-posed linear system, exponential stability, strong stability, conservative system, exact controllability, beam equation, wave equation.

1. Introduction and main results

This paper is a continuation of our paper [35] in which we have investigated a class of conservative linear systems with a special structure, which occur often in applications. These systems are described by a second order differential equation (in a Hilbert space) and an output equation, and they may have unbounded control and observation operators. The main aim of [35] was to prove the well-posedness, conservativity and other regularity properties of such systems. Here we investigate conditions under which such systems are exponentially stable or strongly stable. It turns out that these stability properties are equivalent to certain controllability and observability properties, as well as to certain estimates.

We recall the construction from the paper [35], in order to be able to state the new results. Let H be a Hilbert space, and let $A_0 : \mathcal{D}(A_0) \to H$ be a self-adjoint, positive and boundedly invertible operator. We introduce the scale of Hilbert spaces H_α, $\alpha \in \mathbb{R}$, as follows: for every $\alpha \geq 0$, $H_\alpha = \mathcal{D}(A_0^\alpha)$, with the norm $\|z\|_\alpha = \|A_0^\alpha z\|_H$. The space $H_{-\alpha}$ is defined by duality with respect to the pivot space H as follows: $H_{-\alpha} = H_\alpha^*$ for $\alpha > 0$. Equivalently, $H_{-\alpha}$ is the completion of H with respect to the norm $\|z\|_{-\alpha} = \left\|A_0^{-\alpha} z\right\|_H$. The operator A_0 can be extended (or restricted) to each H_α, such that it becomes a bounded operator

$$A_0 : H_\alpha \to H_{\alpha-1} \qquad \forall\, \alpha \in \mathbb{R}.$$

Let C_0 be a bounded linear operator from $H_{\frac{1}{2}}$ to U, where U is another Hilbert space. We identify U with its dual, so that $U = U^*$. We denote $B_0 = C_0^*$, so that $B_0 \in \mathcal{L}(U, H_{-\frac{1}{2}})$. The class of systems studied in [35] and also here is described by

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t) + A_0 z(t) + \frac{1}{2} B_0 \frac{\mathrm{d}}{\mathrm{d}t} C_0 z(t) \;=\; B_0 u(t), \tag{1.1}$$

$$z(0) \;=\; z_0, \qquad \dot z(0) \;=\; w_0, \tag{1.2}$$

$$y(t) \;=\; -\frac{\mathrm{d}}{\mathrm{d}t} C_0 z(t) + u(t), \tag{1.3}$$

where $t \in [0,\infty)$ is the time. The equation (1.1) is understood as an equation in $H_{-\frac{1}{2}}$, i.e., all the terms are in $H_{-\frac{1}{2}}$. Most of the linear equations modelling the damped vibrations of elastic structures can be written in the form (1.1), where z stands for the displacement field and the term $B_0 \frac{\mathrm{d}}{\mathrm{d}t} C_0 z(t)$, informally written as $B_0 C_0 \dot z(t)$, represents a viscous feedback damping. The signal $u(t)$ is an external input with values in U (often a displacement, a force or a moment acting on the boundary) and the signal $y(t)$ is the output (measurement) with values in U as well. The state $x(t)$ of this system and its state space X are defined by

$$x(t) \;=\; \begin{bmatrix} z(t) \\ \dot z(t) \end{bmatrix}, \qquad X \;=\; H_{\frac{1}{2}} \times H.$$

We will use some fairly standard notation for certain function spaces: we refer to [35, Section 1] for the meaning of $\mathcal{H}^p(0,\infty;W)$, $\mathcal{H}^p_{loc}(0,\infty;W)$ (with $p \in \mathbb{N}$), $C^n(0,\infty;W)$ and $BC^n(0,\infty;W)$ (with $n \in \{0,1,2,\ldots\}$). We write C instead of C^0.

We assume that the reader understands the concepts of a well-posed linear system and of a conservative linear system. These were explained in [35, Sections 1,3,4] with suitable references to the literature. We will often use results from [35], which we refer to as "Part I". In such cases, we put the prefix I in front of the number of the item quoted. For example, Theorem I.1.4 refers to Theorem 1.4 in Part I, and (I.4.2) refers to formula (4.2) in Part I. The first main result of [35] has been the following (Theorem I.1.1):

Theorem 1.1. *With the above assumptions, the equations* (1.1)–(1.3) *determine a conservative linear system* Σ*, in the following sense:*

There exists a conservative linear system Σ *whose input and output spaces are both* U *and whose state space is* X*. If* $u \in L^2([0,\infty), U)$ *is the input function,* $x_0 = \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in X$ *is the initial state,* $x = \begin{bmatrix} z \\ w \end{bmatrix}$ *is the corresponding state trajectory and* y *is the corresponding output function, then*

(1)
$$z \in BC(0,\infty; H_{\frac{1}{2}}) \cap BC^1(0,\infty; H) \cap \mathcal{H}^2_{loc}(0,\infty; H_{-\frac{1}{2}}).$$

(2) *The two components of* x *are related by* $w = \dot{z}$.

(3) $C_0 z \in \mathcal{H}^1(0,\infty; U)$ *and the equations* (1.1) *(in* $H_{-\frac{1}{2}}$*) and* (1.3) *(in* U*) hold for almost every* $t \geq 0$ *(hence,* $y \in L^2([0,\infty), U)$*).*

If $\dot{z}$ is a continuous function of t, with values in $H_{\frac{1}{2}}$ (see Theorems I.1.2 and I.1.4 for sufficient conditions for this to be true), then (1.1) and (1.3) can be rewritten as

$$\ddot{z}(t) + A_0 z(t) + \frac{1}{2} B_0 C_0 \dot{z}(t) = B_0 u(t), \tag{1.4}$$

$$y(t) = -C_0 \dot{z}(t) + u(t). \tag{1.5}$$

We introduce the space $Z_0 = H_1 + A_0^{-1} B_0 U$, which is a Hilbert space if we define on it a suitable norm, see Theorem I.1.2. We can rewrite the equations (1.4), (1.5) as a first order system as follows:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= \overline{C}x(t) + u(t), \end{cases} \tag{1.6}$$

where

$$A = \begin{bmatrix} 0 & I \\ -A_0 & -\frac{1}{2}B_0 C_0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \tag{1.7}$$

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \;\middle|\; A_0 z + \frac{1}{2} B_0 C_0 w \in H \right\}, \tag{1.8}$$

$$\overline{C} : Z_0 \times H_{\frac{1}{2}} \to U, \qquad \overline{C} = [0 \quad -C_0]. \tag{1.9}$$

We denote by C the restriction of $\overline{C}$ to $\mathcal{D}(A)$. A is the generator of a strongly continuous semigroup of contractions on X, denoted $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$. For the concepts of semigroup generator, control operator, observation operator and transfer function of a well-posed linear system, we refer to Weiss [31], [32], or to Section I.3. We denote by $\mathbb{C}_\omega$ the open right half-plane in $\mathbb{C}$ where $\operatorname{Re} s > \omega$. We know from Proposition I.5.3 that for any $s \in \rho(A)$ (in particular, for any $s \in \mathbb{C}_0$), the operator $s^2 I + A_0 + \frac{s}{2} B_0 C_0 \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$ has a bounded inverse denoted $V(s)$:

$$V(s) = \left(s^2 I + A_0 + \frac{s}{2} B_0 C_0\right)^{-1} \in \mathcal{L}(H_{-\frac{1}{2}}, H_{\frac{1}{2}}). \tag{1.10}$$

The following proposition is a restatement of a part of Theorem I.1.3.

Proposition 1.2. *With the notation of Theorem* 1.1 *and* (1.7)–(1.10), *the semigroup generator of* Σ *is* A, *its control operator is* B *and its observation operator is* C. *The transfer function of* Σ *is given for all* $s \in \mathbb{C}_0$ *by*

$$\mathbf{G}(s) = \overline{C}(sI - A)^{-1} B + I = I - C_0 s V(s) B_0$$

and we have $\|\mathbf{G}(s)\| \leq 1$ *for all* $s \in \mathbb{C}_0$.

Now we have all the necessary ingredients to state the new results of this paper. The following theorems use various controllability, observability and stability concepts. The precise definition of these concepts is given in Section 2.

Theorem 1.3. *With the above notation, the following assertions are equivalent:*

(1) *The pair* (A, B) *is exactly controllable (in some finite time).*

(2) *The pair* (A, C) *is exactly observable (in some finite time).*

(3) *The semigroup* $\mathbb{T}$ *is exponentially stable.*

(4) *The pair* (A, B) *is optimizable.*

(5) *The pair* (A, C) *is estimatable.*

(6) *We have* $\sup_{s\in\mathbb{C}_0} \|A_0^{\frac{1}{2}} V(s)\|_{\mathcal{L}(H)} < \infty$.

(7) *We have* $\sup_{s\in\mathbb{C}_0} \|sV(s)\|_{\mathcal{L}(H)} < \infty$.

(8) *For a dense subset* E *of* $\mathbb{R}$, *we have* $iE \subset \rho(A)$ *and* $\sup_{\omega\in E} \|A_0^{\frac{1}{2}} V(i\omega)\|_{\mathcal{L}(H)} < \infty$.

(9) *For a dense subset* E *of* $\mathbb{R}$, *we have* $iE \subset \rho(A)$ *and* $\sup_{\omega\in E} \|\omega V(i\omega)\|_{\mathcal{L}(H)} < \infty$.

A more precise statement concerning the equivalence of (1), (2) and (3), which gives some information on the time of exact controllability and observability, and which is valid for any conservative system, is Proposition 3.2. The equivalence of (1)–(5) remains valid for *every* conservative system, see Proposition 3.3.

By a well-known theorem of Jan Prüss and Huang Falun, an operator semigroup $\mathbb{T}$ with generator A is exponentially stable if and only if $(sI - A)^{-1}$ is uniformly bounded on $\mathbb{C}_0$. We refer to Section 2 for precise references, further comments and related results (Propositions 2.5 and 2.6). In the specific case of the semigroup generated by A from (1.7)–(1.8), the resolvent $(sI-A)^{-1}$ can be written as a 2×2 matrix of operators, see Proposition I.5.3 (or formula (4.1) later in this paper). Thus, to verify the exponential stability of $\mathbb{T}$, we would have to verify that the four entries of this 2×2 matrix are all uniformly bounded on $\mathbb{C}_0$. However, conditions (6) and (7) in Theorem 1.3 tell us that, in fact, we only have to verify one of the two entries in the second column of the matrix of $(sI-A)^{-1}$. Conditions (8) and (9) tell us that, in fact, it suffices to check the boundedness of one of these entries on a dense subset of the imaginary axis, and we can still conclude exponential stability.

The version of this theorem corresponding to bounded B and C, i.e., with $C_0\in\mathcal{L}(H,U)$, is in Liu [22, Sections 2-3], but without conditions (4)–(7). Using the boundedness of C_0 (and hence also of B_0), Liu was able to give in [22, Theorem 3.4] also other, Hautus-type conditions which are equivalent to the exponential stability of $\mathbb{T}$. For unbounded C_0, we were only able to obtain a Hautus-type estimate as a necessary condition for exponential stability, see Proposition 4.1.

We mention that semigroups of the type discussed in this paper do not necessarily satisfy the spectrum determined growth condition. For a counterexample (a damped wave equation on a compact manifold) see Lebeau [19].

In the proof of Theorem 1.3 (more precisely, to show that (6)$\Longrightarrow$(3)) we use the following proposition, which is of independent interest. For bounded C_0, this proposition follows easily from [22, Theorem 3.4], but for unbounded C_0 the proof is more delicate (see Section 4). Related results for a bounded (possibly not positive) operator in place of $C_0^*C_0$ were given in Liu, Liu and Rao [23].

Proposition 1.4 *With the above notation, suppose that C_0 is bounded from below, in the sense that there exists a $c>0$ such that $\|C_0z\|_U\geq c\|z\|_H$, for all $z\in H_{\frac{1}{2}}$.*

Then $\mathbb{T}$ is exponentially stable.

A result similar to Theorem 1.3 holds also for strong stability, with an additional assumption on the spectrum $\sigma(A_0)$:

Theorem 1.5. *With the above notation, assume that $\sigma(A_0)$ is countable (this happens, e.g., if A_0^{-1} is compact). Then the following assertions are equivalent:*

(1) $\mathbb{T}$ *is strongly stable.*

(2) *The pair (A,C) is exactly observable in infinite time.*

(3) *The pair (A,C) is approximately observable in infinite time.*

(4) $\mathbb{T}$ *is weakly stable (equivalently, $\mathbb{T}^*$ is weakly stable).*

(5) $\mathbb{T}^*$ *is strongly stable.*

(6) *The pair (A,B) is exactly controllable in infinite time.*

(7) *The pair (A,B) is approximately controllable in infinite time.*

(8) *For any $z\in H_1$, if z is an eigenvector of A_0, then $C_0z\neq 0$.*

Note that the statement "A_0^{-1} is compact" does *not* imply that $(sI - A)^{-1}$ is compact (to see this, take $U = H$ and $C_0 = A_0^{\frac{1}{2}}$). This theorem follows from a more general result concerning all conservative systems, see Proposition 3.4 here. In the proof, we use also the famous strong stability theorem of Arendt and Batty [2].

Systems with A and B as above have been studied in Guo and Luo [10, 11], establishing connections between the exponential stability of A and the exact controllability of the undamped system $\ddot{z}(t) + A_0 z(t) = B_0 u(t)$, under the additional hypothesis that the undamped system is well-posed. (Unfortunately, the main result on diagonal systems in [10] (Theorem 4) is incorrectly formulated, and it is also incorrectly quoted in [11].) In [11] the emphasis is on eigenvalues and eigenvectors of A, assuming that the eigenvalues of A_0 satisfy a gap condition and $u(t)$ is scalar.

In Section 2 we give the background needed here. Section 3 concerns the stability properties of conservative systems, so that the results there refer to a more general context than the main results stated earlier. In Section 4 we prove our main results, while Section 5 is devoted to two examples: a system involving the beam equation and another one based on the wave equation.

