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**Integral Equations and Operator Theory**

# Transfer functions of regular linear systems PART III: INVERSIONS AND DUALITY

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**Abstract.** We study four transformations which lead from one well-posed linear system to another: time-inversion, flow-inversion, time-flow-inversion and duality. Time-inversion means reversing the direction of time, flow-inversion means interchanging inputs with outputs, while time-flow-inversion means doing both of the inversions mentioned before. A well-posed linear system $\Sigma$ is time-invertible if and only if its operator semigroup extends to a group. The system $\Sigma$ is flow-invertible if and only if its input-output map has a bounded inverse on some (hence, on every) finite time interval $[0, \tau]$ $(\tau > 0)$. This is true if and only if the transfer function of $\Sigma$ has a uniformly bounded inverse on some right half-plane. The system $\Sigma$ is time-flow-invertible if and only if on some (hence, on every) finite time interval $[0, \tau]$, the combined operator $\Sigma_\tau$ from the initial state and the input function to the final state and the output function is invertible. This is the case, for example, if the system is conservative, since then $\Sigma_\tau$ is unitary. Time-flow-inversion can sometimes, but not always, be reduced to a combination of time- and flow-inversion. We derive a surprising necessary and sufficient condition for $\Sigma$ to be time-flow-invertible: its system operator must have a uniformly bounded inverse on some left half-plane. Finally, the duality transformation is always possible. We show by some examples that none of these transformations preserves regularity in general. However, the duality transformation does preserve weak regularity. For all the transformed systems mentioned above, we give formulas for their system operators, transfer functions and, in the regular case and under additional assumptions, for their generating operators.

**Key words.** Well-posed linear system, regular linear system, operator semigroup, system operator, time-inversion, flow-inversion, time-flow-inversion, dual system, conservative system, Lax–Phillips semigroup.

## 1. Introduction

This is a continuation of the papers Weiss [22] and Staffans and Weiss [19] (Part I and Part II), which addressed some fundamental questions about the representation of well-posed linear systems and, in particular, regular linear systems. A *well-posed linear system* is a linear system whose input, state and output spaces are Hilbert spaces, input and output functions are locally $L^2$, and on any finite time-interval, the final state and the output function depend continuously on the initial state and the input function. Certain functional equations must be satisfied, which express time-invariance and causality. If the transfer function of a well-posed system has a strong (or weak) limit at $+\infty$, then the system is called *regular* (or *weakly regular*). The precise definitions of these and other concepts used in the Introduction were given in [22] and again in [19], and we will not formulate them again in this paper. For historical comments we refer to Section 1 of [19].

In this paper we investigate three types of inversions of a well-posed system: time-inversion, flow-inversion, and time-flow-inversion. (The third inversion can sometimes, but not always, be reduced to a combination of the first two.) We also investigate duality of well-posed systems. Each of these transformations, if applicable, leads to a new well-posed system. We are particularly interested in characterizing the system operator of the various inverted systems and of the dual system, with emphasis on the simpler case when the systems are regular. (In the regular case, the system operator has a natural decomposition into four blocks which correspond to the generating operators of the system, as explained in Part II.)

To make our aims more easily understood, we explain what our main results mean for a finite-dimensional linear system $\Sigma$ described by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t).\end{aligned} \tag{1.1}$$

Here $u(\cdot)$ is the *input function*, $x(t)$ is the *state* at time $t$, and $y(\cdot)$ is the *output function*. We call the matrices $A$, $B$, $C$, $D$ the *generating operators* of $\Sigma$ and

$$S_\Sigma(s) = \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \qquad (\text{where} \quad s \in \mathbb{C}) \tag{1.2}$$

is called the *system operator* of $\Sigma$. The transfer function of this system is

$$\mathbf{G}(s) = C(sI - A)^{-1}B + D \qquad (\text{for} \quad s \in \rho(A)). \tag{1.3}$$

Take $\tau \geq 0$. We are usually interested in the solutions of (1.1) for $t \in [0, \tau]$, but of course, the solutions exist on the whole real line. Given an initial state $x(0)$ and the restriction of $u$ to $[0, \tau]$, denoted by $\mathbf{P}_\tau u$, we can solve (1.1) to compute $x(\tau)$ and the restriction of $y$ to $[0, \tau]$, denoted by $\mathbf{P}_\tau y$. Formally, we have

$$\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \end{bmatrix} = \Sigma_\tau \begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \end{bmatrix} = \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix} \begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \end{bmatrix}. \tag{1.4}$$

The operators appearing in the block $2 \times 2$ matrix $\Sigma_\tau$ above are given by

$$\begin{aligned} \mathbb{T}_\tau &= e^{A\tau}, & \Phi_\tau u &= \int_0^\tau e^{A(\tau-\sigma)} Bu(\sigma)\,\mathrm{d}\sigma, \\ (\Psi_\tau x_0)(t) &= Ce^{At}x_0, & (\mathbb{F}_\tau u)(t) &= C\int_0^t e^{A(t-\sigma)}Bu(\sigma)\,\mathrm{d}\sigma + Du(t), \end{aligned} \tag{1.5}$$

where $t \in [0, \tau]$. These families of operators (parametrized by $\tau \geq 0$) constitute an alternative description of the system $\Sigma$. This is, of course, a much more cumbersome description of $\Sigma$ than (1.1), but for infinite-dimensional systems these operator families are the natural starting point, see [22] or [16].

In the finite-dimensional situation which we are now discussing, we can reverse the direction of the time by changing $t$ to $\tau - t$ in (1.1) to get the *time-inverted system* $\Sigma^{Я}$. This system (with input $v(t) = u(\tau - t)$, state $z(t) = x(\tau - t)$ and output $w(t) = y(\tau - t)$) is described by

$$\begin{aligned}\dot{z}(t) &= -Az(t) - Bv(t),\\ w(t) &= \phantom{-}Cz(t) + Dv(t).\end{aligned} \tag{1.6}$$

The generating operators and the transfer function of this system are

$$\begin{bmatrix} A^{Я} & B^{Я} \\ C^{Я} & D^{Я} \end{bmatrix} = \begin{bmatrix} -A & -B \\ C & D \end{bmatrix}, \qquad \mathbf{G}^{Я}(s) = \mathbf{G}(-s). \tag{1.7}$$

By *flow-inversion* we mean an interchange of the roles of $u$ and $y$ in (1.1), so that $y$ becomes the input and $u$ the output. This is possible if and only if $D$ is invertible, in particular, the input and output dimensions must be the same. The resulting system $\Sigma^\times$ (the superscript $\times$ stands for flow-inversion) is then given by

$$\begin{aligned}\dot{x}(t) &= (A - BD^{-1}C)x(t) + BD^{-1}y(t),\\ u(t) &= -D^{-1}Cx(t) + D^{-1}y(t).\end{aligned} \tag{1.8}$$

The generating operators and the transfer function of $\Sigma^\times$ are given by

$$\begin{bmatrix} A^\times & B^\times \\ C^\times & D^\times \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C & D \end{bmatrix}^{-1}, \qquad \mathbf{G}^\times(s) = \mathbf{G}^{-1}(s). \tag{1.9}$$

*Time-flow-inversion* means that we perform both of the transformations described above at the same time, i.e., $w$ from (1.6) becomes the input and $v$ from (1.6) becomes the output. This is possible if and only if $D$ is invertible. The resulting system $\Sigma^{\leftarrow}$ is described by

$$\begin{aligned} \dot{z}(t) &= (-A + BD^{-1}C)z(t) - BD^{-1}w(t), \\ v(t) &= -D^{-1}Cz(t) + D^{-1}w(t), \end{aligned} \tag{1.10}$$

and its generating operators and transfer function are

$$\begin{bmatrix} A^{\leftarrow} & B^{\leftarrow} \\ C^{\leftarrow} & D^{\leftarrow} \end{bmatrix} = \begin{bmatrix} -A & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C & D \end{bmatrix}^{-1}, \qquad \mathbf{G}^{\leftarrow}(s) = \mathbf{G}^{-1}(-s). \tag{1.11}$$

The *dual system* of $\Sigma$ from (1.1), denoted by $\Sigma^d$, is given by

$$\begin{aligned} \dot{x}^d(t) &= A^* x^d(t) + C^* y^d(t), \\ u^d(t) &= B^* x^d(t) + D^* y^d(t), \end{aligned} \tag{1.12}$$

where $y^d$ is the input function, $x^d(t)$ is the state at time $t$, and $u^d$ is the output function. Thus, the generating operators and the transfer function of $\Sigma^d$ are

$$\begin{bmatrix} A^d & B^d \\ C^d & D^d \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^*, \qquad \mathbf{G}^d(s) = \mathbf{G}^*(\overline{s}). \tag{1.13}$$

We now indicate how the transformations defined above carry over to the general case of an (infinite-dimensional) well-posed linear system. The generating operators $A$, $B$, $C$ and $D$ are well defined for weakly regular systems (see Section 4 of Part II), but for well-posed linear systems in general there is a problem: $A$, $B$ and $C$ are still well defined, but $D$ is not uniquely determined. The operators $A$, $B$ and $C$ may be unbounded, while $D$ is always bounded. A local (in time) representation of a well-posed system $\Sigma$ (similar to (1.1)) uses $S_\Sigma(0)$, the system operator evaluated at zero (see Theorem 3.1 in Part II). The system operator is of the form

$$S_\Sigma(s) = \begin{bmatrix} A \& B \\ C \& D \end{bmatrix} - \begin{bmatrix} sI & 0 \\ 0 & 0 \end{bmatrix},$$

where $C\&D$ is the so-called *combined observation/feedthrough operator*, see Section 3 of Part II. In the finite-dimensional case, using the notation from (1.2), $C\&D$ would be the matrix $\begin{bmatrix} C & D \end{bmatrix}$, but in the general infinite-dimensional framework $C\&D$ is a densely defined unbounded operator. Thus, to get local (in time) representations of the various inverted systems and of the dual system, we express their system operators evaluated at zero in terms of the original $S_\Sigma(0)$.

In short, the situation with the various transformations is as follows. It is always possible to pass from a given system $\Sigma$ to its dual $\Sigma^d$, and the formulas given in (1.13) remain valid if we replace $\begin{bmatrix} C & D \end{bmatrix}$ by $C\&D$ and $\begin{bmatrix} C^d & D^d \end{bmatrix}$ by $[C\&D]^d$. Here $[C\&D]^d$ is the combined observation/feedthrough operator of $\Sigma^d$. This and other results about duality are presented in Section 3.

Time-inversion of a well-posed system $\Sigma$ is possible if and only if the underlying semigroup $\mathbb{T}$ is invertible, i.e., $\mathbb{T}_\tau$ is invertible for some (hence, for all) $\tau > 0$. According to a recent result of Zwart [26], denoting the generator of $\mathbb{T}$ by $A$, $\mathbb{T}$ is invertible if and only if $(sI - A)^{-1}$ is uniformly bounded on some left half-plane. In this case, the formulas in (1.7) remain valid with similar modifications as for (1.13) if we extend the transfer function $\mathbf{G}$ appropriately to $\rho(A)$ (in most cases this part of the transfer function can be obtained from the original transfer function by analytic continuation). The details of time-inversion can be found in Section 4.

Flow-inversion of a well-posed system is possible if and only $\mathbb{F}_\tau$ is invertible for some (hence, for all) $\tau > 0$. Recall that $\mathbb{F}_\tau$ is the input-output map of $\Sigma$ on the interval $[0, \tau]$, see (1.4). The flow-invertibility of $\Sigma$ is equivalent to the condition that the transfer function $\mathbf{G}$ has a uniformly bounded inverse on some right half-plane. In this case, (1.9) remains valid with similar modifications as for (1.13). A necessary condition for the flow-invertibility of $\Sigma$ is that its input and output spaces have the same dimension (finite or infinite). Flow-inversion has been used in Rebarber and Townley

[12] for the feedback stabilization and robustness analysis of boundary control systems. We discuss flow-inversion in Section 5.

If the original system $\Sigma$ is time-invertible, and the time-inverted system is flow-invertible, then we get the time-flow-inverted system $\Sigma^{\leftarrow}$ by performing these two operations in sequel. Likewise, if the original system is flow-invertible, and the flow-inverted system is time-invertible, then a combination of these two inversions will give the time-flow-inverted system $\Sigma^{\leftarrow}$. However, *it is possible for a system to be time-flow-invertible even in the case where it is neither time-invertible nor flow-invertible.* The simplest example of such a system is a delay line, and additional examples will be provided. Time-flow-invertibility of a system does not even force the dimensions of the input and output spaces to be the same. The exact necessary and sufficient contition for the time-flow-invertibility of $\Sigma$ is that the operator matrix $\Sigma_\tau$ in (1.4) is invertible for some (hence, for every) $\tau > 0$. In particular, this condition is true whenever the system is *conservative*, which means that $\Sigma_\tau$ is unitary (e.g., a delay line is conservative). The first part of (1.11) remains valid if we replace $\begin{bmatrix} C & D \end{bmatrix}$ by $C\&D$ and $\begin{bmatrix} C^{\leftarrow} & D^{\leftarrow} \end{bmatrix}$ by $[C\&D]^{\leftarrow}$. The second part of (1.11) remains valid for all $s \in \rho(A^{\leftarrow}) \cap \rho(-A)$ (this set is usually large enough for (1.11) to determine $\mathbf{G}^{\leftarrow}$, but it can even be empty in some pathological cases, as we show by an example). An important criterion for time-flow-invertibility is the following:

**Theorem 1.1.** *The well-posed linear system $\Sigma$ is time-flow-invertible if and only if $S_\Sigma(s)$ has a uniformly bounded inverse for all $s$ in some left half-plane.*

We prove this theorem by applying the semigroup inversion result of Zwart [26] mentioned earlier to the Lax-Phillips semigroup induced by $\Sigma$. For a more detailed statement of the above theorem and for other facts on time-flow-inversion we refer to Section 6. Recently, time-flow-inversion for the wave equation has been used by Bardos and Fink [2] to focalize acoustic waves in a cavity.

We will also discuss the preservation of (weak) regularity under the various transformations. As shown in Section 3 of Part II, it is always possible to split $C\&D$ in a non-unique manner into an operator matrix $\begin{bmatrix} \overline{C} & D \end{bmatrix}$, where $\overline{C}$ is an extension of $C$. If the system is weakly regular, then it is possible to carry out this splitting in such a way that $D$ is the weak limit of $\mathbf{G}$ at $+\infty$, and this is the standard splitting of $C\&D$ that we adopt for weakly regular systems. In the duality transform, if the original system is weakly regular, then the dual system is also weakly regular, and $D^d = D^*$. In the case of flow-inversion a similar result is true: if the original system is regular, then the flow-inverted system is regular if and only if $D$ is invertible, in which case $D^\times = D^{-1}$. We do not know if regularity is preserved under flow inversion in general. Weak regularity need not be preserved under flow-inversion (as we show with an example). In the cases of time-inversion and time-flow-inversion the situation is even more complicated, due to the 180° rotation of the frequency domain (as manifested in the formulas for the transfer functions given in (1.7) and (1.11)). We show via examples that the time-inverted system $\Sigma^{Я}$ is not necessarily weakly regular (even if $\Sigma$ is regular). Even in the case where $\Sigma^{Я}$ is (weakly) regular, the backward feedthrough operator $D^{Я}$ may be different from the forward feedthrough operator $D$, as we show by another example. Similar comments apply to the time-flow-inverted system.

We will often use the terminology and results from [22] and from [19], which we refer to as "Part I" and "Part II". In such cases, we put the prefix I or II in front of the number of the item quoted. For example, Definition I.2.1 refers to Definition 2.1 in Part I, and (II.3.2) refers to formula (3.2) in Part II.

## 2. Some useful facts about well-posed linear systems

In this section we recall the notation from Part II that is needed in this paper. We also give a reformulation of the concept of a well-posed linear system: we rewrite the functional equations from Definition I.2.1 in a form that is more suitable for our computations. For easy reference, we also write down some formulas and one theorem from Part II which will be needed frequently.