2. Background on controllability, observability, optimizability, estimatability and stability

In this section we recall some controllability, observability and stability concepts, quoting the relevant literature. Throughout this section, U, X and Y are Hilbert spaces and $A : \mathcal{D}(A) \to X$ is the generator of a strongly continuous semigroup $\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$ on X. The space X_1 is $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, where $\beta \in \rho(A)$ is fixed, while X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1} z\|$. We assume that the reader understands the concept of an admissible (in particular, infinite-time admissible) control operator for $\mathbb{T}$. This has been presented in Section I.2, with suitable references. If $B \in \mathcal{L}(U, X_{-1})$ is admissible, then for every $\tau \geq 0$ we denote by Φ_τ the operator

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{t-\sigma} B u(\sigma)\, \mathrm{d}\sigma, \tag{2.1}$$

as in (I.2.3). We have $\Phi_\tau \in \mathcal{L}(L^2([0,\infty), U), X)$. If B is admissible, then for every $x_0 \in X$ and every $u \in L^2([0,\infty), U)$, the function $x(t) = \mathbb{T}_t x_0 + \Phi_t u$ is called the *state trajectory* corresponding to the initial state x_0 and the input function u. We have $x \in \mathcal{H}^1_{loc}(0, \infty; X)$ and $\dot{x}(t) = Ax(t) + Bu(t)$ (equality in X_{-1}) for almost every $t \geq 0$. If, moreover, B is infinite-time admissible, then we denote, as in (I.2.5),

$$\tilde{\Phi} u = \lim_{\tau \to \infty} \int_0^\tau \mathbb{T}_t B u(t)\, \mathrm{d}t, \tag{2.2}$$

and we have $\tilde{\Phi} \in \mathcal{L}(L^2([0,\infty), U), X)$.

Similarly, we assume that the reader understands the concepts of an admissible (in particular, infinite-time admissible) observation operator for $\mathbb{T}$, also presented in Section I.2. If $C \in \mathcal{L}(X_1, Y)$ is admissible, then we denote by Ψ the unique continuous operator from X to $L^2_{loc}([0,\infty), Y)$ such that

$$(\Psi x_0)(t) = C\mathbb{T}_t x_0 \qquad \forall\, x_0 \in \mathcal{D}(A). \tag{2.3}$$

In particular, if C is infinite-time admissible, then $\Psi \in \mathcal{L}(X, L^2([0,\infty), Y))$. Recall that B is an (infinite-time) admissible control operator for $\mathbb{T}$ if and only if B^* is an (infinite-time) admissible observation operator for $\mathbb{T}^*$.

Definition 2.1. Let A be the generator of a strongly continuous semigroup $\mathbb{T}$ on X and let $B \in \mathcal{L}(U, X_{-1})$ be an admissible control operator for $\mathbb{T}$.

The pair (A, B) is *exactly controllable* in time $T > 0$, if for every $x_0 \in X$ there exists a $u \in L^2([0,T], U)$ such that $\Phi_T u = x_0$.

(A, B) is *exactly controllable* if the above property holds for some $T > 0$.

(A, B) is *exactly controllable in infinite time* if B is infinite-time admissible and the operator $\tilde{\Phi}$ from (2.2) is onto.

(A, B) is *approximately controllable* in time $T > 0$ if $\operatorname{Ran}\Phi_T$ is dense in X.

(A, B) is *approximately controllable in infinite time* if $\cup_{\tau>0}\operatorname{Ran}\Phi_\tau$ is dense in X.

(A, B) is *optimizable* if for any $x_0 \in X$, there exists $u \in L^2([0,\infty), U)$ such that the state trajectory corresponding to x_0 and u is in $L^2([0,\infty), X)$.

Note that the exact (or approximate) controllability in infinite time of (A, B) does *not* imply its exact (or approximate) controllability in time T, for some $T > 0$. Clearly, exact controllability implies optimizability and also approximate controllability in some finite time. Optimizability is one possible generalization of the concept of stabilizability, as known from finite-dimensional control theory.

Remark 2.2. Let $B \in \mathcal{L}(U, X_{-1})$ be an infinite-time admissible control operator for $\mathbb{T}$. Then (A, B) is approximately controllable in infinite time if and only if the range of $\tilde{\Phi}$ from (2.2) is dense in X. The proof is easy.

Now we introduce the corresponding observability concepts via duality.

Definition 2.3. Suppose that $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$ (equivalently, C^* is an admissible control operator for the adjoint semigroup $\mathbb{T}^*$). We say that (A, C) is *exactly observable* (in time T) (in infinite time) if (A^*, C^*) is exactly controllable (in time T) (in infinite time). Similarly, (A, C) is *approximately observable* (in time T) (in infinite time) if (A^*, C^*) is approximately controllable (in time T) (in infinite time). Finally, the pair (A, C) is called *estimatable* if (A^*, C^*) is optimizable.

Let Ψ be the operator defined in (2.3) and for every $\tau \geq 0$ put $\Psi_\tau = \mathbf{P}_\tau \Psi$. Then (A, C) is exactly observable in time $T > 0$ if and only if Ψ_T is bounded from below. (A, C) is exactly observable in infinite time if and only if C is infinite-time admissible and Ψ is bounded from below. (A, C) is approximately observable in time T (or in infinite time) if and only if $\Psi_T x_0 = 0$ (or $\Psi x_0 = 0$) implies $x_0 = 0$.

Recall that the *growth bound* of a strongly continuous semigroup $\mathbb{T}$ is $\omega_0(\mathbb{T}) = \lim_{t\to\infty} \frac{1}{t} \log \|\mathbb{T}_t\| = \inf_{t>0} \frac{1}{t} \log \|\mathbb{T}_t\|$, see for example Pazy [24].

Definition 2.4. The semigroup $\mathbb{T}$ is *exponentially stable* if its growth bound is negative: $\omega_0(\mathbb{T}) < 0$. $\mathbb{T}$ is *strongly stable* if

$$\lim_{t\to\infty} \|\mathbb{T}_t x_0\| = 0 \qquad \forall\, x_0 \in X\,.$$

Finally, $\mathbb{T}$ is *weakly stable* if $\lim\limits_{t\to\infty} \langle \mathbb{T}_t x_0, y_0\rangle = 0$, for all $x_0, y_0 \in X$.

Let $\mathbb{T}$ be a strongly continuous semigroup on X, with generator A. A well-known spectral mapping result of Prüss [25, p. 852] implies that if the function $\|(sI - A)^{-1}\|$ is bounded on $\mathbb{C}_0$, then $\mathbb{T}$ is exponentially stable. A little later and independently, this result was explicitly stated and proved by Huang Falun [13]. A short proof was given in Weiss [30, Section 4]. Here we need a result which is closely related to the one just mentioned, without being an obvious consequence of it. The result is very slightly more general than another result of Huang Falun, see [13, Theorem 3]. Moreover, the proposition below gives an estimate for the growth bound $\omega_0(\mathbb{T})$.

Proposition 2.5. *Let $\mathbb{T}$ be a strongly continuous semigroup on X with generator A. Assume that $\omega_0(\mathbb{T}) \leq 0$ and E is a dense subset of $\mathbb{R}$ such that $iE \subset \rho(A)$ and*

$$\|(i\omega I - A)^{-1}\| \leq M \qquad \forall\, \omega \in E\,,$$

for some $M > 0$. Then $\mathbb{T}$ is exponentially stable, more precisely, $\omega_0(\mathbb{T}) \leq -\frac{1}{M}$.

Proof. By a result in Butzer and Berens [7, p. 31] all numbers $s \in \mathbb{C}$ with $|\mathrm{Re}\, s| < \frac{1}{M}$ (this is a vertical strip) belong to $\rho(A)$ and we have

$$\|(sI - A)^{-1}\| \leq \frac{M}{1 - |\mathrm{Re}\, s| \cdot M} \quad \text{for} \quad |\mathrm{Re}\, s| < \frac{1}{M}. \tag{2.4}$$

On the other hand, we know from the Hille-Yosida theorem that $\|(sI - A)^{-1}\|$ is bounded on any half-plane $\mathbb{C}_\gamma$ with $\gamma > 0$. This fact, combined with (2.4), shows that $\|(sI - A)^{-1}\|$ is bounded on any half-plane $\mathbb{C}_\alpha$ with $\alpha > -\frac{1}{M}$. Now by yet another result of Huang, [13, Theorem 4], we conclude that $\omega_0(\mathbb{T}) \leq -\frac{1}{M}$. (The last step is equivalent to applying the Prüss-Huang result mentioned before the proposition for the semigroups generated by $A + \lambda I$ with $\lambda < \frac{1}{M}$.) ∎

Proposition 2.6. *Suppose that X, $\mathbb{T}$, A, U and B are as in Definition* 2.1. *Then the following three statements are equivalent:*

(1) $\mathbb{T}$ *is exponentially stable.*

(2) (A, B) *is optimizable,* $\mathbb{C}_0 \subset \rho(A)$ *and, for some* $M > 0$,

$$\left\|(sI - A)^{-1}B\right\|_{\mathcal{L}(U,X)} \leq M \qquad \forall\, s \in \mathbb{C}_0 .$$

(3) (A, B) *is optimizable,* $\omega_0(\mathbb{T}) \leq 0$, *there exists a dense subset of* $\mathbb{R}$, *denoted* E, *such that* $iE \subset \rho(A)$ *and, for some* $M > 0$,

$$\left\|(i\omega I - A)^{-1}B\right\|_{\mathcal{L}(U,X)} \leq M \qquad \forall\, \omega \in E .$$

Proof. The equivalence of (1) and (2) is exactly Proposition 5.1 in Weiss and Rebarber [33]. It is easy to see that (1) implies (3), with $E = \mathbb{R}$ (by using also (2) and limits as $s \to i\omega$). Now suppose that (3) holds (this implies $\mathbb{C}_0 \subset \rho(A)$). We argue as in the proof of [33, Proposition 5.1] obtaining formula (5.1) from [33], valid for all $s \in \mathbb{C}_0$. Taking limits, we see that this formula holds also with $i\omega$ in place of $s \in \mathbb{C}_0$, where $\omega \in E$. Continuing to reason as in [33], we obtain that $(i\omega I - A)^{-1}$ is uniformly bounded as a function of $\omega \in E$. Since $\omega_0(\mathbb{T}) \leq 0$, we can apply Proposition 2.5 to conclude that $\mathbb{T}$ is exponentially stable. ■

Observability (or dually, controllability) and strong stability concepts are linked to properties of Lyapunov equations and we state in dual form the following result from Section 3 of Hansen and Weiss [12].

Proposition 2.7. *Let A be the generator of the strongly continuous semigroup $\mathbb{T}$ on X and let $C \in \mathcal{L}(X_1, Y)$ be an admissible observation operator for $\mathbb{T}$. Then the following three statements are equivalent:*

(a) *There exist operators* $\Pi \in \mathcal{L}(X), \Pi \geq 0$ *which satisfy the following equation:*

$$A^*\Pi z + \Pi A z = \,-\, C^*C z \qquad \forall\, z \in \mathcal{D}(A) . \tag{2.5}$$

(b) C *is an infinite-time admissible observation operator for* $\mathbb{T}$.

(c) *There exists an operator* $P \in \mathcal{L}(X)$ *such that for any* $z \in \mathcal{D}(A)$,

$$Pz = \lim_{\tau \to \infty} \int_0^\tau \mathbb{T}_t^* C^* C \mathbb{T}_t z \,\mathrm{d}t . \tag{2.6}$$

Moreover, if C is infinite-time admissible, then the following statements hold:

(d) P *from* (2.6) *is the smallest nonnegative solution of* (2.5).

(e) *If P is invertible, then $\mathbb{T}$ is strongly stable.*

(f) *If $\mathbb{T}$ is strongly stable, then P is the unique self-adjoint solution of* (2.5).

(g) *If $\mathbb{T}$ is uniformly bounded and $P > 0$, then $\mathbb{T}$ is weakly stable.*

The operator P introduced above is called *the observability Gramian* of (A, C), and (2.5) is called a *Lyapunov equation.* Note that, in terms of the operator Ψ from (2.3), we have $P = \Psi^*\Psi$. The following is well known and easy to prove.

Proposition 2.8. *Suppose that C is an infinite-time admissible observation operator for the semigroup $\mathbb{T}$ generated by A. Then (A, C) is approximately observable in infinite time if and only if $P > 0$ (where P is the observability Gramian of (A, C)).*

The *controllability Gramian* of (A, B) is, by definition, the observability Gramian of (A^*, B^*). Thus, the controllability Gramian of (A, B) is defined by

$$Rx = \lim_{\tau\to\infty} \int_0^\tau \mathbb{T}_t BB^*\mathbb{T}_t^* x \,\mathrm{d}t \qquad \forall\, x \in \mathcal{D}(A^*), \tag{2.7}$$

we have $R = \tilde{\Phi}\tilde{\Phi}^*$ and the Lyapunov equation satisfied by R is

$$RA^*z + ARz = -BB^*z \qquad \forall\, z \in \mathcal{D}(A^*).$$

The dual version of Proposition 2.7 is straightforward.

For more details on Gramians we refer to Hansen and Weiss [12], Jacob and Partington [14], Russell and Weiss [28] and Grabowski [8]. For more details on exact controllability in an operator-theoretic setting we refer also to Avdonin and Ivanov [3], Jacob and Zwart [15], Rebarber and Weiss [27], Tucsnak and Weiss [29] and the references therein. In the PDE setting, the relevant literature is overwhelming, and we mention the books of Lions [20], Lagnese and Lions [17], Bensoussan *et al* [6] and Komornik [16] and the paper of Bardos, Lebeau and Rauch [5].

3. Conservative linear systems

Recall from Section I.3 that for any well-posed linear system Σ with input function u, state trajectory x and output function y,

$$\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \end{bmatrix} = \Sigma_\tau \begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \end{bmatrix}, \tag{3.1}$$

where $\mathbf{P}_\tau$ denotes the truncation of a function to $[0, \tau]$ and

$$\Sigma_\tau = \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix}. \tag{3.2}$$

We denote the input, state and output spaces of Σ by U, X and Y, respectively. Then the operators Σ_τ appearing above are bounded from $X \times L^2([0, \tau], U)$ to $X \times L^2([0, \tau], Y)$, which means that for some $c_\tau \geq 0$

$$\|x(\tau)\|^2 + \int_0^\tau \|y(t)\|^2 \,\mathrm{d}t \leq c_\tau^2 \left(\|x(0)\|^2 + \int_0^\tau \|u(t)\|^2 \,\mathrm{d}t \right).$$

As explained in Section I.1, the system Σ is *conservative* if the operators Σ_τ are unitary, from $X \times L^2([0,\tau],U)$ to $X \times L^2([0,\tau],Y)$. This implies that for any input function $u \in \mathcal{H}^1(0,\infty;U)$ and any initial state $x(0)=x_0 \in X$ with $Ax_0+Bu(0) \in X$, the function $\|x(t)\|^2$ is in $C^1[0,\infty)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2 \qquad \forall\, t \geq 0, \tag{3.3}$$

see Proposition I.4.3. Conversely, if (3.3) holds for both the system Σ and for its dual system Σ^d, then Σ is conservative, see Corollary I.4.4.

Proposition 3.1. *Let Σ be a conservative linear system with input space U, state space X, output space Y, semigroup $\mathbb{T}$, control operator B, observation operator C and transfer function $\mathbf{G}$. Then the following statements are true:*

(1) *$\mathbb{T}$ is a semigroup of contractions.*

(2) *B is infinite-time admissible.*

(3) *C is infinite-time admissible.*

(4) *$\|\mathbf{G}(s)\| \leq 1$ for all $s \in \mathbb{C}_0$.*

Indeed, the above proposition is an immediate consequence of Proposition I.4.5 (which concerns the larger class of dissipative linear systems). The following proposition is probably well known (especially the equivalence of (2) and (3)) but we are not aware of a reference which states the equivalence of all three conditions.

Proposition 3.2. *With the notation of Proposition* 3.1 *and denoting the generator of $\mathbb{T}$ by A, for each $\tau > 0$, the following statements are equivalent:*

(1) *The pair (A,B) is exactly controllable in time τ.*

(2) *The pair (A,C) is exactly observable in time τ.*

(3) *$\|\mathbb{T}_\tau\| < 1$ (in particular, $\mathbb{T}$ is exponentially stable).*

Proof. (3) $\Longrightarrow$ (2): With the notation from (3.2), it is clear that for all $x_0 \in X$

$$\|\mathbb{T}_\tau x_0\|^2 + \|\Psi_\tau x_0\|^2 = \|x_0\|^2 \qquad \forall\, \tau \geq 0. \tag{3.4}$$

If (3) holds, then $\|\mathbb{T}_\tau\|^2 = 1-\varepsilon^2$, with $\varepsilon > 0$. Now (3.4) implies that $\|\Psi_\tau x_0\|^2 \geq \varepsilon^2\|x_0\|^2$, so that Σ is exactly observable in time τ.

(2) $\Longrightarrow$ (3): If (2) holds, then there exists $\varepsilon > 0$ such that $\|\Psi_\tau x_0\| \geq \varepsilon\|x_0\|$, for all $x_0 \in X$. Now (3.4) implies that $\|\mathbb{T}_\tau x_0\|^2 \leq (1-\varepsilon^2)\|x_0\|^2$, hence $\|\mathbb{T}_\tau\| < 1$.

(3) $\Longleftrightarrow$ (1): (3) is equivalent to the fact that $\|\mathbb{T}_\tau^*\| < 1$. (1) is equivalent to the fact that (A^*,B^*) is exactly observable in time τ. Since the dual system Σ^d is conservative, according to Proposition I.4.2 and the equivalence of (2) and (3) proved earlier, we get that (1) is equivalent to (3). ∎

The equivalence of (1)–(5) in Theorem 1.3 is an immediate consequence of Theorem 1.1, Proposition 1.2 and the following simple result about conservative systems:

Proposition 3.3. *With the notation of Proposition* 3.2*, the following five statements are equivalent:*

(1) *The pair* (A, B) *is exactly controllable.*

(2) *The pair* (A, C) *is exactly observable.*

(3) $\mathbb{T}$ *is exponentially stable.*

(4) *The pair* (A, B) *is optimizable.*

(5) *The pair* (A, C) *is estimatable.*

Proof. The equivalence of (1)–(3) follows from the previous proposition. It is well known and easy to see that (1) implies (4) and (2) implies (5) (for any well-posed system). Suppose that (5) holds. We know from Proposition 3.1 that C is infinite-time admissible. Now it follows from [33, Proposition 5.5] that (3) holds. The proof of (4)$\Longrightarrow$(3) is similar, by using the dual version of [33, Proposition 5.5]. ∎

Another result linking strong stability, observability and controllability of conservative systems is the following. It is related to Proposition 6 in [34].

Proposition 3.4. *Let* Σ, $\mathbb{T}$, A, B *and* C *be as in Proposition* 3.2*. Assume that the intersection* $\sigma(A) \cap i\mathbb{R}$ *is countable (this happens, for example, if* $(\beta I - A)^{-1}$ *is compact for some* $\beta \in \rho(A)$*). Then the following seven assertions are equivalent:*

(1) $\mathbb{T}$ *is strongly stable.*

(2) *The pair* (A, C) *is exactly observable in infinite time.*

(3) *The pair* (A, C) *is approximately observable in infinite time.*

(4) $\mathbb{T}$ *is weakly stable (equivalently,* $\mathbb{T}^*$ *is weakly stable).*

(5) $\mathbb{T}^*$ *is strongly stable.*

(6) *The pair* (A, B) *is exactly controllable in infinite time.*

(7) *The pair* (A, B) *is approximately controllable in infinite time.*

Proof. (1) $\Longrightarrow$ (2): If $\mathbb{T}$ is strongly stable, then we see from (3.4) that $\lim_{\tau \to \infty} \|\Psi_\tau x_0\| = \|x_0\|$. This implies that Ψ is an isometry from X to $L^2([0, \infty), Y)$.

(2) $\Longrightarrow$ (3): This implication is obvious.

(3) $\Longrightarrow$ (4): The fact that Σ is conservative implies that C is infinite-time admissible. According to Proposition 2.8, (3) means that $P > 0$, where P is the observability Gramian. By the last part of Proposition 2.7, $\mathbb{T}$ is weakly stable.

(4) $\Longrightarrow$ (1): Since $\mathbb{T}$ is weakly stable, A has no eigenvalues on $i\mathbb{R}$. Together with the assumption that $\sigma(A) \cap i\mathbb{R}$ is countable, this means that the conditions of the famous stability theorem of Arendt and Batty [2] are satisfied. According to this theorem, $\mathbb{T}$ is strongly stable.

(4) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6) $\Longleftrightarrow$ (7): This is similar to the equivalence of (1) – (4), but with the dual system Σ^d in place of Σ (recall that Σ^d is also conservative). ∎

4. Proof of the main results

In this section we prove Theorems 1.3, 1.4, 1.5 as well as other related results. We use the assumptions and the notation from Section 1: The conservative linear system Σ is the one constructed in Theorem 1.1 from the operators $A_0 \in \mathcal{L}(H_1, H)$ and $C_0 \in \mathcal{L}(H_{\frac{1}{2}}, U)$. The spaces H_α with $\alpha \in \mathbb{R}$ are constructed from the fractional powers of A_0. The notation $\|z\|_\alpha$ means the norm of z in H_α, in particular, $\|z\|_0$ is the norm of z in H. We put $B_0 = C_0^*$. The operators A and B are defined in (1.7), (1.8) and $\overline{C}$ is defined in (1.9). C is the restriction of $\overline{C}$ to $\mathcal{D}(A)$. The semigroup of contractions generated by A on $X = H_{\frac{1}{2}} \times H$ is denoted by $\mathbb{T}$, and the transfer function of Σ is denoted by $\mathbf{G}$. Recall also the $\mathcal{L}(H_{-\frac{1}{2}}, H_{\frac{1}{2}})$-valued function $V(s)$ from (1.10) and the space Z_0 defined after (1.5).

Proposition 4.1. *With the above notation, if $\mathbb{T}$ is exponentially stable, then denoting $M = \sup_{\omega\in\mathbb{R}} \|(i\omega I - A)^{-1}\|_{\mathcal{L}(X)}$ we have, for every $z \in H_{\frac{1}{2}}$,*

$$\|(\omega^2 I - A_0)z\|_{-\frac{1}{2}} + \frac{\omega}{2}\|B_0 C_0 z\|_{-\frac{1}{2}} \geq \frac{1}{M}\|z\|_0 \qquad \forall\, \omega \in [0,\infty).$$

Proof. We know from Proposition I.5.3 that on $H_{\frac{1}{2}} \times H_{-\frac{1}{2}}$ (in particular, on X) we have, for all $s \in \rho(A)$ with $s \neq 0$,

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s}[I - V(s)A_0] & V(s) \\ -V(s)A_0 & sV(s) \end{bmatrix}. \tag{4.1}$$

For $s = 0$ the formula remains valid, if we replace the left upper block in the matrix with $\frac{1}{2}A_0^{-1}B_0C_0$, see (I.5.4). Since $\mathbb{T}$ is exponentially stable, any point $i\omega$ (with $\omega \in \mathbb{R}$) is in $\rho(A)$ and $(i\omega I - A)^{-1}$ is uniformly bounded. Looking at the left lower block of $(i\omega I - A)^{-1}$, we have

$$\sup_{\omega\in\mathbb{R}} \|V(i\omega)A_0^{\frac{1}{2}}\|_{\mathcal{L}(H)} = \sup_{\omega\in\mathbb{R}} \|V(i\omega)A_0\|_{\mathcal{L}(H_{\frac{1}{2}},H)} \leq \sup_{\omega\in\mathbb{R}} \|(i\omega I - A)^{-1}\|_{\mathcal{L}(X)} = M.$$

The last estimate means that for any $g \in H$ and any $\omega \in \mathbb{R}$,

$$\|V(i\omega)A_0^{\frac{1}{2}}g\|_0 \leq M\|g\|_0.$$

If we choose $g = A_0^{-\frac{1}{2}}\left(-\omega^2 I + A_0 + \frac{i\omega}{2}B_0C_0\right)z$, with $z \in H_{\frac{1}{2}}$ fixed, then we get that for all $\omega \in \mathbb{R}$

$$M\left\|A_0^{-\frac{1}{2}}\left(-\omega^2 I + A_0 + \frac{i\omega}{2}B_0C_0\right)z\right\|_0 \geq \|z\|_0.$$

From here, using the triangle inequality, we get the estimate in the proposition. ■

Proof of Proposition 1.4. Recall that the inner product on X is defined by

$$\left\langle \begin{bmatrix} z_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ w_2 \end{bmatrix} \right\rangle_X = \langle A_0^{\frac{1}{2}}z_1, A_0^{\frac{1}{2}}z_2\rangle_H + \langle w_1, w_2\rangle_H.$$

In the sequel, we drop the subscript H when writing the inner product on H (but we use subscripts for other spaces). We define a new inner product on X by

$$\left\langle \begin{bmatrix} z_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ w_2 \end{bmatrix} \right\rangle_{\text{new}} = \langle A_0^{\frac{1}{2}} z_1, A_0^{\frac{1}{2}} z_2 \rangle + \delta \langle w_1, z_2 \rangle + \delta \langle z_1, w_2 \rangle + \langle w_1, w_2 \rangle$$

$$= \left\langle \begin{bmatrix} I & \delta A_0^{-1} \\ \delta I & I \end{bmatrix} \begin{bmatrix} z_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ w_2 \end{bmatrix} \right\rangle_X ,$$

where $\delta > 0$ is such that $\delta \|A_0^{-\frac{1}{2}}\| < 1$ (later we shall impose further restrictions on δ). The 2×2 matrix J appearing above defines a self-adjoint and positive bounded operator on X. Indeed, $J \geq 0$ follows from the Cauchy-Schwarz inequality:

$$\left\langle \begin{bmatrix} I & \delta A_0^{-1} \\ \delta I & I \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} \right\rangle_X \geq \|A_0^{\frac{1}{2}} z\|^2 - 2\delta \|z\| \cdot \|w\| + \|w\|^2$$

$$\geq \|A_0^{\frac{1}{2}} z\|^2 - 2\delta \|A_0^{-\frac{1}{2}}\| \cdot \|A_0^{\frac{1}{2}} z\| \cdot \|w\| + \|w\|^2$$

$$\geq \|A_0^{\frac{1}{2}} z\|^2 - 2\|A_0^{\frac{1}{2}} z\| \cdot \|w\| + \|w\|^2 = \left(\|A_0^{\frac{1}{2}} z\| - \|w\| \right)^2 \geq 0.$$

It is easy to check that J is boundedly invertible, hence $J > 0$, which shows that our definition of a new inner product is correct, and the new norm on X defined by $\|x\|_{\text{new}} = \sqrt{\langle x, x \rangle_{\text{new}}}$ is equivalent to the original norm. Thus, it will suffice to prove that $\mathbb{T}$ is exponentially stable with respect to the new norm.