**Notation 2.1.** We recall the notation from Section II.2 that will be used again here. Let $W$ be a Hilbert space and $J$ an interval. The spaces $L^2(J;W)$ and $L^2_{\mathrm{loc}}(J;W)$ are defined in the usual way, and $\mathbf{P}_J$ is the projection from $L^2_{\mathrm{loc}}((-\infty,\infty);W)$ onto $L^2_{\mathrm{loc}}(J;W)$ by truncation. We abbreviate $\mathbf{P}_\tau = \mathbf{P}_{[0,\tau]}$ (where $\tau \geq 0$), $\mathbf{P}_- = \mathbf{P}_{(-\infty,0]}$ and $\mathbf{P}_+ = \mathbf{P}_{[0,\infty)}$. The operator $\mathbf{S}_\tau$ is the (unilateral) right shift by $\tau$ on $L^2_{\mathrm{loc}}([0,\infty);W)$, and $\mathbf{S}^*_\tau$ is the left shift by $\tau$ on the same space. For any $u, v \in L^2_{\mathrm{loc}}([0,\infty);W)$ and any $\tau \geq 0$, the concatenation $u \underset{\tau}{\diamondsuit} v$ is defined by

$$u \underset{\tau}{\diamondsuit} v = \mathbf{P}_\tau u + \mathbf{S}_\tau v.$$

For every $u \in L^2_{\mathrm{loc}}([0,\infty);W)$ and every $\tau \geq 0$ we have

$$u = \mathbf{P}_\tau u + \mathbf{P}_{[\tau,\infty)} u = \mathbf{P}_\tau u + \mathbf{S}_\tau \mathbf{S}^*_\tau u = u \underset{\tau}{\diamondsuit} \mathbf{S}^*_\tau u.$$

This implies that $\begin{bmatrix} \mathbf{P}_\tau & \mathbf{S}_\tau \end{bmatrix}$ is a bijection from $L^2([0,\tau];W) \times L^2([0,\infty);W)$ to $L^2([0,\infty);W)$, with inverse $\begin{bmatrix} \mathbf{P}_\tau \\ \mathbf{S}^*_\tau \end{bmatrix}$, formally expressed as follows:

$$\begin{bmatrix} \mathbf{P}_\tau & \mathbf{S}_\tau \end{bmatrix} \begin{bmatrix} \mathbf{P}_\tau \\ \mathbf{S}^*_\tau \end{bmatrix} = I, \qquad \begin{bmatrix} \mathbf{P}_\tau \\ \mathbf{S}^*_\tau \end{bmatrix} \begin{bmatrix} \mathbf{P}_\tau & \mathbf{S}_\tau \end{bmatrix} = \begin{bmatrix} \mathbf{P}_\tau & 0 \\ 0 & I \end{bmatrix}, \tag{2.1}$$

where $I$ is the identity on $L^2([0,\infty);W)$.

Let $U$, $X$ and $Y$ be Hilbert spaces and put $\mathcal{U} = L^2([0,\infty);U)$, $\mathcal{Y} = L^2([0,\infty);Y)$. The concept of a *well-posed linear system* $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ on $\mathcal{U}$, $X$ and $\mathcal{Y}$ has been defined in Definition II.2.2 (or I.2.1). If $\Sigma$ is such a system, then we call $U$ its *input space*, $X$ its *state space* and $Y$ its *output space*. For $\tau \geq 0$ we denote

$$\Sigma_\tau = \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix},$$

as in (1.4). It follows from formula (II.2.1) in the definition of a well-posed system that $\Phi$ is causal: $\Phi_\tau \mathbf{P}_\tau = \Phi_\tau$ for all $\tau \geq 0$ (see (II.2.4)). In particular, we have $\Phi_0 = 0$. This, together with some of the other assumptions in the definition implies that $\Sigma_\tau$ satisfies the initial conditions

$$\Sigma_0 = \begin{bmatrix} \mathbb{T}_0 & \Phi_0 \\ \Psi_0 & \mathbb{F}_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.2}$$

The following reformulation of the definition will be useful when we prove that the various inverted systems and the dual system are indeed well-posed systems.

**Proposition 2.2.** *Assume that $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$, where the four components are families of operators indexed by $\tau \geq 0$ such that $\mathbb{T}_\tau \in \mathcal{L}(X)$, $\Phi_\tau \in \mathcal{L}(\mathcal{U};X)$, $\Psi_\tau \in \mathcal{L}(X;\mathcal{Y})$ and $\mathbb{F}_\tau \in \mathcal{L}(\mathcal{U};\mathcal{Y})$. Then $\Sigma$ is a well-posed linear system on $\mathcal{U}$, $X$ and $\mathcal{Y}$ if and only if*

(I) *$\mathbb{T}$ is strongly continuous at zero, i.e., $\lim_{t\downarrow 0} \mathbb{T}_t x_0 = x_0$,*
(II) *the initial conditions* (2.2) *hold,*
(III) *the following functional equation holds for all $\tau, t \geq 0$ :*

$$\begin{bmatrix} \mathbb{T}_{\tau+t} & \Phi_{\tau+t} \\ \Psi_{\tau+t} & \mathbb{F}_{\tau+t} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & \mathbf{P}_\tau & \mathbf{S}_\tau \end{bmatrix} \begin{bmatrix} \mathbb{T}_t & 0 & \Phi_t \\ 0 & I & 0 \\ \Psi_t & 0 & \mathbb{F}_t \end{bmatrix} \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau & 0 \\ \Psi_\tau & \mathbb{F}_\tau & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{P}_\tau \\ 0 & \mathbf{S}^*_\tau \end{bmatrix}. \tag{2.3}$$

*Proof.* The equations (II.2.1)–(II.2.3) can be rewritten (using (2.1)) in the form

$$\Phi_{\tau+t} = \begin{bmatrix} \mathbb{T}_t \Phi_\tau & \Phi_t \end{bmatrix} \begin{bmatrix} \mathbf{P}_\tau \\ \mathbf{S}^*_\tau \end{bmatrix}, \tag{2.4}$$

$$\Psi_{\tau+t} = \begin{bmatrix} \mathbf{P}_\tau & \mathbf{S}_\tau \end{bmatrix} \begin{bmatrix} \Psi_\tau \\ \Psi_t \mathbb{T}_\tau \end{bmatrix}, \tag{2.5}$$

$$\mathbb{F}_{\tau+t} = \begin{bmatrix} \mathbf{P}_\tau & \mathbf{S}_\tau \end{bmatrix} \begin{bmatrix} \mathbb{F}_\tau & 0 \\ \Psi_t \Phi_\tau & \mathbb{F}_t \end{bmatrix} \begin{bmatrix} \mathbf{P}_\tau \\ \mathbf{S}^*_\tau \end{bmatrix}. \tag{2.6}$$

A convenient way of rewriting (2.4)–(2.6) and the semigroup identity $\mathbb{T}_{\tau+t} = \mathbb{T}_t\mathbb{T}_\tau$ in block matrix form is (2.3). The remaining conditions in the definition are the strong continuity of $\mathbb{T}$, as in (I), and the initial conditions, which are a subset of those in (2.2). However, the extra initial condition $\Phi_0 = 0$ in (2.2) is a consequence of (II.2.1), as we have seen earlier, so that the two lists of conditions are equivalent. □

For any well-posed system $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$, the *extended output map* $\Psi_\infty$ and the *extended input-output* map $\mathbb{F}_\infty$ are defined as in Section II.2 (or I.2):

$$\Psi_\infty x_0 = \lim_{t\to\infty} \Psi_t x_0 , \qquad \mathbb{F}_\infty u = \lim_{t\to\infty} \mathbb{F}_t u ,$$

the limits being taken in the Fréchet space $L^2_{\rm loc}([0,\infty);Y)$.

For any $x_0 \in X$ and any $u \in L^2_{\rm loc}([0,\infty);U)$, the *state trajectory* $x\colon [0,\infty) \to X$ and the *output function* $y \in L^2_{\rm loc}([0,\infty);Y)$ of $\Sigma$ corresponding to the initial state $x_0$ and the input function $u$ are defined by

$$\begin{aligned} x(t) &= \mathbb{T}_t x_0 + \Phi_t u, \qquad t \geq 0, \\ y &= \Psi_\infty x_0 + \mathbb{F}_\infty u. \end{aligned} \tag{2.7}$$

**Remark 2.3.** Using the functions from (2.7), the functional equation (2.3) has a natural intuitive explanation: if we apply the right-hand side of (2.3) to $\left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right]$, then the successive intermediate results, from right to left, are

$$\begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \\ \mathbf{S}^*_\tau u \end{bmatrix}, \quad \begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \\ \mathbf{S}^*_\tau u \end{bmatrix}, \quad \begin{bmatrix} x(\tau+t) \\ \mathbf{P}_\tau y \\ \mathbf{P}_t \mathbf{S}^*_\tau y \end{bmatrix}, \quad \begin{bmatrix} x(\tau+t) \\ \mathbf{P}_{\tau+t} y \end{bmatrix}.$$

**Notation 2.4.** We need some more notation from Sections II.2 and II.5: for any Hilbert space $W$, any interval $J$ and any $\omega \in \mathbb{R}$ we put

$$L^2_\omega(J;W) = e_\omega L^2(J;W),$$

where $(e_\omega v)(t) = e^{\omega t}v(t)$. We denote by $H^1_{\rm loc}(J;W)$ the space of all those continuous functions on $J$ whose derivatives (in the sense of distributions) are in $L^2_{\rm loc}(J;W)$. $H^1(J;W)$ is the space of those continuous $v \in L^2(J;W)$ for which $v' \in L^2(J;W)$ (see Section II.2 for more detail). We denote by $\omega_{\mathbb{T}}$ the growth bound of the semigroup $\mathbb{T}$ and by $\mathbb{C}_\omega$ the half-plane of those $s \in \mathbb{C}$ for which $\operatorname{Re} s > \omega$.

For the remainder of this section, we assume that $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ is a well-posed linear system with input space $U$, state space $X$, output space $Y$ and transfer function $\mathbf{G}$. Denote the generator of $\mathbb{T}$ by $A$. Recall that the space $X_1$ is defined as $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, where $\beta \in \rho(A)$, and $X_{-1}$ is the completion of $X$ with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1}z\|$. There exists a unique $B \in \mathcal{L}(U;X_{-1})$, called the *control operator* of $\Sigma$, such that for all $t \geq 0$,

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma)\, {\rm d}\sigma . \tag{2.8}$$

For any initial state $x_0 \in X$ and any input $u \in L^2_{\rm loc}([0,\infty);U)$, the state trajectory $x$ defined in (2.7) is the unique strong solution in $X_{-1}$ of

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad t \geq 0, \\ x(0) &= x_0 , \end{aligned} \tag{2.9}$$

i.e., $x$ is continuous with values in $X$, $x \in H^1_{\rm loc}([0,\infty);X_{-1})$ and its derivative $\dot{x}$ satisfies the first equation in (2.9) for almost every $t \geq 0$.

There exists a unique $C \in \mathcal{L}(X_1;Y)$, called the *observation operator* of $\Sigma$, such that for every $x_0 \in X_1$ and all $t \geq 0$,

$$(\Psi_\infty x_0)(t) = C\mathbb{T}_t x_0 \tag{2.10}$$

(this determines $\Psi_\infty$). The $\Lambda$-*extension* of $C$, denoted $C_\Lambda$, is defined by

$$C_\Lambda x_0 = \lim_{\lambda\to+\infty} C\lambda(\lambda I - A)^{-1}x_0 . \tag{2.11}$$

Its domain $\mathcal{D}(C_\Lambda)$ consists of all $x_0 \in X$ for which the above limit exists. The *weak* $\Lambda$-*extension* of $C$, denoted $C_{\Lambda w}$, is defined in the same way, but with the strong limit replaced by a weak limit (hence, $C_{\Lambda w}$ is an extension of $C_\Lambda$).

For every $x_0 \in X$ and $u \in L^2_\omega([0,\infty);U)$ with $\omega > \omega_{\mathbb{T}}$, the corresponding output function $y$ (see (2.7)) is in $L^2_\omega([0,\infty);Y)$ and its Laplace transform is

$$\hat{y}(s) = C(sI-A)^{-1}x_0 + \mathbf{G}(s)\hat{u}(s), \qquad \operatorname{Re} s > \omega\,.$$

$\mathbf{G}$ satisfies for all $s, \beta \in \mathbb{C}_{\omega_{\mathbb{T}}}$

$$\mathbf{G}(s) - \mathbf{G}(\beta) = C\left[(sI-A)^{-1} - (\beta I - A)^{-1}\right]B\,. \tag{2.12}$$

We recall some facts from Section II.4. The system $\Sigma$ is called *weakly regular* if the following weak limit exists in $Y$, for every $u_0 \in U$:

$$\text{weak}\lim_{\lambda\to+\infty} \mathbf{G}(\lambda)u_0 = Du_0\,. \tag{2.13}$$

$\Sigma$ is called *regular* if the above limit exists in the norm topology. In either case, the operator $D \in \mathcal{L}(U;Y)$ defined by (2.13) is called the *feedthrough operator* of $\Sigma$. If $Y$ is finite-dimensional, then weak regularity equals regularity, of course. In general, this is not true, as demonstrated by Example 8.1.

If $\Sigma$ is weakly regular, then its output $y$ (defined in (2.7)) is given by

$$y(t) = C_{\Lambda w}x(t) + Du(t), \tag{2.14}$$

for almost every $t \geq 0$ (in particular, $x(t) \in \mathcal{D}(C_{\Lambda w})$ for almost every $t \geq 0$). If $\Sigma$ is weakly regular, then we also have that

$$\mathbf{G}(s) = C_{\Lambda w}(sI-A)^{-1}B + D\,, \qquad \operatorname{Re} s > \omega_{\mathbb{T}}\,. \tag{2.15}$$

If $\Sigma$ is regular, then (2.14) and (2.15) remain true with $C_\Lambda$ in place of $C_{\Lambda w}$. The operators $A, B, C$ and $D$ defined as above are called the *generating operators* of the (weakly) regular linear system $\Sigma$, since they determine $\Sigma$ via (2.9) and (2.14).

Now we recall some concepts and formulas from Section II.3. We define the dense subspace $V$ of $X \times U$ by

$$V = \left\{ \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in X \times U \;\middle|\; Ax_0 + Bu_0 \in X \right\}. \tag{2.16}$$

We define the operator $C\&D : V \to Y$ by

$$C\&D \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = C\left[x_0 - (\beta I - A)^{-1}Bu_0\right] + \mathbf{G}(\beta)u_0\,,$$

where $\beta \in \mathbb{C}_{\omega_{\mathbb{T}}}$ is arbitrary (i.e., the the result is independent of $\beta$ as long as $\operatorname{Re}\beta > \omega_{\mathbb{T}}$). We call $C\&D$ the *combined observation/feedthrough operator* of $\Sigma$. We also introduce the *system operator* of $\Sigma$, $S_\Sigma(s) : V \to X \times Y$ by

$$S_\Sigma(s) = \begin{bmatrix} A \;\; B \\ C\&D \end{bmatrix} - \begin{bmatrix} sI & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{for all } \; s \in \mathbb{C},$$

which can be regarded as a densely defined and closed operator from $X \times U$ to $X \times Y$. If $\Sigma$ is weakly regular, then $C\&D = [C_{\Lambda w} \;\, D]$ and hence

$$S_\Sigma(s) = \begin{bmatrix} A - sI & B \\ C_{\Lambda w} & D \end{bmatrix}, \qquad \text{for all } \; s \in \mathbb{C}.$$

If $\Sigma$ is regular, then in the above formula $C_{\Lambda w}$ may be replaced by $C_\Lambda$. We recall Theorem II.3.1 for easy reference, since we need it often:

**Theorem 2.5.** (i) *Assume that* $u \in H^1_{\text{loc}}([0,\infty);U)$ *and* $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in V$*. The state trajectory* $x$ *and the output function* $y$ *are defined as in* (2.7)*. Then*

$$x \in C^1([0,\infty);X), \quad \begin{bmatrix} x \\ u \end{bmatrix} \in C([0,\infty);V), \quad y \in H^1_{\text{loc}}([0,\infty);Y),$$

*and for every $t \geq 0$ we have that*

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S_\Sigma(0) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \tag{2.17}$$

*If $u \in H^1_\omega([0,\infty);U)$ with $\omega > \omega_{\mathbb{T}}$, then $y \in H^1_\omega([0,\infty);Y)$.*

(ii) *The transfer function $\mathbf{G}$ of $\Sigma$ is given for $\operatorname{Re} s > \omega_{\mathbb{T}}$ by*

$$\mathbf{G}(s) = C\&D \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix}. \tag{2.18}$$

In the sequel, *we shall use* (2.18) *as the definition of* $\mathbf{G}(s)$ *for all* $s \in \rho(A)$ (not just for $\operatorname{Re} s > \omega_{\mathbb{T}}$). The same extension of the transfer function will be used for the dual system and for the various inverted systems. If $\rho(A)$ is connected, then this extension of $\mathbf{G}$ coincides with its analytic continuation to $\rho(A)$. If $\rho(A)$ is not connected, then $\mathbf{G}$ may still have an analytic continuation to a part of $\rho(A)$, but this continuation may be different from $\mathbf{G}$ given by (2.18), see Remark I.4.8 and also Example 8.3. Our extension of $\mathbf{G}$ satisfies (2.12) for all $s$, $\beta \in \rho(A)$.