We shall estimate $\langle Ax, x \rangle_{\text{new}}$, where $x = \begin{bmatrix} z \\ w \end{bmatrix} \in \mathcal{D}(A)$. We have

$$\operatorname{Re} \langle Ax, x \rangle_{\text{new}} = \operatorname{Re} \langle (J - I)Ax, x \rangle_X + \operatorname{Re} \langle Ax, x \rangle_X .$$

We know from (I.5.3) that $\operatorname{Re} \langle Ax, x \rangle_X = -\frac{1}{2} \|C_0 w\|_U^2$. Computing the product $(J - I)A$, we get that

$$\operatorname{Re} \langle Ax, x \rangle_{\text{new}} = -\delta \left\| A_0^{\frac{1}{2}} z \right\|^2 - \frac{\delta}{2} \operatorname{Re} \langle C_0 w, C_0 z \rangle_U + \delta \|w\|^2 - \frac{1}{2} \|C_0 w\|_U^2 .$$

Now remember that C_0 is bounded from below, so that $\|w\| \leq \frac{1}{c} \|C_0 w\|_U$. Therefore, the above estimate and the Cauchy-Schwarz inequality imply

$$\operatorname{Re} \langle Ax, x \rangle_{\text{new}} \leq -\delta \left\| A_0^{\frac{1}{2}} z \right\|^2 + \frac{\delta}{2} \|C_0 w\|_U \cdot \|C_0 z\|_U - \left(\frac{1}{2} - \frac{\delta}{c^2} \right) \|C_0 w\|_U^2 .$$

Let $k > 0$ be such that $\|C_0 z\|_U \leq k \|A_0^{\frac{1}{2}} z\|$ for all $z \in H_{\frac{1}{2}}$. Then

$$\operatorname{Re} \langle Ax, x \rangle_{\text{new}} \leq -\delta \left\| A_0^{\frac{1}{2}} z \right\|^2 + \frac{\delta k}{2} \|C_0 w\|_U \cdot \|A_0^{\frac{1}{2}} z\| - \left(\frac{1}{2} - \frac{\delta}{c^2} \right) \|C_0 w\|_U^2 .$$

The right-hand side above is a quadratic form in the two numbers $\left\|A_0^{\frac{1}{2}} z\right\|$ and $\|C_0 w\|_U$. The matrix of this quadratic form is

$$Q = -\begin{bmatrix} \delta & -\frac{\delta k}{4} \\ -\frac{\delta k}{4} & \frac{1}{2} - \frac{\delta}{c^2} \end{bmatrix}.$$

This Q will be negative definite if

$$\frac{1}{2} - \frac{\delta}{c^2} > 0 \quad \text{and} \quad 16\left(\frac{1}{2} - \frac{\delta}{c^2}\right) > \delta k^2.$$

Both of these conditions can be satisfied if we choose δ sufficiently small. Suppose that δ has been correctly chosen, so that $Q \leq -\gamma I$ for some $\gamma > 0$. Then we obtain

$$\operatorname{Re}\langle Ax, x\rangle_{\text{new}} \leq -\gamma\left(\left\|A_0^{\frac{1}{2}} z\right\|^2 + \|C_0 w\|_U^2\right)$$

$$\leq -\gamma\left(\left\|A_0^{\frac{1}{2}} z\right\|^2 + c^2\|w\|^2\right) \leq -\gamma \min(1, c^2)\|x\|_X^2.$$

Recall that the two norms on X are equivalent, so that $\|x\|_X \geq m\|x\|_{\text{new}}$ for some $m > 0$. Denoting $\eta = \gamma \min(1, c^2) m^2$ (so that $\eta > 0$), we obtain

$$\operatorname{Re}\langle Ax, x\rangle_{\text{new}} \leq -\eta\|x\|_{\text{new}}^2,$$

so that $A + \eta I$ is dissipative with respect to the new inner product. Hence, the growth bound of $\mathbb{T}$ (which does not depend on the norm) is $\omega_0(\mathbb{T}) \leq -\eta$. ∎

Lemma 4.2. *If we define* $B_b = \begin{bmatrix} 0 \\ I \end{bmatrix} \in \mathcal{L}(H, X)$, *then* (A, B_b) *is optimizable.*

Proof. Consider a new conservative linear system $\tilde{\Sigma}$ obtained from the same operator A_0 on the same Hilbert space H, but with a larger input space $\tilde{U}$ and with C_0 replaced by $\tilde{C}_0$, which are defined as follows:

$$\tilde{U} = U \times H, \qquad \tilde{C}_0 = \begin{bmatrix} C_0 \\ I \end{bmatrix}.$$

Thus, following the standard construction from Section 1, B_0 will be replaced by $\tilde{B}_0 = \tilde{C}_0^* = [B_0 \ I]$. According to (1.7), the semigroup $\tilde{\mathbb{T}}$ of $\tilde{\Sigma}$ is generated by

$$\tilde{A} = \begin{bmatrix} 0 & I \\ -A_0 & -\frac{1}{2}B_0 C_0 - \frac{1}{2}I \end{bmatrix},$$

with $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$, as defined in (1.8). It is clear that $\tilde{C}_0$ is bounded from below, so that $\tilde{\mathbb{T}}$ is exponentially stable according to Proposition 1.4.

Now consider the system Σ_b with input space H, state space X and output space H described by

$$\begin{cases} \dot{x}(t) &= Ax(t) + B_b u(t), \\ y(t) &= B_b^* x(t). \end{cases}$$

Clearly Σ_b is well-posed, since B_b is bounded. The static output feedback $u = -\frac{1}{2}y$ applied to this system leads to a closed-loop system whose semigroup generator is $A - \frac{1}{2}B_b^* B_b = \tilde{A}$, which (as we already know) is exponentially stable. In particular, it follows that for any initial state $x_0 \in X$, the functions u and x defined by $u(t) = -\frac{1}{2}B_b^* \tilde{\mathbb{T}}_t x_0$ and $x(t) = \tilde{\mathbb{T}}_t x_0$ are both in L^2. Thus, (A, B_b) is optimizable. ■

Proof of Theorem 1.3. According to Theorem 1.1 and Proposition 1.2, Σ is a conservative linear system with semigroup generator A, control operator B and observation operator C. Now the equivalence of (1)–(5) in Theorem 1.3 follows from Proposition 3.3. It is also easy to see that (3) implies (6), (7), (8) and (9). Indeed, if $\mathbb{T}$ is exponentially stable, then $(sI - A)^{-1}$ exists and is uniformly bounded on $\mathbb{C}_\alpha$ for some $\alpha < 0$, see the proof of Proposition 2.5. Looking at the right column of $(sI - A)^{-1}$ in (4.1), we obtain that (6)–(9) all hold.

We prove the equivalence of (6) and (7). Suppose that (7) is false, i.e., there is a sequence (s_n) in $\mathbb{C}_0$ such that $\|s_n V(s_n)\| \to \infty$. Since $\mathbb{T}$ is uniformly bounded, $\|(sI - A)^{-1}\|$ is bounded on any right half-plane $\mathbb{C}_\gamma$ with $\gamma > 0$. Since $sV(s)$ is one of the entries of $(sI - A)^{-1}$, it follows that for large n, the sequence (s_n) must be outside $\mathbb{C}_\gamma$. Since this is true for each $\gamma > 0$, we must have $\operatorname{Re} s_n \to 0$. Since $0 \in \rho(A)$, $\|sV(s)\|$ is bounded on a neighborhood of 0. Thus, without loss of generality, we may assume that $|\operatorname{Im} s_n| \geq \varepsilon > 0$.

By the uniform boundedness theorem, there exists a vector $x \in H$ such that

$$\lambda_n = \|s_n V(s_n) x\| \to \infty .$$

Denote $z_n = \frac{1}{\lambda_n} V(s_n) x$, then clearly $z_n \in H_{\frac{1}{2}}$, $\|s_n z_n\| = 1$ (hence $\|z_n\|$ is bounded) and

$$\frac{1}{\lambda_n} x = s_n^2 z_n + A_0 z_n + \frac{s_n}{2} B_0 C_0 z_n \to 0 \quad \text{in} \quad H .$$

Taking inner products with z_n, we obtain

$$\frac{1}{\lambda_n} \langle x, z_n \rangle = s_n^2 \|z_n\|^2 + \left\| A_0^{\frac{1}{2}} z_n \right\|^2 + \frac{s_n}{2} \|C_0 z_n\|^2 \to 0 . \tag{4.2}$$

Looking here only at the imaginary parts and dividing by $\operatorname{Im} s_n$, we obtain

$$2(\operatorname{Re} s_n) \|z_n\|^2 + \frac{1}{2} \|C_0 z_n\|^2 \to 0 , \tag{4.3}$$

in particular, $\|C_0 z_n\| \to 0$. Now we look at the real parts of the terms in (4.2):

$$-|s_n|^2 \|z_n\|^2 + 2(\operatorname{Re} s_n)^2 \|z_n\|^2 + \left\| A_0^{\frac{1}{2}} z_n \right\|^2 + \frac{\operatorname{Re} s_n}{2} \|C_0 z_n\|^2 \to 0 . \tag{4.4}$$

Recalling that $\operatorname{Re} s_n \to 0$ and $\|s_n z_n\| = 1$, we conclude that $\lim_{n\to\infty} \|A_0^{\frac{1}{2}} z_n\| = 1$. Thus, $\lim_{n\to\infty} \frac{1}{\lambda_n}\|A_0^{\frac{1}{2}} V(s_n)x\| = 1$, so that $\|A_0^{\frac{1}{2}} V(s_n)\| \to \infty$. We have obtained that assertion (6) is false, so that (6) implies (7). The proof of the fact that (7) implies (6) is similar, with the following modifications: x and λ_n are now chosen such that $\lambda_n = \|A_0^{\frac{1}{2}} V(s_n)x\| \to \infty$. We take again $z_n = \frac{1}{\lambda_n} V(s_n)x$, and now $\|A_0^{\frac{1}{2}} z_n\| = 1$ (instead of $\|s_n z_n\| = 1$). Now the reasoning up to (4.4) remains the same, and from (4.4) we conclude that $\lim_{n\to\infty} \|s_n z_n\| = 1$, which implies that $\|s_n V(s_n)\| \to \infty$.

To prove the equivalence of (8) and (9), we argue similarly as in the proof of the equivalence of (6) and (7), but now $\operatorname{Re} s_n = 0$. This makes the proof simpler, since now we do not need (4.3) and in (4.4) two terms disappear.

We prove that (6) implies (3). If (6) (and hence also (7)) holds, then we see from (4.1) that $(sI - A)^{-1} B_b$ is uniformly bounded on $\mathbb{C}_0$, where B_b is the operator from Lemma 4.2. Since, by the same lemma, (A, B_b) is optimizable, we can apply Proposition 2.6 (the equivalence of points (1) and (2) in that proposition) to conclude that $\mathbb{T}$ is exponentially stable, i.e., (3) holds.

Finally, we prove that (8) implies (3). If (8) (and hence also (9)) holds, then we see from (4.1) that $(i\omega I - A)^{-1} B_b$ is uniformly bounded for $\omega \in E$, where B_b is the operator from Lemma 4.2. Since (A, B_b) is optimizable, and since $\mathbb{T}$ is uniformly bounded, the conditions in point (3) of Proposition 2.6 are satisfied. According to Proposition 2.6, $\mathbb{T}$ is exponentially stable. ■

In order to prove Theorem 1.5, we have to prove several preliminary results. We denote by $\sigma_p(A)$ the set of eigenvalues (the point spectrum) of A.

Lemma 4.3. *If $\lambda \in \sigma_p(A)$ and $x \in \mathcal{D}(A)$ is a corresponding eigenvector (i.e., $(\lambda I - A)x = 0$ and $x \neq 0$), then x is of the form*

$$x = \begin{bmatrix} z \\ \lambda z \end{bmatrix}, \qquad z \in Z_0, \tag{4.5}$$

where $Z_0 = H_1 + A_0^{-1} B_0 U$ and

$$\left(\lambda^2 I + A_0 + \frac{\lambda}{2} B_0 C_0\right) z = 0. \tag{4.6}$$

If $\lambda \notin \mathbb{R}$, then this implies

$$\|C_0 z\|^2 = 4|\operatorname{Re}\lambda| \cdot \|z\|^2, \qquad \langle A_0 z, z\rangle = |\lambda|^2 \cdot \|z\|^2. \tag{4.7}$$

Proof. The formulas (4.5) and (4.6) are an immediate consequence of $\mathcal{D}(A) \subset Z_0 \times H_{\frac{1}{2}}$ (which follows from (1.8)), and of $(\lambda I - A)x = 0$. If we take the scalar product of the sides of (4.6) with z and use the extension of the scalar product to the duality pairing between $H_{-\frac{1}{2}}$ and $H_{\frac{1}{2}}$, we obtain

$$\left\langle \left(\lambda^2 I + A_0\right) z, z\right\rangle + \frac{\lambda}{2}\|C_0 z\|^2 = 0.$$

Since $\mathbb{T}$ is a contraction semigroup, λ must be in the closed left half-plane. Denoting $\lambda = -\sigma + i\omega$, with $\sigma \geq 0$ and $\omega \in \mathbb{R}$, this means

$$\left\langle \left((\sigma^2 - \omega^2) I + A_0 \right) z, z \right\rangle - 2i\sigma\omega \|z\|^2 + \frac{-\sigma + i\omega}{2} \|C_0 z\|^2 = 0. \tag{4.8}$$

Looking at the imaginary part of this, we see that $\omega \neq 0$ implies

$$-2\sigma \|z\|^2 + \frac{1}{2} \|C_0 z\|^2 = 0,$$

which is the same as the first equality in (4.7). Now we look at the real part of (4.8), using the expression for $\|C_0 z\|^2$ that we have just found, obtaining (after a short computation) the second equality in (4.7). ■

Lemma 4.4. *Suppose that $\omega \in \mathbb{R}$ is such that $i\omega \in \sigma_p(A)$. Then also $-i\omega \in \sigma_p(A)$ and $\omega^2 \in \sigma_p(A_0)$. In this case, z from (4.5) is an eigenvector of A_0 corresponding to the eigenvalue ω^2 (in particular, $z \in H_\alpha$ for all $\alpha > 0$) and we have $C_0 z = 0$.*

Proof. Suppose that $i\omega \in \sigma_p(A)$ and let $z \in Z_0$ be the first component of a corresponding eigenvector, as in (4.5). We know from (I.5.4) that $0 \in \rho(A)$, so that $\omega \neq 0$. According to the first part of (4.7) we have $C_0 z = 0$. Now (4.6) (with $\lambda = i\omega$) shows that z is an eigenvector of A_0, corresponding to the eigenvalue ω^2. It is now easy to see that the vector with components z and $-i\omega z$ is also an eigenvector of A, corresponding to the eigenvalue $-i\omega$. ■