## 3. The dual system

The meaning of the dual system for finite-dimensional linear systems was explained in the introduction, in terms of the matrices $A$, $B$, $C$ and $D$ appearing in (1.1). For infinite-dimensional systems it seems preferable to define the dual system in terms of the operator families $\mathbb{T}$, $\Phi$, $\Psi$, and $\mathbb{F}$ appearing in the definition of a well-posed system. To do this we must first introduce the *reflection operators* $\text{Я}$ and $\text{Я}_\tau$.

**Notation 3.1.** Let $W$ be a Hilbert space. For every $u \in L^2_{\text{loc}}((-\infty,\infty);W)$ and all $\tau \geq 0$, we define

$$(\text{Я}u)(t) = u(-t) \quad \text{for } t \in \mathbb{R}, \qquad (\text{Я}_\tau u)(t) = \begin{cases} u(\tau - t) & \text{for } t \in [0,\tau], \\ 0 & \text{for } t \notin [0,\tau]. \end{cases}$$

Clearly, for every $u, v \in L^2_{\text{loc}}([0,\infty);U)$ and for all $\tau, t \geq 0$,

$$\begin{aligned} \text{Я}_\tau &= \text{Я}_\tau \mathbf{P}_\tau = \mathbf{P}_\tau \text{Я}_\tau, \qquad \text{Я}_\tau^2 = \mathbf{P}_\tau, \\ \text{Я}_{\tau+t} &= \begin{bmatrix} \mathbf{S}_t & \text{Я}_t \end{bmatrix} \begin{bmatrix} \text{Я}_\tau \\ \mathbf{S}_\tau^* \end{bmatrix} = \begin{bmatrix} \text{Я}_\tau & \mathbf{S}_\tau \end{bmatrix} \begin{bmatrix} \mathbf{S}_t^* \\ \text{Я}_t \end{bmatrix}, \\ \begin{bmatrix} \mathbf{S}_t \mathbf{P}_\tau & \mathbf{P}_t \end{bmatrix} &= \text{Я}_{\tau+t} \begin{bmatrix} \mathbf{P}_\tau & \mathbf{S}_\tau \end{bmatrix} \begin{bmatrix} \text{Я}_\tau & 0 \\ 0 & \text{Я}_t \end{bmatrix}. \end{aligned} \tag{3.1}$$

To prove that the dual system, the time-inverted system and the time-flow inverted system are indeed well-posed linear systems, it will be handy to have the concept of an anti-causal well-posed system, defined as follows:

**Definition 3.2.** We use the notation $U$, $X$, $Y$, $\mathcal{U}$, $\mathcal{Y}$ from Proposition 2.2. An *anti-causal well-posed linear system* on $\mathcal{U}$, $X$ and $\mathcal{Y}$ is a quadruple of operator families $\Sigma^a = (\mathbb{T}^a, \Phi^a, \Psi^a, \mathbb{F}^a)$ parameterized by $\tau \geq 0$ such that, if we define

$$\begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \text{Я}_\tau \end{bmatrix} \begin{bmatrix} \mathbb{T}^a_\tau & \Phi^a_\tau \\ \Psi^a_\tau & \mathbb{F}^a_\tau \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \text{Я}_\tau \end{bmatrix},$$

then $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ is a well-posed linear system on $\mathcal{U}$, $X$ and $\mathcal{Y}$. In this case, we call $\Sigma$ the *causal version* of $\Sigma^a$, and we call $\Sigma^a$ the *anti-causal version* of $\Sigma$.

Anti-causal systems have the following simple algebraic characterization:

**Proposition 3.3.** *The quadruple $\Sigma^a = (\mathbb{T}^a, \Phi^a, \Psi^a, \mathbb{F}^a)$ of families of operators $\mathbb{T}^a_\tau \in \mathcal{L}(X)$, $\Phi^a_\tau \in \mathcal{L}(\mathcal{U};X)$, $\Psi^a_\tau \in \mathcal{L}(X;\mathcal{Y})$, $\mathbb{F}^a_\tau \in \mathcal{L}(\mathcal{U};\mathcal{Y})$, indexed by $\tau \geq 0$, forms an anti-causal well-posed linear* *system on $\mathcal{U}$, $X$ and $\mathcal{Y}$ if and only if $\mathbb{T}^a$ is strongly continuous and the following equalities hold: for all $\tau, t \geq 0$,*

$$\begin{bmatrix} \mathbb{T}_0^a & \Phi_0^a \\ \Psi_0^a & \mathbb{F}_0^a \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.2}$$

$$\begin{bmatrix} \mathbb{T}_{\tau+t}^a & \Phi_{\tau+t}^a \\ \Psi_{\tau+t}^a & \mathbb{F}_{\tau+t}^a \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & \mathbf{P}_\tau & \mathbf{S}_\tau \end{bmatrix} \begin{bmatrix} \mathbb{T}_\tau^a & \Phi_\tau^a & 0 \\ \Psi_\tau^a & \mathbb{F}_\tau^a & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathbb{T}_t^a & 0 & \Phi_t^a \\ 0 & I & 0 \\ \Psi_t^a & 0 & \mathbb{F}_t^a \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{P}_\tau \\ 0 & \mathbf{S}_\tau^* \end{bmatrix}. \tag{3.3}$$

*Proof.* This can be checked by direct computation based on (2.3) and (3.1). A key obsevation which simplifies this computation is that several of the block matrices that appear in these equations commute with each other because of their block structure. For example, if we use a star to represent an irrelevant entry, then $\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & \mathbf{Я}_\tau & 0 \\ 0 & 0 & I \end{smallmatrix}\right]$ commutes with every operator with the structure $\left[\begin{smallmatrix} * & 0 & * \\ 0 & I & 0 \\ * & 0 & * \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathbf{Я}_t \end{smallmatrix}\right]$ commutes with every operator with the structure $\left[\begin{smallmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & I \end{smallmatrix}\right]$. □

Observe that (3.3) is almost identical to (2.3): the only difference is that two of the factors have changed places in (3.3) compared to (2.3).