We denote by $\sigma_a(A)$ the set of those $\lambda \in \sigma(A)$ for which λ is not an eigenvalue of A, but $\lambda I - A$ is not bounded from below. In other words, $\lambda \in \sigma_a(A)$ if $\lambda \notin \sigma_p(A)$ and there exists a sequence (x_n) in $\mathcal{D}(A)$ with

$$\|x_n\|_X = 1 \quad \text{and} \quad \lim_{n \to \infty} \|(\lambda I - A) x_n\|_X = 0. \tag{4.9}$$

Lemma 4.5. *If $\omega \in \mathbb{R}$ is such that $i\omega \in \sigma(A)$, then*

$$i\omega \in \sigma_p(A) \cup \sigma_a(A).$$

Proof. Suppose that $i\omega \in \sigma(A)$. We prove that $i\omega \in \sigma_p(A) \cup \sigma_a(A)$ by showing that the contrary statement leads to a contradiction. Indeed, the contrary statement means that $i\omega I - A$ is bounded from below. In this case, the range of $i\omega I - A$ is not dense in X (because if it were dense, then it were all of X and hence $i\omega I - A$ would have a bounded inverse). Let N be the orthogonal complement of the range of $i\omega I - A$, then it is easy to see that N is invariant under $\mathbb{T}^*$: $\mathbb{T}_t^* N \subset N$ for all $t \geq 0$. Considering the restriction of $\mathbb{T}^*$ to N, we see that $\mathcal{D}(A^*) \cap N$ must be dense in N, so that in particular there exist elements $q \in \mathcal{D}(A^*) \cap N$ with $q \neq 0$. From the definition of N we now see that for such q we have $(-i\omega I - A^*) q = 0$, so that $-i\omega \in \sigma_p(A^*)$. Introduce the isomorphism $J \in \mathcal{L}(X)$ defined by the matrix

$$J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

We have $J^{-1} = J$ and $A^* = JAJ$, see the fourth step in the proof of Theorems I.1.1 and I.1.3 (in Section I.6). Thus, A and A^* have the same eigenvalues, so that $-i\omega \in \sigma_p(A)$. According to Lemma 4.4, we obtain that $i\omega \in \sigma_p(A)$, which contradicts our "contrary statement" at the beginning of this proof. ■

Lemma 4.6. *If $\omega \in \mathbb{R}$ is such that $i\omega \in \sigma_a(A)$, then also $-i\omega \in \sigma_a(A)$ and moreover $\omega^2 \in \sigma_p(A_0) \cup \sigma_a(A_0)$.*

Proof. Assume that $i\omega \in \sigma_a(A)$, so that for some sequence (x_n) in $\mathcal{D}(A)$, we have (4.9) (with $\lambda = i\omega$). Denoting $x_n = \begin{bmatrix} z_n \\ w_n \end{bmatrix}$ and $(i\omega I - A)x_n = \begin{bmatrix} \nu_n \\ \varepsilon_n \end{bmatrix}$, so that $\nu_n \to 0$ (in $H_{\frac{1}{2}}$) and $\varepsilon_n \to 0$ (in H), we have

$$\begin{bmatrix} i\omega I & -I \\ A_0 & i\omega I + \frac{1}{2}B_0 C_0 \end{bmatrix} \begin{bmatrix} z_n \\ w_n \end{bmatrix} = \begin{bmatrix} \nu_n \\ \varepsilon_n \end{bmatrix}.$$

From the first row we have $w_n = i\omega z_n - \nu_n$. Substituting this into the equation representing the second row, we get

$$A_0 z_n - \omega^2 z_n - i\omega\nu_n + \frac{i\omega}{2} B_0 C_0 z_n - \frac{1}{2} B_0 C_0 \nu_n = \varepsilon_n . \tag{4.10}$$

The two sides of this equation are in H, but some terms are in $H_{-\frac{1}{2}}$. Taking the scalar product with z_n and using the duality pairing between $H_{-\frac{1}{2}}$ and $H_{\frac{1}{2}}$, we get

$$\left\langle \left(A_0 - \omega^2 I\right) z_n, z_n \right\rangle - i\omega\langle \nu_n, z_n\rangle + \frac{i\omega}{2}\|C_0 z_n\|^2 - \frac{1}{2}\langle C_0\nu_n, C_0 z_n\rangle = \langle \varepsilon_n, z_n\rangle ,$$

which shows that

$$\lim_{n\to\infty} \left[\left\langle \left(A_0 - \omega^2 I\right) z_n, z_n \right\rangle + \frac{i\omega}{2}\|C_0 z_n\|^2 \right] = 0 .$$

Remember that A is invertible (see (I.5.4)), so that $\omega \neq 0$. Taking the imaginary part of the last limit, we conclude that $\lim_{n\to\infty} C_0 z_n = 0$. Now going back to (4.10), we conclude that

$$\lim_{n\to\infty} \left\| \left(\omega^2 I - A_0\right) z_n \right\|_{-\frac{1}{2}} = 0 . \tag{4.11}$$

Now recall from (4.9) that

$$\|x_n\|_X^2 = \|z_n\|_{\frac{1}{2}}^2 + \|w_n\|_0^2 = 1 .$$

Since $w_n = i\omega z_n - \nu_n$ with $\lim_{n\to\infty}\|\nu_n\|_{\frac{1}{2}} = 0$, we have $\lim_{n\to\infty}(\|w_n\|_0^2 - \omega^2\|z_n\|_0^2) = 0$, so that $\lim_{n\to\infty}\left(\|z_n\|_{\frac{1}{2}}^2 + \omega^2\|z_n\|_0^2\right) = 1$. Since $\|z_n\|_0 \leq \|A_0^{-\frac{1}{2}}\| \cdot \|z_n\|_{\frac{1}{2}}$ and $\|A_0^{-\frac{1}{2}}\|^2 = \|A_0^{-1}\|$, we obtain $\liminf_{n\to\infty}\left(1 + \omega^2\|A_0^{-1}\|\right) \cdot \|z_n\|_{\frac{1}{2}}^2 \geq 1$, so that the sequence $(\|z_n\|_{\frac{1}{2}})$ is eventually bounded from below:

$$\|z_n\|_{\frac{1}{2}} \geq m > 0 \quad \text{for} \quad n \geq n_0 .$$

We have for $n \geq n_0$

$$\begin{aligned} m \leq \|z_n\|_{\frac{1}{2}} &= \|A_0 z_n\|_{-\frac{1}{2}} = \|(\omega^2 I - A_0)z_n - \omega^2 z_n\|_{-\frac{1}{2}} \\ &\leq \|(\omega^2 I - A_0)z_n\|_{-\frac{1}{2}} + \omega^2 \|z_n\|_{-\frac{1}{2}} . \end{aligned}$$

Now (4.11) implies that the sequence $(\|z_n\|_{-\frac{1}{2}})$ is eventually bounded from below. This, together with (4.11) implies that $\omega^2 I - A_0$ is not bounded from below as an (unbounded) operator on $H_{-\frac{1}{2}}$ (with domain $H_{\frac{1}{2}}$). Since $\omega^2 I - A_0$ as an operator on $H_{-\frac{1}{2}}$ is isomorphic to $\omega^2 I - A_0$ as an operator on H (with domain H_1), we conclude that the latter is also not bounded from below. Thus, $\omega^2 \in \sigma_p(A_0) \cup \sigma_a(A_0)$.

It remains to prove that $-i\omega \in \sigma_a(A)$. Define

$$\xi_n = \begin{bmatrix} z_n \\ -w_n \end{bmatrix} \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} ,$$

so that $\|\xi_n\|_X = \|x_n\|_X = 1$. We will use the extension of A which maps X into X_{-1}. In particular, this extension maps $H_{\frac{1}{2}} \times H_{\frac{1}{2}} \subset X$ into $H_{\frac{1}{2}} \times H_{-\frac{1}{2}} \subset X_{-1}$, as it is easy to see (see Proposition I.5.2 for the inclusion $H_{\frac{1}{2}} \times H_{-\frac{1}{2}} \subset X_{-1}$). We have

$$(-i\omega I - A)\xi_n = \begin{bmatrix} -i\omega I & -I \\ A_0 & -i\omega I + \frac{1}{2} B_0 C_0 \end{bmatrix} \begin{bmatrix} z_n \\ -w_n \end{bmatrix} = \begin{bmatrix} -\nu_n \\ \varphi_n \end{bmatrix} ,$$

where $\varphi_n \in H_{-\frac{1}{2}}$ is given by a formula similar to (4.10):

$$\varphi_n = (A_0 - \omega^2 I)z_n - i\omega \nu_n - \frac{i\omega}{2} B_0 C_0 z_n + \frac{1}{2} B_0 C_0 \nu_n .$$

Remember that $\nu_n \to 0$ (in $H_{\frac{1}{2}}$) and $C_0 z_n \to 0$ (in U). Using also (4.11), we conclude from the above formula that $\varphi_n \to 0$ (in $H_{-\frac{1}{2}}$). From here we see that

$$\lim_{n \to \infty} (-i\omega I - A)\xi_n = 0 \quad \text{in} \quad H_{\frac{1}{2}} \times H_{-\frac{1}{2}} .$$

Since the sequence (ξ_n) is bounded from below in $H_{\frac{1}{2}} \times H_{\frac{1}{2}}$, it follows that $(-i\omega I - A)$ is not bounded from below (as an operator from $H_{\frac{1}{2}} \times H_{\frac{1}{2}}$ to $H_{\frac{1}{2}} \times H_{-\frac{1}{2}}$). According to point (1) of Proposition I.5.3, it follows that $-i\omega \in \sigma(A)$. Now we can apply Lemma 4.5 to conclude that $-i\omega \in \sigma_p(A) \cup \sigma_a(A)$. If we would have $-i\omega \in \sigma_p(A)$, then it would follow from Lemma 4.4 that also $i\omega \in \sigma_p(A)$, which would contradict our assumption that $i\omega \in \sigma_a(A)$. Thus, we must have $-i\omega \in \sigma_a(A)$. ■

Proof of Theorem 1.5. According to Lemmas 4.4, 4.5 and 4.6, if $\omega \in \mathbb{R}$ is such that $i\omega \in \sigma(A)$, then $\omega^2 \in \sigma(A_0)$. Thus, if $\sigma(A_0)$ is countable, as assumed in the theorem, then also $\sigma(A) \cap i\mathbb{R}$ must be countable, as required in Proposition 3.4. Now the equivalence of points (1)–(7) in the theorem follows from Proposition 3.4.

It remains to prove the equivalence between point (8) and the other points. Suppose that (8) holds. We claim that A has no eigenvalues on the imaginary axis. Indeed, if $\omega \in \mathbb{R}$ were such that $i\omega \in \sigma_p(A)$, then according to Lemma 4.4, $\omega^2 \in \sigma_p(A_0)$

and for a corresponding eigenvector $z \in H_1$ we would have $C_0 z = 0$, which would contradict (8). Now we can apply the main theorem of Arendt and Batty [2]: our earlier claim together with the fact that $\sigma(A) \cap i\mathbb{R}$ is countable implies that $\mathbb{T}$ is strongly stable. Thus, point (1) of the theorem holds and with it all the others.

Conversely, suppose that point (8) is false, i.e., there exists an $\omega \in \mathbb{R}$ such that $\omega^2 \in \sigma_p(A_0)$, and for a corresponding eigenvector $z \in H_1$ we have $C_0 z = 0$. Let x be defined by (4.5), with $\lambda = i\omega$. Then it is easy to verify that $x \in \mathcal{D}(A)$ and $Ax = i\omega x$, whence $\mathbb{T}_t x = e^{i\omega t} x$. This shows that $\mathbb{T}$ is not strongly stable, so that in this case the points (1)–(7) are all false. ■

5. Examples

The aim of this section is to apply the general stability results derived earlier to some models based on partial differential equations. We will consider an Euler-Bernoulli beam with an exponentially stabilizing feedback acting in one point, followed by an n-dimensional wave equation with boundary control and a nonlocal feedback term entering a Dirichlet boundary condition.

5.1. An Euler-Bernoulli beam with pointwise control

The physical system that we have in mind consists of two rigidly joined beams, with both velocity and angular velocity damping at the joint. The other end of both beams is hinged. The inputs are a force and a torque acting at the joint (in addition to the damping force and torque) and the measurements depend linearly on the velocity and the angular velocity at the joint, as well as on the inputs. The fact that we apply both a force and a torque feedback at the joint prevents the well-known lack of robustness of stability concerning the location of the joint: we obtain exponential stability for any location. For the case when only force feedback or only torque feedback is applied, lack of robustness of exponential stability was demonstrated in Rebarber [26] (we refer to [26] for earlier references on this subject).

Our system consists of two homogeneous Euler-Bernoulli beams, situated along the intervals $[0,\xi]$ and $[\xi,\pi]$, with the joint at the point $\xi \in (0,\pi)$. Denoting by $[f]_\xi$ the jump of the function f at $x=\xi$, the equations are

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial^4 z}{\partial x^4} = 0, \quad x \in (0,\pi) \setminus \{\xi\}, \quad [z]_\xi = 0, \quad \left[\frac{\partial z}{\partial x}\right]_\xi = 0, \tag{5.1}$$

$$\left[\frac{\partial^3 z}{\partial x^3}\right]_\xi + \frac{\alpha^2}{2}\frac{\partial z}{\partial t}(\xi,t) = \alpha u_1(t), \quad -\left[\frac{\partial^2 z}{\partial x^2}\right]_\xi + \frac{\beta^2}{2}\frac{\partial^2 z}{\partial x \partial t}(\xi,t) = \beta u_2(t), \tag{5.2}$$

$$\left[\frac{\partial^3 z}{\partial x^3}\right]_\xi - \frac{\alpha^2}{2}\frac{\partial z}{\partial t}(\xi,t) = \alpha y_1(t), \quad -\left[\frac{\partial^2 z}{\partial x^2}\right]_\xi - \frac{\beta^2}{2}\frac{\partial^2 z}{\partial x \partial t}(\xi,t) = \beta y_2(t), \tag{5.3}$$

$$z(x,0) = z_0(x), \quad \frac{\partial z}{\partial t}(x,0) = w_0(x), \quad x \in (0,\pi) \setminus \{\xi\}, \tag{5.4}$$

$$z(0,t) = z(\pi,t) = 0, \quad \frac{\partial^2 z}{\partial x^2}(0,t) = \frac{\partial^2 z}{\partial x^2}(\pi,t) = 0. \tag{5.5}$$

Here, z stands for the transverse displacement of the beam and $\alpha, \beta > 0$ are damping coefficients. The external force is αu_1 and the external torque is βu_2, both acting at ξ. The output signals are y_1 and y_2.