**Theorem 3.4.** *Let $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ be a well-posed linear system with input space $U$, state space $X$ and output space $Y$. Define $\Sigma_\tau^d$ (for all $\tau \geq 0$) by*

$$\Sigma_\tau^d = \begin{bmatrix} \mathbb{T}_\tau^d & \Phi_\tau^d \\ \Psi_\tau^d & \mathbb{F}_\tau^d \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathbf{Я}_\tau \end{bmatrix} \begin{bmatrix} \mathbb{T}_\tau^* & \Psi_\tau^* \\ \Phi_\tau^* & \mathbb{F}_\tau^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{Я}_\tau \end{bmatrix}. \tag{3.4}$$

*Then $\Sigma^d = (\mathbb{T}^d, \Phi^d, \Psi^d, \mathbb{F}^d)$ is a well-posed linear system with input space $Y^*$, state space $X^*$ and output space $U^*$. Let $x_0 \in X$, $x_0^d \in X^*$, $u \in L^2_{\mathrm{loc}}([0,\infty); U)$ and $y^d \in L^2_{\mathrm{loc}}([0,\infty); Y^*)$. Let $x$ and $y$ be the state trajectory and the output function of $\Sigma$ corresponding to the initial state $x_0$ and the input function $u$ (see (2.7)). Let $x^d$ and $u^d$ be the state trajectory and the output function of $\Sigma^d$ corresponding to the intial state $x_0^d$ and the input function $y^d$. Then, for every $\tau \geq 0$,*

$$\begin{aligned} &\langle x_0, x^d(\tau) \rangle + \int_0^\tau \langle u(\sigma), u^d(\tau - \sigma) \rangle \,\mathrm{d}\sigma \\ &\qquad = \langle x(\tau), x_0^d \rangle + \int_0^\tau \langle y(\sigma), y^d(\tau - \sigma) \rangle \,\mathrm{d}\sigma. \end{aligned} \tag{3.5}$$

The system $\Sigma^d$ introduced above is called the *dual system* corresponding to $\Sigma$. It is easy to verify (from (3.4)) that applying the duality transformation twice (and identifying the bidual space of any Hilbert space with the original Hilbert space), we get back the original system: $(\Sigma^d)^d = \Sigma$. Cleary $\omega_{\mathbb{T}} = \omega_{\mathbb{T}^d}$ (since $\mathbb{T}_\tau^d = \mathbb{T}_\tau^*$).

*Proof of Theorem* 3.4. We denote by $\Sigma^* = (\mathbb{T}^*, \Psi^*, \Phi^*, \mathbb{F}^*)$ the families of operators obtained by taking the adjoints of the corresponding operators of $\Sigma$, so that

$$\Sigma_\tau^* = \begin{bmatrix} \mathbb{T}_\tau^* & \Psi_\tau^* \\ \Phi_\tau^* & \mathbb{F}_\tau^* \end{bmatrix}.$$

If we take adjoints in (2.3), then we get (3.3) with $\Sigma^a$ replaced by $\Sigma^*$. In addition, $\Sigma^*$ satisfies (3.2) and $\mathbb{T}^*$ is strongly continuous. Thus, by Proposition 3.3, $\Sigma^*$ is an anti-causal well-posed system with input space $Y^*$, state space $X^*$ and output space $U^*$. By its definition (3.4), $\Sigma^d$ is the causal version of $\Sigma^*$, so that $\Sigma^d$ is a (causal) well-posed system on the stated spaces.

To prove (3.5), we rewrite it in the form

$$\left\langle \begin{bmatrix} x_0 \\ \mathbf{P}_\tau u \end{bmatrix}, \begin{bmatrix} x^d(\tau) \\ \mathbf{Я}_\tau u^d \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \end{bmatrix}, \begin{bmatrix} x_0^d \\ \mathbf{Я}_\tau y^d \end{bmatrix} \right\rangle.$$

This version of the formula is a consequence of the facts that

$$\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \end{bmatrix} = \Sigma_\tau \begin{bmatrix} x_0 \\ \mathbf{P}_\tau u \end{bmatrix}, \qquad \begin{bmatrix} x^d(\tau) \\ \mathbf{Я}_\tau u^d \end{bmatrix} = \Sigma_\tau^* \begin{bmatrix} x_0^d \\ \mathbf{Я}_\tau y^d \end{bmatrix}.$$

□

In the above theorem we have not specified if by $X^*$ (and the other dual spaces) we mean the linear dual or the antilinear dual (which is usually identified with $X$). Accordingly, the adjoint operators have two possible meanings. It seems that the dual system in the linear sense is more natural to generalize to the Banach space context, while the dual system in the antilinear sense is more useful when discussing optimal control. Unless otherwise specified, we will have the antilinear dual in mind, which is the usual way to proceed in the Hilbert space case, and accordingly, in the sequel we identify $U^* = U$, $X^* = X$ and $Y^* = Y$.

Formula (3.5) is equivalent to [13, formula (3.4)] and also to [16, Lemma 2.15] (stated without proof). Our following theorem describes the system operator of the dual system in terms of the original system operator.

**Theorem 3.5.** *With the assumptions and the notation of Theorem* 3.4*, denote the semigroup generator of* $\Sigma$ *by* $A$*, its control operator by* $B$*, its observation operator by* $C$*, its combined observation/feedthrough operator by* $C\&D$ *and its system operator by* $S_\Sigma(s)$*. The corresponding operators for* $\Sigma^d$ *are denoted* $A^d$*,* $B^d$*,* $C^d$*,* $[C\&D]^d$ *and* $S_\Sigma^d(s)$*. Then we have*

$$S_\Sigma^d(0) = \begin{bmatrix} A^d & B^d \\ [C\&D]^d \end{bmatrix} = \begin{bmatrix} A & B \\ C\&D \end{bmatrix}^* = S_\Sigma^*(0), \tag{3.6}$$

*where the adjoint of* $S_\Sigma(0)$ *is computed by regarding it as a densely defined and closed operator from* $X \times U$ *to* $X \times Y$*. Moreover,* $A^d = A^*$*,* $B^d = C^*$*,* $C^d = B^*$*, and the transfer functions of* $\Sigma$ *and* $\Sigma^d$*, denoted by* $\mathbf{G}$ *and* $\mathbf{G}^d$*, are related by*

$$\mathbf{G}^d(s) = \mathbf{G}^*(\overline{s}), \qquad s \in \rho(A^d) \tag{3.7}$$

*(for the linear dual the relationship is* $\mathbf{G}^d(s) = \mathbf{G}^*(s)$*).*

Some clarifications may be needed. The spaces $X_1$ and $X_{-1}$ are as in Section 2 (before (2.8)). The corresponding spaces that we get by replacing $A$ by $A^d = A^*$ are denoted $X_1^d$ and $X_{-1}^d$. Thus, we have the continuous and dense embeddings $X_1^d \subset X \subset X_{-1}^d$. The scalar product of $X$ has continuous extensions to $X_1 \times X_{-1}^d$ and to $X_1^d \times X_{-1}$, and $X_{-1}^d$ (respectively $X_{-1}$) may be regarded as the dual of $X_1$ (respectively of $X_1^d$) (see [20] for more detail). Then we have $B^* \in \mathcal{L}(X_1^d, U)$ and $C^* \in \mathcal{L}(Y, X_{-1}^d)$, these operators being computed as the adjoints of the bounded operators $B \in \mathcal{L}(U; X_{-1})$ and $C \in \mathcal{L}(X_1; Y)$. By contrast, the adjoint of $S_\Sigma(0)$ is computed in (3.6) by regarding it as an unbounded operator, as explained in the theorem. Finally, recall that $\mathbf{G}^d$ is defined on $\rho(A^d) = \rho(A^*) = \overline{\rho(A)}$ according to our convention at the end of Section 2.

*Proof of Theorem* 3.5. We introduce the space $V^d$ as the analogue of $V$ for the system $\Sigma^d$:

$$V^d = \left\{ \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in X \times Y \;\middle|\; A^d x_0 + B^d y_0 \in X \right\}.$$

Let $x_0$, $u$, $x$, $y$, $x_0^d$, $y^d$, $x^d$ and $u^d$ be as in the last part of Theorem 3.4, but now we reqire that $u \in H^1_{\mathrm{loc}}([0,\infty); U)$ and $y^d \in H^1_{\mathrm{loc}}([0,\infty); Y)$. Moreover, denoting $u_0 = u(0)$ and $y_0^d = y^d(0)$, we require that $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in V$ and $\left[\begin{smallmatrix} x_0^d \\ y_0^d \end{smallmatrix}\right] \in V^d$. Then by Theorem 3.4 (after subtracting $\langle x_0, x_0^d\rangle$ from both sides of (3.5)) we obtain that

$$\begin{aligned} \langle x_0, x^d(\tau) - x_0^d\rangle &+ \int_0^\tau \big\langle u(\sigma), u^d(\tau-\sigma)\big\rangle \,\mathrm{d}\sigma \\ &= \langle x(\tau) - x_0, x_0^d\rangle + \int_0^\tau \big\langle y(\sigma), y^d(\tau-\sigma)\big\rangle \,\mathrm{d}\sigma, \end{aligned}$$

for all $\tau > 0$. Divide by $\tau$ and let $\tau \downarrow 0$, using that (according to Theorem 2.5) $x, x^d \in C^1([0,\infty); X)$ and all the functions are continuous. Then we obtain

$$\langle x_0, \dot{x}^d(0)\rangle + \langle u_0, u^d(0)\rangle = \langle \dot{x}(0), x_0^d\rangle + \langle y(0), y_0^d\rangle,$$

and now using (2.17) and its dual version, both for $t = 0$, this becomes

$$\left\langle S_\Sigma(0) \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \begin{bmatrix} x_0^d \\ y_0^d \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, S_\Sigma^d(0) \begin{bmatrix} x_0^d \\ y_0^d \end{bmatrix} \right\rangle. \tag{3.8}$$

This being true for all $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in V$ and $\left[\begin{smallmatrix} x_0^d \\ y_0^d \end{smallmatrix}\right] \in V^d$, we conclude that $V^d \subset \mathcal{D}(S_\Sigma^*(0))$, and that $S_\Sigma^d(0)$ is the restriction of $S_\Sigma^*(0)$ to $V^d$. To prove (3.6), it remains to show the opposite inclusion $\mathcal{D}(S_\Sigma^*(0)) \subset V^d$. However, we postpone this for a while, and instead look at the other assertions in Theorem 3.5.