An equivalent formulation of (5.1)-(5.5) can be obtained by considering a single homogeneous Euler-Bernoulli beam, situated along the interval $[0,\pi]$, with a force and a torque acting at the point $\xi \in (0,\pi)$ (the equivalence is proved in Proposition 5.2 below). In this case the equations (5.1)-(5.2) are replaced by

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial^4 z}{\partial x^4} + \frac{\alpha^2}{2}\frac{\partial}{\partial t}(z(\xi,t))\,\delta_\xi - \frac{\beta^2}{2}\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}(\xi,t)\right)\frac{\mathrm{d}\delta_\xi}{\mathrm{d}x} =$$

$$= \alpha u_1(t)\delta_\xi - \beta u_2(t)\frac{\mathrm{d}\delta_\xi}{\mathrm{d}x}. \tag{5.6}$$

Here, δ_ξ is the Dirac mass concentrated at ξ and (5.6) is understood as an equation in $\mathcal{D}'(0,\pi)$ (distributions on $(0,\pi)$). The outputs are equivalently given by

$$y_1(t) = -\alpha\frac{\partial z}{\partial t}(\xi,t) + u_1(t), \quad y_2(t) = -\beta\frac{\partial^2 z}{\partial t\partial x}(\xi,t) + u_2(t). \tag{5.7}$$

The well-posedness of the system described by (5.4)-(5.7) can be obtained by using Theorem 1.1, if we introduce the appropriate spaces and operators. We start by defining $H = L^2[0,\pi]$ and the operator $A_0 : \mathcal{D}(A_0) \to H$ by

$$\mathcal{D}(A_0) = \left\{\phi \in \mathcal{H}^4(0,\pi) \;\middle|\; \phi(0) = \phi(\pi) = 0,\ \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(0) = \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(\pi) = 0\right\},$$

$$A_0\phi = \frac{\mathrm{d}^4\phi}{\mathrm{d}x^4} \qquad \forall\, \phi \in \mathcal{D}(A_0).$$

It is well known that A_0 is self-adjoint, positive and boundedly invertible. As in Section 1, we put $H_1 = \mathcal{D}(A_0)$ and we introduce the spaces H_α ($\alpha \in \mathbb{R}$) by considering fractional powers of A_0 and duality. A simple calculation shows that

$$H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}}) = \mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi),$$

with the norm

$$\|z\|^2_{\frac{1}{2}} = \int_0^\pi \left|\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}(x)\right|^2 \mathrm{d}x.$$

Proposition 5.1. *The equations* (5.4)–(5.7) *determine a conservative linear system* Σ *with input and output space* $\mathbb{C}^2$ *and with state space* $X = H_{\frac{1}{2}} \times H$. *For* $z_0 \in H_{\frac{1}{2}}$, $w_0 \in H$ *and* $u_1, u_2 \in L^2[0,\infty)$, *the equations* (5.4)–(5.6) *admit a unique solution*

$$z \in BC\left(0,\infty; H_{\frac{1}{2}}\right) \cap BC^1\left(0,\infty; H\right) \cap \mathcal{H}^2_{loc}\left(0,\infty; H_{-\frac{1}{2}}\right).$$

Moreover, we have $z(\xi,\cdot), \dfrac{\partial z}{\partial x}(\xi,\cdot) \in \mathcal{H}^1(0,\infty)$.

Proof. We take $U = \mathbb{C}^2$ and $B_0 \in \mathcal{L}(U, H_{-\frac{1}{2}})$ is defined by

$$B_0 \begin{bmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \end{bmatrix} = \alpha \mathrm{v}_1 \delta_\xi - \beta \mathrm{v}_2 \frac{\mathrm{d}\delta_\xi}{\mathrm{d}x} \qquad \forall \begin{bmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \end{bmatrix} \in \mathbb{C}^2.$$

By the definition of B_0, for each $\mathrm{v}_1, \mathrm{v}_2 \in \mathbb{C}$, $B_0 \begin{bmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \end{bmatrix}$ is a bounded linear functional acting on $H_{\frac{1}{2}}$, so that indeed B_0 maps $\mathbb{C}^2$ into $H_{-\frac{1}{2}}$. The adjoint of B_0 is

$$C_0 \phi = B_0^* \phi = \begin{bmatrix} \alpha \phi(\xi) \\ \beta \frac{\mathrm{d}\phi}{\mathrm{d}x}(\xi) \end{bmatrix} \qquad \forall \, \phi \in H_{\frac{1}{2}}. \tag{5.8}$$

Now it is clear that the problem (5.4)–(5.7) can be written in the form (1.1)–(1.3). Thus, this proposition follows from Theorem 1.1. ■

Note that, since Σ is conservative, for every $t \geq 0$ we have

$$\left\| \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} \right\|_X^2 - \left\| \begin{bmatrix} z(0) \\ \dot{z}(0) \end{bmatrix} \right\|_X^2 = \int_0^t \left[|u_1(\sigma)|^2 + |u_2(\sigma)|^2 \right] \mathrm{d}\sigma$$

$$- \int_0^t \left[|y_1(\sigma)|^2 + |y_2(\sigma)|^2 \right] \mathrm{d}\sigma.$$

The space Z_0 defined after (1.5) (see also Theorem I.1.2) is now given by

$$Z_0 = \left\{ z \in H_{\frac{1}{2}} \;\middle|\; z|_{(0,\xi)} \in \mathcal{H}^4(0,\xi), \; z|_{(\xi,1)} \in \mathcal{H}^4(\xi,1) \right\}.$$

The systems (5.1)–(5.5) and (5.4)-(5.7) are equivalent, in the following sense:

Proposition 5.2. *Consider the functions*

$$z \in C\left(0,\infty; Z_0\right) \cap C^1\left(0,\infty; H_{\frac{1}{2}}\right) \cap C^2\left(0,\infty; H\right), \qquad u, y \in C\left(0,\infty; U\right).$$

Then z, u and y satisfy (5.1)-(5.5) *if and only if they satisfy* (5.4)-(5.7).

Proof. We denote by $z^{(\mathrm{iii})}$ (respectively by $z^{(\mathrm{iv})}$) the third (respectively the fourth) derivative of z, computed in $\mathcal{D}'\left((0,\pi) \setminus \{\xi\}\right)$. Roughly speaking, these derivatives are calculated almost everywhere, ignoring the possible jumps at $x = \xi$. Hence they are, in general, different from $\frac{\mathrm{d}^3 z}{\mathrm{d}x^3}$ and $\frac{\mathrm{d}^4 z}{\mathrm{d}x^4}$ computed in the sense of distributions, i.e., in $\mathcal{D}'(0,\pi)$. We define $L_0 \in \mathcal{L}\left(Z_0, L^2[0,\pi]\right)$ and $G_0 \in \mathcal{L}(Z_0, U)$ by

$$L_0 z = z^{(\mathrm{iv})} \;\text{ and }\; G_0 z = \begin{bmatrix} \frac{1}{\alpha}\left[z^{\mathrm{iii}}\right]_\xi & -\frac{1}{\beta}\left[\frac{d^2 z}{dx^2}\right]_\xi \end{bmatrix}^T \tag{5.9}$$

(T stands for transpose). Simple calculations show that

$$L_0 = A_0 - B_0 G_0, \quad G_0 H_1 = \{0\}, \quad G_0 A_0^{-1} B_0 = I. \tag{5.10}$$

By Theorem I.1.4 we obtain that the functions z, u, y satisfy (5.4)–(5.7) (which are the same, in this example, as (1.1)–(1.3)) if and only if they satisfy

$$\ddot{z}(t) + L_0 z(t) = 0, \quad G_0 z(t) + \frac{1}{2} C_0 \dot{z}(t) = u(t), \quad G_0 z(t) - \frac{1}{2} C_0 \dot{z}(t) = y(t),$$

(which are the same as (5.1)-(5.3) and (5.5)) as well as (5.4). ■

The main result of this section is the following:

Theorem 5.3. *For all $\xi \in (0, \pi)$, (5.4)–(5.7) define an exponentially stable well-posed system with input and output space $U = \mathbb{C}^2$ and state space X.*

Proof. Let $A : \mathcal{D}(A) \to X$ be defined as in (1.7), (1.8). If $\psi = \begin{bmatrix} z \\ w \end{bmatrix} \in \mathcal{D}(A)$, then $w \in H_{\frac{1}{2}}$ and z is piecewise in H^4 (on $(0, \xi)$ and on (ξ, π)). Thus, $\mathcal{D}(A)$ is compactly included in X, so that A has compact resolvent, hence $\sigma(A)$ consists of eigenvalues of A. We prove that $\sigma(A) \cap i\mathbb{R} = \emptyset$. Arguing by contradiction, suppose that $i\beta \in \sigma(A)$, with $\beta \in \mathbb{R}$. According to Lemma 4.4 if ψ is an eigenvector of A corresponding to $i\beta$, then $\psi = \begin{bmatrix} z \\ i\beta z \end{bmatrix}$ where $(\beta^2 I - A_0) z = 0$. This implies that

$$\frac{\mathrm{d}^4 z}{\mathrm{d}x^4} - \beta^2 z = 0, \quad z(0) = z(\pi) = 0, \quad \frac{\mathrm{d}^2 z}{\mathrm{d}x^2}(0) = \frac{\mathrm{d}^2 z}{\mathrm{d}x^2}(\pi) = 0.$$

It follows that $z(x) = c \sin(nx)$ for some $c \in \mathbb{C}$ and $n \in \mathbb{N}$ (with $n^4 = \beta^2$). Moreover, by Lemma 4.4 we also have $C_0 z = 0$ which, by (5.8), yields that $z(\xi) = 0$, $\frac{\mathrm{d}z}{\mathrm{d}x}(\xi) = 0$. This implies that $z = 0$, which is a contradiction. Thus, $\sigma(A) \cap i\mathbb{R} = \emptyset$.

By the equivalence of (3) and (9) in Theorem 1.3, it suffices to show that

$$\sup_{\omega \in \mathbb{R}} \|\omega V(i\omega)\|_{\mathcal{L}(H)} < \infty. \tag{5.11}$$

Suppose that this condition is false. By the uniform boundedness theorem, there exist a sequence of real numbers (β_n), with $|\beta_n| \to \infty$, and $h \in H$ such that $\lambda_n = \|\beta_n V(i\beta_n) h\| \to \infty$. Denoting $z_n = \frac{1}{\lambda_n} V(i\beta_n) h$ and $g_n = \frac{1}{\lambda_n} h$ (so that $g_n \to 0$ in H), we have

$$\|\beta_n z_n\|_H = 1 \qquad \forall\, n \in \mathbb{N}, \tag{5.12}$$

$$-\beta_n^2 z_n + A_0 z_n + \frac{i\beta_n}{2} B_0 C_0 z_n = g_n \in H. \tag{5.13}$$

We show (in four steps) that (5.12) and (5.13) lead to a contradiction.

First step. Taking the inner product in H of the sides of (5.13) with $\beta_n z_n$, taking the imaginary parts and using (5.12), we obtain that $\beta_n C_0 z_n \to 0$. According to (5.8) this means

$$|\beta_n z_n(\xi)| \to 0, \qquad \left|\beta_n \frac{\mathrm{d}z_n}{\mathrm{d}x}(\xi)\right| \to 0. \tag{5.14}$$

On the other hand (5.13) implies that $z_n \in Z_0$, so that, by (5.10), we have

$$A_0 z_n = L_0 z_n + B_0 G_0 z_n .$$

Substituting this into (5.13) and using the fact that $B_0 u \in H$ iff $u = 0$, we obtain

$$-\beta_n^2 z_n + L_0 z_n = g_n \quad \text{and} \quad G_0 z_n = \frac{-i\beta_n}{2} C_0 z_n .$$

The above relations together with (5.8) and (5.9) imply that

$$-\beta_n^2 z_n + z_n^{(\mathrm{iv})} = g_n \to 0, \quad \text{in } L^2[0,\pi], \tag{5.15}$$

$$z_n^{(\mathrm{iii})}(\xi^+) - z_n^{(\mathrm{iii})}(\xi^-) = -i\beta_n \frac{\alpha^2}{2} z_n(\xi), \tag{5.16}$$

$$\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^+) - \frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-) = i\beta_n \frac{\beta^2}{2} \frac{\mathrm{d}z_n}{\mathrm{d}x}(\xi). \tag{5.17}$$

Second step. Define $f_n \in H$ by $f_n(x) = x\frac{\mathrm{d}z_n}{\mathrm{d}x}(x)$ for $x \in [0,\xi]$ and $f_n(x) = 0$ for $x > \xi$. We take the inner product of the sides of (5.15) with f_n to get

$$\int_0^\xi \left(-\beta_n^2 z_n(x) + z_n^{(\mathrm{iv})}(x)\right) x \frac{\mathrm{d}\bar{z}_n}{\mathrm{d}x}(x)\mathrm{d}x = \int_0^\xi x\, g_n(x) \frac{\mathrm{d}\bar{z}_n}{\mathrm{d}x}(x)\mathrm{d}x . \tag{5.18}$$