It is clear from (3.4) that $\mathbb{T}_\tau^d = \mathbb{T}_\tau^*$, so that its generator is $A^d = A^*$, as is well-known from semigroup theory, see for instance Pazy [10]. Take $s \in \rho(A)$, $u_0 \in U$, $x_0 = (sI - A)^{-1}Bu_0$, $x_0^d \in X_1^d$ and $y_0^d = 0$ in (3.8). We simplify this formula, using that (by (2.18) and the definitions of $C\&D$ and $[C\&D]^d$)

$$S_\Sigma(0)\begin{bmatrix}(sI-A)^{-1}B \\ I\end{bmatrix} = \begin{bmatrix} s(sI-A)^{-1}B \\ \mathbf{G}(s)\end{bmatrix}, \qquad S_\Sigma^d(0)\begin{bmatrix} x_0^d \\ 0 \end{bmatrix} = \begin{bmatrix} A^d x_0^d \\ C^d x_0^d\end{bmatrix}, \tag{3.9}$$

to get

$$\left\langle s(sI-A)^{-1}Bu_0, x_0^d\right\rangle = \left\langle \begin{bmatrix}(sI-A)^{-1}B \\ I\end{bmatrix} u_0, \begin{bmatrix} A^d x_0^d \\ C^d x_0^d\end{bmatrix}\right\rangle.$$

Using that $A^d = A^*$, this simplifies to $\langle Bu_0, x_0^d\rangle = \langle u_0, C^d x_0^d\rangle$ for all $u_0 \in U$ and all $x_0^d \in X_1^d$. Thus $C^d = B^*$. A similar computation where we interchange the roles of $\Sigma$ and $\Sigma^d$ shows that $C = B^{d*}$, hence $B^d = C^*$.

Finally, we take $s, u_0$ and $x_0$ as above, $y_0^d \in Y$, and $x_0^d = (\overline{s}I - A^d)^{-1}B^d y_0^d$ in (3.8). Using the first formula in (3.9) and its dual version, we obtain

$$\begin{aligned}&\left\langle \begin{bmatrix} s(sI-A)^{-1}B \\ \mathbf{G}(s)\end{bmatrix} u_0, \begin{bmatrix}(\overline{s}I - A^d)^{-1}B^d \\ I\end{bmatrix} y_0^d\right\rangle \\ &\qquad = \left\langle \begin{bmatrix}(sI-A)^{-1}B \\ I\end{bmatrix} u_0, \begin{bmatrix}\overline{s}(\overline{s}I - A^d)^{-1}B^d \\ \mathbf{G}^d(\overline{s})\end{bmatrix} y_0^d\right\rangle.\end{aligned}$$

After simplification, this gives $\langle \mathbf{G}(s)u_0, y_0^d\rangle = \langle u_0, \mathbf{G}^d(\overline{s})y_0^d\rangle$ for all $u_0 \in U$ and $y_0^d \in Y$. Thus $\mathbf{G}^d(\overline{s}) = \mathbf{G}^*(s)$ for $s \in \rho(A)$, which is equivalent to (3.7).

We now return to prove the inclusion $\mathcal{D}(S_\Sigma^*(0)) \subset V^d$. By the definition of the adjoint of an unbounded operator, $\left[\begin{smallmatrix} x_0^d \\ y_0^d \end{smallmatrix}\right] \in \mathcal{D}(S_\Sigma^*(0))$ if and only if the expression $\left\langle S_\Sigma(0)\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], \left[\begin{smallmatrix} x_0^d \\ y_0^d \end{smallmatrix}\right]\right\rangle$, regarded as a function of $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in V$, can be extended to a bounded linear functional on $X \times U$. Suppose that this is the case, i.e., $\left[\begin{smallmatrix} x_0^d \\ y_0^d \end{smallmatrix}\right] \in \mathcal{D}(S_\Sigma^*(0))$. If we take $u_0 = 0$, we obtain that the functional $x_0 \mapsto \langle Ax_0, x_0^d\rangle + \langle Cx_0, y_0^d\rangle$, originally defined on $X_1$, can be extended to a bounded linear functional on $X$. We can interpret this expression as $\langle x_0, A^*x_0^d + C^*y_0^d\rangle$, where the first term belongs to $X_1$ and the second to $X_{-1}^d$. Since this functional has a bounded extension to $X$, we must have $A^*x_0^d + C^*y_0^d = A^d x_0^d + B^d y_0^d \in X$, i.e., $\left[\begin{smallmatrix} x_0^d \\ y_0^d \end{smallmatrix}\right] \in V^d$. □

**Remark 3.6.** Salamon in [13, Sect. 3] and Arov and Nudelman [1, Sect. 3] define the dual system in terms of $A, B, C$ and $\mathbf{G}$, i.e., taking as a starting point what appears here as Theorem 3.5. Their definitions are equivalent to our definition (after a time-inversion in the case of [13]; Salamon's dual system is anti-causal). The anti-causal dual is also used by Staffans [15, 16], who defines it via the adjoints of $\mathbb{T}$ and of the extended operators $\widetilde{\Phi}_0$, $\Psi_\infty$ and $\mathcal{F}$ ($\widetilde{\Phi}_0$ and $\mathcal{F}$ have been defined in Section II.5). If we rewrite his definition for the causal dual in our notation, then we get

$$\begin{bmatrix} \mathbb{T}^d & \widetilde{\Phi}_0^d \\ \Psi_\infty^d & \mathcal{F}^d \end{bmatrix} = \begin{bmatrix} \mathbb{T}^* & \Psi_\infty^* \boldsymbol{Я} \\ \boldsymbol{Я}\widetilde{\Phi}_0^* & \boldsymbol{Я}\mathcal{F}^*\boldsymbol{Я}\end{bmatrix}.$$

Our interest in weak regularity (see Part II) is motivated by the following:

**Proposition 3.7.** *If the well-posed system $\Sigma$ is weakly regular, then its dual system $\Sigma^d$ is weakly regular as well, and their feedthrough operators, denoted by $D$ and $D^d$, are related by*

$$D^d = D^*.$$

*Proof.* Weak regularity of $\Sigma$ is equivalent to the fact that $\lim_{\lambda\to+\infty}\mathbf{G}(\lambda) = D$ in the weak operator topology (see Section 2), and similarly weak regularity of $\Sigma^d$ means that $\lim_{\lambda\to+\infty}\mathbf{G}^d(\lambda) = D^d$ in the weak operator topology. By (3.7) and since $\lambda$ is real, $\mathbf{G}^d(\lambda) = \mathbf{G}^*(\lambda)$, and weak operator convergence is preserved under duality. Thus, weak regularity of $\Sigma$ and $\Sigma^d$ are equivalent and $D^d = D^*$. ☐

**Remark 3.8.** Proposition 3.7 implies that if $\Sigma$ is weakly regular and its input space $U$ is finite-dimensional, then $\Sigma^d$ is regular. Unfortunately, there are regular systems whose dual is not regular, as we shall see in Example 8.1.