By a straightforward calculation, the terms on the left-hand side become

$$\mathrm{Re}\left\{\int_0^\xi -\beta_n^2 z_n(x) x \frac{\mathrm{d}\bar{z}_n}{\mathrm{d}x}(x)\mathrm{d}x\right\} = -\frac{1}{2}\xi|\beta_n z_n(\xi)|^2 + \frac{1}{2}\int_0^\xi |\beta_n z_n(x)|^2 \mathrm{d}x ,$$

$$\begin{aligned}\mathrm{Re}\left\{\int_0^\xi z_n^{(\mathrm{iv})}(x) x \frac{\mathrm{d}\bar{z}_n}{\mathrm{d}x}(x)\mathrm{d}x\right\} &= \mathrm{Re}\left[\left(\xi z_n^{(\mathrm{iii})}(\xi^-) - \frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-)\right)\frac{\mathrm{d}\bar{z}_n}{\mathrm{d}x}(\xi)\right] \\ &\quad - \frac{\xi}{2}\left|\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-)\right|^2 + \frac{3}{2}\int_0^\xi \left|\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(x)\right|^2 \mathrm{d}x .\end{aligned}$$

The two relations above and (5.18) yield

$$\begin{aligned}&-\frac{1}{2}\xi|\beta_n z_n(\xi)|^2 + \frac{1}{2}\int_0^\xi |\beta_n z_n(x)|^2 \mathrm{d}x + \mathrm{Re}\left[\left(\xi z_n^{(\mathrm{iii})}(\xi^-) - \frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-)\right)\frac{\mathrm{d}\bar{z}_n}{\mathrm{d}x}(\xi)\right] \\ &-\frac{\xi}{2}\left|\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-)\right|^2 + \frac{3}{2}\int_0^\xi \left|\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(x)\right|^2 \mathrm{d}x = \mathrm{Re}\int_0^\xi x\, g_n(x)\frac{\mathrm{d}\bar{z}_n}{\mathrm{d}x}(x)\mathrm{d}x .\end{aligned}$$

Since g_n converges to zero in $L^2[0,\xi]$, we deduce that the right-hand side of the last formula converges to zero. This implies that

$$\begin{aligned}&-\frac{1}{2}\xi|\beta_n z_n(\xi)|^2 + \frac{1}{2}\int_0^\xi |\beta_n z_n(x)|^2 \mathrm{d}x + \mathrm{Re}\left[\left(\xi z_n^{(\mathrm{iii})}(\xi^-) - \frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-)\right)\frac{\mathrm{d}\bar{z}_n}{\mathrm{d}x}(\xi)\right] \\ &-\frac{\xi}{2}\left|\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-)\right|^2 + \frac{3}{2}\int_0^\xi \left|\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(x)\right|^2 \mathrm{d}x \to 0 .\end{aligned} \tag{5.19}$$

On the other hand, from (5.15) and the fact that $\|\beta_n z_n\|_H = 1$ we deduce that $\left(z_n^{(\mathrm{iv})}/\beta_n\right)$ is a bounded sequence in $L^2[0,\xi]$. This further implies the boundedness of the sequences of complex numbers $\left(z_n^{(\mathrm{iii})}(\xi^-)/\beta_n\right)$ and $\left(\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-)/\beta_n\right)$. Then, by (5.14) we obtain that

$$-\frac{1}{2}\xi|\beta_n z_n(\xi)|^2 + \mathrm{Re}\left[\left(\xi z_n^{(\mathrm{iii})}(\xi^-) - \frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-)\right)\frac{\mathrm{d}\bar{z}_n}{\mathrm{d}x}(\xi)\right] \to 0. \tag{5.20}$$

Relations (5.19) and (5.20) lead to

$$\frac{1}{2}\int_0^\xi |\beta_n z_n(x)|^2\,\mathrm{d}x + \frac{3}{2}\int_0^\xi \left|\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}\right|^2 \mathrm{d}x - \frac{\xi}{2}\left|\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-)\right|^2 \to 0.$$

Similarly, we take the inner product of the sides of (5.15) with $(x-\pi)\frac{\mathrm{d}z_n}{\mathrm{d}x}$ truncated to $[\xi,\pi]$, and then we repeat the above argument. This gives

$$\frac{1}{2}\int_\xi^\pi |\beta_n z_n(x)|^2\,\mathrm{d}x + \frac{3}{2}\int_\xi^\pi \left|\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}\right|^2 \mathrm{d}x - \frac{\pi-\xi}{2}\left|\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^+)\right|^2 \to 0.$$

If we add the last two formulas and we use (5.13), we obtain

$$\int_0^\pi |\beta_n z_n(x)|^2\,\mathrm{d}x + 3\int_0^\pi \left|\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(x)\right|^2 \mathrm{d}x - \frac{\xi}{2}\left|\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-)\right|^2 - \frac{\pi-\xi}{2}\left|\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^+)\right|^2 \to 0. \tag{5.21}$$

Third step. We show that both $\left|\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-)\right|$ and $\left|\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^+)\right|$ converge to zero. We use an idea of Z. Liu, see for instance [1]: we take the inner product of the sides of (5.15) with $\frac{1}{\phi_n}e^{-\phi_n(\xi-x)}$ truncated to $[0,\xi]$, where $\phi_n = \sqrt{|\beta_n|}$. Thus, we obtain

$$-\int_0^\xi \phi_n^3 z_n(x)e^{-\phi_n(\xi-x)}\mathrm{d}x + \int_0^\xi \frac{z_n^{(\mathrm{iv})}(x)}{\phi_n}e^{-\phi_n(\xi-x)}\mathrm{d}x = \int_0^\xi g_n(x)\frac{e^{-\phi_n(\xi-x)}}{\phi_n}\mathrm{d}x.$$

By using again (5.12), (5.15) and the fact that $\phi_n e^{-\phi_n(\xi-x)}$ is bounded in $L^1[0,\pi]$, we obtain that

$$-\int_0^\xi \phi_n^3 z_n(x)e^{-\phi_n(\xi-x)}\mathrm{d}x + \int_0^\xi \frac{z_n^{(\mathrm{iv})}(x)}{\phi_n}e^{-\phi_n(\xi-x)}\mathrm{d}x \to 0. \tag{5.22}$$

Integrating by parts the second term above four times, we obtain

$$\int_0^\xi \frac{z_n^{(\mathrm{iv})}(x)}{\phi_n}e^{-\phi_n(\xi-x)}\mathrm{d}x =$$

$$\left(\frac{z_n^{(\mathrm{iii})}(x)}{\phi_n} - \frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(x) + \phi_n\frac{\mathrm{d}z_n}{\mathrm{d}x}(x) - \phi_n^2 z_n\right)e^{-\phi_n(\xi-x)}\Bigg|_0^\xi + \int_0^\xi \phi_n^3 z_n e^{-\phi_n(\xi-x)}\mathrm{d}x.$$

By using (5.22) we obtain from here that

$$\left(\frac{z_n^{(\mathrm{iii})}(x)}{\phi_n} - \frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(x) + \phi_n\frac{\mathrm{d}z_n}{\mathrm{d}x}(x) - \phi_n^2 z_n\right)e^{-\phi_n(\xi-x)}\Bigg|_0^\xi \to 0. \tag{5.23}$$

Among the boundary terms in (5.23), those at $x = 0$ all converge to zero due to the exponential decay of $e^{-\phi_n \xi}$ and to the boundedness of $\left(z_n^{(\mathrm{iv})}/\beta_n\right)$ in $L^2[0,\xi]$; the terms containing $z_n(\xi)$, $\frac{\mathrm{d}z_n}{\mathrm{d}x}(\xi)$ also converge to zero due to (5.14). Thus, (5.23) yields

$$\frac{z_n^{(\mathrm{iii})}}{\phi_n}(\xi^-) - \frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-) \to 0. \tag{5.24}$$

Similarly, we take the inner product of the sides of (5.15) with $\frac{1}{\phi_n} e^{-\phi_n(x-\xi)}$ in $L^2[\xi,\pi]$. Repeating the above analysis, we obtain

$$\frac{z_n^{(\mathrm{iii})}}{\phi_n}(\xi^+) + \frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^+) \to 0. \tag{5.25}$$

On the other hand, by using (5.14) and (5.16) we obtain

$$z_n^{(\mathrm{iii})}(\xi^+) - z_n^{(\mathrm{iii})}(\xi^-) \to 0. \tag{5.26}$$

Then the difference of (5.24) and (5.25) leads, by using (5.26), to

$$\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^+) + \frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-) \to 0.$$

Recall that, by (5.14) and (5.17), we have $\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^+) - \frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-) \to 0$. Therefore,

$$\frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^-) \to 0, \qquad \frac{\mathrm{d}^2 z_n}{\mathrm{d}x^2}(\xi^+) \to 0. \tag{5.27}$$

Fourth step. Relations (5.21) and (5.27) imply that $\|\beta_n z_n\|_H \to 0$, which clearly contradicts (5.12). This contradiction shows that (5.11) holds. ■

Theorem 5.3 can be generalized for coupled beams described by a version of (5.6) containing variable coefficients, see Ammari, Liu and Tucsnak [1].

Theorem 5.3 together with Theorem 1.3 imply the following:

Proposition 5.4. *The system from Theorem* 5.3 *is exactly controllable.*

5.2. The wave equation with Dirichlet type nonlocal boundary feedback

The physical system that we have in mind consists of a vibrating membrane which is fixed on a part of the boundary, while on the other part of the boundary the vibrations are damped by a feedback control acting on the Dirichlet boundary condition. The input is the displacement field on the controlled part of the boundary and the measurement depends linearly on the velocity field, as well as on the input. A membrane could be modelled in a domain in $\mathbb{R}^2$, but we consider a more general wave equation on an n-dimensional (possibly unbounded) domain Ω. The boundary

Γ of $\Omega \subset \mathbb{R}^n$ is assumed to be of class C^2 and to have a decomposition as $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, where Γ_0, Γ_1 are disjoint open parts of Γ, with $\Gamma_1 \neq \emptyset$. We also assume that the Poincaré inequality holds for all $f \in \mathcal{H}_0^1(\Omega)$, which is always true if Ω is bounded, but it also holds for some unbounded domains (see Section I.7). The operator $G : \mathcal{H}^{-1}(\Omega) \to \mathcal{H}_0^1(\Omega)$ is defined by

$$Gf = \phi \text{ if and only if } \phi \in \mathcal{H}_0^1(\Omega) \text{ and } -\Delta\phi = f.$$

Thus, in a certain sense, $G = -\Delta^{-1}$. We denote by $\gamma_0 : \mathcal{H}^1(\Omega) \to \mathcal{H}^{\frac{1}{2}}(\Gamma)$ the Dirichlet trace operator, which is onto, and by $\gamma_1 : \mathcal{H}^2(\Omega) \to \mathcal{H}^{\frac{1}{2}}(\Gamma)$ the outward normal derivative operator. Using Green's formula

$$\langle \gamma_1 f, \gamma_0 g \rangle_{L^2} = \langle \Delta f, g \rangle_{L^2} + \langle \nabla f, \nabla g \rangle_{L^2}, \tag{5.28}$$

for all $f \in \mathcal{H}^2(\Omega)$ and $g \in \mathcal{H}^1(\Omega)$, γ_1 can be extended so that $\gamma_1 f$ is defined as a distribution in $\mathcal{H}^{-\frac{1}{2}}(\Gamma)$, for every $f \in \mathcal{H}^1(\Omega)$ for which $\Delta f \in L^2(\Omega)$. Here, Δf denotes the Laplacian of f in the sense of distributions on Ω, i.e., in the space $\mathcal{D}'(\Omega)$.

We consider the system described by the equations

$$\ddot{z} = \Delta z \quad \text{in} \quad \Omega \times (0, \infty), \tag{5.29}$$

$$z = 0 \quad \text{on} \quad \Gamma_0 \times (0, \infty), \tag{5.30}$$

$$z - \frac{1}{2}\gamma_1(G\dot{z}) = u \quad \text{on} \quad \Gamma_1 \times (0, \infty), \tag{5.31}$$

$$z(x, 0) = z_0(x), \quad \dot{z}(x, 0) = w_0(x) \quad \text{for} \quad x \in \Omega. \tag{5.32}$$

The input of this system is the function u in (5.31). The output associated to this system is

$$y = z + \frac{1}{2}\gamma_1(G\dot{z}) \quad \text{on} \quad \Gamma_1 \times (0, \infty). \tag{5.33}$$

Some comments about the domain Ω: it is not really necessary to assume that Γ is of class C^2. What we really need is that G maps $L^2(\Omega)$ onto $\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$, and that γ_1 maps $\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$ onto $\mathcal{H}^{\frac{1}{2}}(\Gamma)$. These properties hold also for some less regular domains, for example, for convex sets in $\mathbb{R}^2$ (see Grisvard [9, Theorem 3.2.1.2, p. 147]) or for rectangular domains in $\mathbb{R}^n$. We will need the following simple result, a direct consequence of the Riesz representation theorem in $L^2(\Omega)$:

Proposition 5.5. *For every* $\mathrm{v} \in L^2(\Gamma)$*, there exists a unique function* $D\mathrm{v} \in L^2(\Omega)$ *such that*

$$\int_\Omega (D\mathrm{v})(x)\overline{\psi(x)}\,dx = -\int_\Gamma \mathrm{v}\,\overline{\gamma_1(G\psi)}\,d\Gamma \qquad \forall\, \psi \in L^2(\Omega). \tag{5.34}$$

Moreover, the operator D *defined above (called the Dirichlet map) is linear and bounded from* $L^2(\Gamma)$ *into* $L^2(\Omega)$*.*

Due to the Poincaré inequality, the norm on $\mathcal{H}_0^1(\Omega)$ can be defined as $\|f\|_{\mathcal{H}_0^1} = \|\nabla f\|_{L^2}$. Then it is easy to see (using (5.28)) that the corresponding dual norm on $\mathcal{H}^{-1}(\Omega)$ (with respect to the pivot space $L^2(\Omega)$) can be written as

$$\|g\|_{\mathcal{H}^{-1}} = \sup_{\|f\|_{\mathcal{H}_0^1}=1} \langle g, f\rangle_{\mathcal{H}^{-1},\mathcal{H}_0^1} = \|\nabla(Gg)\|_{L^2}.$$

Define $W = \{f \in L^2(\Omega) \mid \Delta f \in \mathcal{H}^{-1}(\Omega)\}$. Since γ_1 maps $\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$ onto $\mathcal{H}^{\frac{1}{2}}(\Gamma)$ (see Lions and Magenes [21, Chapter 2, Theorem 5.4]), the Dirichlet trace operator γ_0 can be extended to an operator $\gamma_0 : W \to \mathcal{H}^{-\frac{1}{2}}(\Gamma)$ by putting

$$\langle \gamma_0 f, \gamma_1(G\psi)\rangle_{\mathcal{H}^{-\frac{1}{2}},\mathcal{H}^{\frac{1}{2}}} = -\langle \Delta f, G\psi\rangle_{\mathcal{H}^{-1},\mathcal{H}_0^1} - \langle f, \psi\rangle_{L^2} \qquad \forall\, \psi \in L^2(\Omega).$$

The operators D and γ_0 defined above are related as follows:

Proposition 5.6. *If* $\mathrm{v} \in L^2(\Gamma)$*, then* $\Delta(D\mathrm{v}) = 0$ *in* $\mathcal{D}'(\Omega)$ *(which implies that* $D\mathrm{v} \in W$*). Moreover, we have that* $\gamma_0(D\mathrm{v}) = \mathrm{v}$*.*

Proof. The fact that $\Delta(D\mathrm{v}) = 0$ (in the sense of distributions) follows directly from (5.34) by taking $\psi = \Delta\varphi$, where $\varphi \in \mathcal{D}(\Omega) = C_0^\infty(\Omega)$. Now using the definition of the extended γ_0 we have that

$$\langle \gamma_0(D\mathrm{v}), \gamma_1(G\psi)\rangle_{\mathcal{H}^{-\frac{1}{2}},\mathcal{H}^{\frac{1}{2}}} = -\langle D\mathrm{v}, \psi\rangle_{L^2} \qquad \forall\, \psi \in L^2(\Omega). \tag{5.35}$$

Since $\gamma_1(G\psi)$ in (5.35) can be any function in $\mathcal{H}^{\frac{1}{2}}(\Gamma)$, and $\mathcal{H}^{\frac{1}{2}}(\Gamma)$ is dense in $L^2(\Omega)$, equations (5.34) and (5.35) imply that $\gamma_0(D\mathrm{v}) = \mathrm{v}$. ∎

To discuss the well-posedness and conservativity of the system (5.29)–(5.33) (using Theorem 1.1), we have to introduce the appropriate spaces and operators. Denote

$$X = L^2(\Omega) \times \mathcal{H}^{-1}(\Omega), \quad U = L^2(\Gamma_1).$$

We also define the Hilbert space

$$Z_0 = \left\{ f \in W \;\middle|\; \gamma_0 f \in L^2(\Gamma) \;\text{ and }\; \gamma_0 f|_{\Gamma_0} = 0 \right\}, \tag{5.36}$$

with the norm $\|f\|_{Z_0}$ given by

$$\|f\|_{Z_0}^2 = \|f\|_{L^2}^2 + \|\Delta f\|_{\mathcal{H}^{-1}}^2 + \|\gamma_0 f\|_{L^2}^2.$$

The precise statement of the well-posedness and conservativity of the system described by (5.29)–(5.33) is given in the following proposition.

Proposition 5.7. *The equations* (5.29)–(5.33) *determine a conservative linear system* Σ *with input and output space* U *and state space* X*. If* $z_0 \in Z_0$*,* $w_0 \in L^2(\Omega)$*,* $u \in \mathcal{H}^1(0,\infty;U)$ *and the compatibility condition*

$$z_0(x) - \frac{1}{2}\gamma_1\left(Gw_0\right)(x) = u(x,0) \quad \text{for} \quad x \in \Gamma_1$$

holds, then (5.29)–(5.33) *have a unique solution* z, y *satisfying*

$$z \in BC\left(0,\infty;Z_0\right) \cap BC^1\left(0,\infty;L^2(\Omega)\right) \cap BC^2\left(0,\infty;\mathcal{H}^{-1}(\Omega)\right),$$

$$y \in \mathcal{H}^1(0,\infty;U).$$

Proof. We define the following spaces and operators:

- The space H is defined by $H = \mathcal{H}^{-1}(\Omega)$, endowed with the norm

$$\|f\|_H = \|\nabla(Gf)\|_{L^2(\Omega)}.$$

- The operator $A_0 : \mathcal{D}(A_0) \to H$ is defined by

$$\mathcal{D}(A_0) = \mathcal{H}_0^1(\Omega), \quad A_0\phi = -\Delta\phi \qquad \forall\, \phi \in \mathcal{D}(A_0).$$

  It is well known that A_0 is self-adjoint, positive and boundedly invertible.

- As in Section 1, we put $H_1 = \mathcal{D}(A_0)$ and we introduce the spaces H_α ($\alpha \in \mathbb{R}$) by considering powers of A_0 and duality, so that $A_0 : H_\alpha \to H_{\alpha-1}$. In order to identify the space $H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}})$, we recall that $H_{\frac{1}{2}}$ is the completion of $\mathcal{D}(A_0)$ with respect to the norm

$$\|z\|_{\frac{1}{2}} = \left\|A_0^{\frac{1}{2}} z\right\|_H = \sqrt{\langle A_0 z, z\rangle_H}.$$

  Since for any $z \in \mathcal{D}(A_0) = \mathcal{H}_0^1(\Omega)$, we have

$$\langle A_0 z, z\rangle_H = -\left\langle \nabla z, \nabla\left(\Delta^{-1} z\right)\right\rangle_{L^2(\Omega)} = \|z\|^2_{L^2(\Omega)},$$

  the space $H_{\frac{1}{2}}$ is given by $H_{\frac{1}{2}} = L^2(\Omega)$.

- Notice that $H_{\frac{3}{2}} = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$. We define the space

$$\widetilde{\mathcal{H}}^{-2}(\Omega) = \left(\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)\right)'$$

  (this dual is computed with respect to the pivot space $L^2(\Omega)$). It can be checked that G can be extended so that $G : \widetilde{\mathcal{H}}^{-2}(\Omega) \to L^2(\Omega)$ and the norm on $\widetilde{\mathcal{H}}^{-2}(\Omega)$ is

$$\|g\|_{\widetilde{\mathcal{H}}^{-2}} = \sup_{\|\Delta f\|_{L^2(\Omega)}=1} \langle g, f\rangle = \|Gg\|_{L^2}.$$

- By definition, the space $H_{-\frac{1}{2}}$ is the completion of H with respect to the norm $\|z\|_{-\frac{1}{2}} = \|A_0^{-\frac{1}{2}} z\|_H$. Since $G = A_0^{-1}$ on H, the last equality and the definition of the norm on H imply that

$$\|z\|_{-\frac{1}{2}} = \left\|\nabla\left(A_0^{-\frac{3}{2}} z\right)\right\|_{L^2}.$$

The above relation and the obvious fact that $\|A_0^{\frac{1}{2}}w\|_{L^2} = \|\nabla w\|_{L^2}$, for any $w \in \mathcal{H}_0^1(\Omega)$, imply that

$$\|z\|_{-\frac{1}{2}} = \|A_0^{-1}z\|_{L^2} = \|Gz\|_{L^2} \qquad \forall\, z \in H.$$

We have thus shown that $H_{-\frac{1}{2}} = \widetilde{\mathcal{H}}^{-2}(\Omega)$.

- We denote by $P \in \mathcal{L}\left(L^2(\Gamma), L^2(\Gamma_1)\right)$ the operator of truncation to Γ_1. Then $P^* \in \mathcal{L}\left(L^2(\Gamma_1), L^2(\Gamma)\right)$ is the operator defined by extending the functions in $L^2(\Gamma_1)$ by zero outside Γ_1. The operator $B_0 \in \mathcal{L}(U, H_{-\frac{1}{2}})$ is defined by

  $$B_0 \mathrm{v} = A_0 D P^* \mathrm{v},$$

  where A_0 is considered as an operator from $H_{\frac{1}{2}}$ to $H_{-\frac{1}{2}}$ and $D : U \to L^2(\Omega)$ is the Dirichlet map defined in Proposition 5.5.

- Let $\phi \in \mathcal{H}_0^1(\Omega) = H_1$ and $\mathrm{v} \in L^2(\Gamma_1)$. Then we have

  $$\langle B_0\mathrm{v}, \phi\rangle_{H_{-\frac{1}{2}},H_{\frac{1}{2}}} = \langle DP^*\mathrm{v}, A_0\phi\rangle_H \;=\; \langle DP^*\mathrm{v}, \phi\rangle_{H_{\frac{1}{2}}}\,.$$

  By using the fact that $H_{\frac{1}{2}} = L^2(\Omega)$, the density of $\mathcal{H}_0^1(\Omega)$ in $L^2(\Omega)$ and (5.34), it follows that, for every $\phi \in L^2(\Omega)$ and for every $\mathrm{v} \in L^2(\Gamma_1)$,

  $$\langle B_0\mathrm{v}, \phi\rangle_{H_{-\frac{1}{2}},H_{\frac{1}{2}}} = \langle DP^*\mathrm{v}, \phi\rangle_{L^2} \;=\; -\int_{\Gamma_1} \mathrm{v}\overline{\gamma_1\left(G\phi\right)}\mathrm{d}\Gamma.$$

  We conclude that the adjoint of B_0 (using the pivot space $H = \mathcal{H}^{-1}(\Omega)$) is $C_0 \in \mathcal{L}(H_{\frac{1}{2}}, U)$ given by

  $$C_0\phi = B_0^*\phi = \;-\,P\gamma_1(G\phi) \qquad \forall\, \phi \in H_{\frac{1}{2}} = L^2(\Omega).$$

It is clear that the spaces U, H, H_α, X and the operators A_0, C_0, B_0 fit into the simple general framework of Section 1. Thus, by Theorem 1.1, they determine via the equations (1.1)–(1.3) a conservative linear system Σ. The state of this system is

$$\xi(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}.$$

To show that Σ is described by (5.29)–(5.33), first we notice that the space $Z_0 = H_1 + A_0^{-1}B_0U$, defined after (1.5) (see also Theorem I.1.2), is given in our case by $Z_0 = H_1 + DP^*U$. By using Lemma 5.6, it can be checked that this space coincides with that defined by (5.36). We define the operator $G_0 \in \mathcal{L}(Z_0, U)$ by

$$G_0 f = P\gamma_0 f \qquad \forall\, f \in Z_0.$$

Clearly we have $G_0H_1 = \{0\}$ and, by Lemma 5.6, we have in $\mathcal{L}(U)$

$$G_0A_0^{-1}B_0 = P\gamma_0A_0^{-1}\left(A_0DP^*\right) = P\gamma_0DP^* = I.$$

Hence, all the assumptions in Theorem I.1.4 are satisfied. We define on Z_0 the operator $L_0 = A_0 - B_0 G_0$, as in Theorem I.1.4, and it is easy see that $L_0 z = -\Delta z$.

If we write the system of equations (I.1.16) in our specific framework, we obtain the system of equations (5.29)–(5.33). Hence, by Theorem I.1.4, the compatibility condition in our proposition is equivalent to (I.1.9) and the equations (5.29)–(5.33) are equivalent to (I.1.11) and (I.1.12). Now by Theorem I.1.2, the equations have a solution z, y with the claimed smoothness properties. ■

This system has also been considered in Lasiecka and Triggiani [18, pp. 669-671] but without considering outputs. They have proved the well-posedness of the mapping from the input function to the state, and they have discussed the exponential stability of the system for suitable Γ_1.

Theorem 5.8. *If Ω is bounded, then the system defined by (5.29)–(5.33) is strongly stable, exactly controllable in infinite time and exactly observable in infinite time.*

Proof. The fact that the equations (5.29)–(5.33) define a conservative linear system with input space U, state space X and output space Y has been said in Proposition 5.7. The boundedness of Ω implies that the spectrum of A_0 is countable. Thus, according to assertion (8) of Theorem 1.5, in order to check the properties claimed in the theorem, it suffices to prove that for any $\phi \in H_1$, if ϕ is an eigenvector of A_0, then $C_0\phi \neq 0$. Due to the particular form of A_0 and C_0, this means that we have to show that if $\phi \in \mathcal{H}_0^1(\Omega)$ is such that, for some $\lambda > 0$,

$$-\Delta\phi = \lambda\phi \ \text{ in } \ \mathcal{H}^{-1}(\Omega), \qquad \frac{\partial}{\partial\nu}(G\phi) = 0 \ \text{ on } \ \Gamma_1, \tag{5.37}$$

then $\phi = 0$. By denoting $\psi = G\phi$, we see that (5.37) is equivalent to

$$-\Delta\psi = \lambda\psi \ \text{ in } \ \mathcal{H}_0^1(\Omega), \qquad \frac{\partial\psi}{\partial\nu} = 0 \ \text{ on } \ \Gamma_1. \tag{5.38}$$

By classical elliptic regularity results, (5.38) implies that $\psi \in \mathcal{H}^2(\Omega)$ (see, for instance, Grisvard [9, Theorem 2.2.2.5]). This fact combined with (5.38) implies, by a classical unique continuation argument (see, for instance, Komornik [16, Corollary 6.2]), that $\psi = 0$. Thus, (5.37) implies that $\phi = \Delta\psi = 0$ in $\mathcal{H}_0^1(\Omega)$. ■

The exponential stability of the system (5.29)–(5.32) was studied in Bardos *et al* [4]. By combining Proposition 5.7 and [4, Theorem 1], we obtain:

Theorem 5.9. *Suppose that Ω is bounded and there exists a time $T_0 > 0$ such that every geometric ray in $\Omega \times (0, T_0)$ intersects $\Gamma_1 \times (0, T_0)$ in a non-diffractive point. Then the equations (5.29)–(5.33) define an exponentially stable, conservative system with input space U, state space X and output space Y. This system is also exactly controllable and exactly observable in time T_0.*

The last sentence of the above theorem follows from the exponential stability (stated in the first part of the theorem) using Theorem 1.3.

Acknowledgments. This research was supported by EPSRC (from the UK) under grant number GR/R05048/01 and by CNRS (from France) under grant number 943706 DRCI. We have had useful discussions on this research with Olof Staffans, Arjan van der Schaft and Peng-Fei Yao.

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