We have explained in Section II.6 that any well-posed linear system can be represented by a strongly continuous semigroup acting on a large product space, and this is related to the way in which systems were represented in the work of Lax and Phillips [5, 6]. As our following result shows, there is a simple connection between the dual system of $\Sigma$ and the adjoint of the Lax–Phillips semigroup induced by $\Sigma$ (for the terminology used here we refer to Section II.6).

**Proposition 3.9.** *We use the assumptions and the notation of Theorem* 3.4. *Let $\omega \in \mathbb{R}$ and let $\mathfrak{T}$ be the Lax–Phillips semigroup of index $\omega$ induced by $\Sigma$. Then the Lax–Phillips semigroup of index $\omega$ induced by $\Sigma^d$ is given by*

$$\mathfrak{T}^d_\tau = \begin{bmatrix} 0 & 0 & \mathbf{Я} \\ 0 & I & 0 \\ \mathbf{Я} & 0 & 0 \end{bmatrix} \mathfrak{T}^*_\tau \begin{bmatrix} 0 & 0 & \mathbf{Я} \\ 0 & I & 0 \\ \mathbf{Я} & 0 & 0 \end{bmatrix}, \qquad \tau \geq 0. \tag{3.10}$$

*Proof.* The semigroup $\mathfrak{T}$ acts on the space $\mathcal{H} = L^2_\omega((-\infty,0];Y) \times X \times L^2_\omega([0,\infty);U)$, so that the adjoint semigroup $\mathfrak{T}^*$ acts on the dual space of $\mathcal{H}$, which we identify with $L^2_{-\omega}((-\infty,0];Y) \times X \times L^2_{-\omega}([0,\infty);U)$. Since for any real $\alpha$, $\mathbf{Я}$ maps $L^2_\alpha((-\infty,0];Y)$ onto $L^2_{-\alpha}([0,\infty);Y)$ and $L^2_\alpha([0,\infty);U)$ onto $L^2_{-\alpha}((-\infty,0];U)$, the right-hand side of (3.10) is a bounded operator acting on the space $L^2_\omega((-\infty,0];U) \times X \times L^2_\omega([0,\infty);Y)$. $\mathfrak{T}^d_\tau$ is defined as in Proposition II.6.2, but with $\Sigma^d$ in place of $\Sigma$. The proof of the identity (3.10) is a straightforward algebraic computation:

$$\begin{aligned} \mathfrak{T}^d_\tau &= \begin{bmatrix} \mathcal{S}_{-\tau}\mathbf{P}_- & \mathcal{S}_{-\tau}\Psi^d_\tau & \mathcal{S}_{-\tau}\mathbb{F}^d_\tau \\ 0 & \mathbb{T}^d_\tau & \Phi^d_\tau \\ 0 & 0 & \mathbf{P}_+\mathcal{S}_{-\tau} \end{bmatrix} = \begin{bmatrix} \mathbf{Я}\mathcal{S}_\tau\mathbf{P}_+\mathbf{Я} & \mathbf{Я}\Phi^*_\tau & \mathbf{Я}\mathbb{F}^*_\tau\mathcal{S}_\tau\mathbf{Я} \\ 0 & \mathbb{T}^*_\tau & \Psi^*_\tau\mathcal{S}_\tau\mathbf{Я} \\ 0 & 0 & \mathbf{Я}\mathbf{P}_-\mathcal{S}_\tau\mathbf{Я} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \mathbf{Я} \\ 0 & I & 0 \\ \mathbf{Я} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P}_-\mathcal{S}_\tau & 0 & 0 \\ \Psi^*_\tau\mathcal{S}_\tau & \mathbb{T}^*_\tau & 0 \\ \mathbb{F}^*_\tau\mathcal{S}_\tau & \Phi^*_\tau & \mathcal{S}_\tau\mathbf{P}_+ \end{bmatrix} \begin{bmatrix} 0 & 0 & \mathbf{Я} \\ 0 & I & 0 \\ \mathbf{Я} & 0 & 0 \end{bmatrix}. \end{aligned}$$

The operator matrix in the middle of the second line of the above formula is in fact $\mathfrak{T}^*_\tau$, so that we have proved (3.10). ☐

## 4. Time-inversion

In this section we introduce the time-inverted system corresponding to a well-posed linear system and we investigate its properties. We assume that $\Sigma = (\mathbb{T},\Phi,\Psi,\mathbb{F})$ is a well-posed linear system with input space $U$, state space $X$, output space $Y$, semigroup generator $A$, control operator $B$, observation operator $C$, combined observation/feedthrough operator $C\&D$, system operator $S_\Sigma(s)$ and transfer function $\mathbf{G}$. We also use $\mathbf{P}_\tau$, $\mathbf{S}_\tau$ and $\mathbf{S}^*_\tau$ from Section 2, as well as the operators $\mathbf{Я}$ and $\mathbf{Я}_\tau$ from Section 3. We denote $\mathbb{T}_{-\tau} = \mathbb{T}_\tau^{-1}$, if the inverse exists.

**Theorem 4.1.** *Suppose that $\mathbb{T}_\tau$ is invertible for some $\tau > 0$ (hence, for all $\tau \geq 0$). Define the operator $\Sigma^{\mathbf{Я}}_\tau$ (for all $\tau \geq 0$) by*

$$\Sigma^{\mathbf{Я}}_\tau = \begin{bmatrix} \mathbb{T}^{\mathbf{Я}}_\tau & \Phi^{\mathbf{Я}}_\tau \\ \Psi^{\mathbf{Я}}_\tau & \mathbb{F}^{\mathbf{Я}}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathbf{Я}_\tau \end{bmatrix} \begin{bmatrix} I & 0 \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix} \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & \mathbf{Я}_\tau \end{bmatrix}. \tag{4.1}$$

*Then* $\Sigma^{\mathbf{Я}} = (\mathbb{T}^{\mathbf{Я}}, \Phi^{\mathbf{Я}}, \Psi^{\mathbf{Я}}, \mathbb{F}^{\mathbf{Я}})$ *is a well-posed linear system. If* $x$ *and* $y$ *are the state trajectory and the output function of* $\Sigma$ *corresponding to the initial state* $x_0 \in X$ *and the input function* $u \in L^2_{\mathrm{loc}}([0,\infty);U)$*, then for all* $\tau \geq 0$,

$$\begin{bmatrix} x(0) \\ \mathbf{Я}_\tau y \end{bmatrix} = \Sigma^{\mathbf{Я}}_\tau \begin{bmatrix} x(\tau) \\ \mathbf{Я}_\tau u \end{bmatrix}. \tag{4.2}$$

Note that $x(0) = x_0$. The system $\Sigma^{\mathbf{Я}}$ introduced above is called the *time-inverted system* corresponding to $\Sigma$. We see from (4.1) that $\Sigma^{\mathbf{Я}}$ exists if and only if $\mathbb{T}$ can be extended to a group, and then $\mathbb{T}^{\mathbf{Я}}_\tau = \mathbb{T}_\tau^{-1}$. It is easy to verify that applying time-inversion twice, we get back the original system: $(\Sigma^{\mathbf{Я}})^{\mathbf{Я}} = \Sigma$.

*Proof of Theorem* 4.1. Let $x$ and $y$ be the state trajectory and the output function of $\Sigma$ corresponding to the initial state $x_0 \in X$ and the input function $u \in L^2_{\mathrm{loc}}([0,\infty);U)$. Then

$$\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau u \end{bmatrix} = \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ 0 & I \end{bmatrix} \begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \end{bmatrix}.$$

The operator matrix $\left[\begin{smallmatrix} \mathbb{T}_\tau & \Phi_\tau \\ 0 & I \end{smallmatrix}\right]$ is invertible whenever $\mathbb{T}_\tau$ is invertible. Therefore, we can express $\left[\begin{smallmatrix} x(0) \\ \mathbf{P}_\tau y \end{smallmatrix}\right]$ in terms of $\left[\begin{smallmatrix} x(\tau) \\ \mathbf{P}_\tau u \end{smallmatrix}\right]$ from (1.4) to get

$$\begin{bmatrix} x(0) \\ \mathbf{P}_\tau y \end{bmatrix} = \begin{bmatrix} I & 0 \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix} \begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \end{bmatrix} = \begin{bmatrix} I & 0 \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix} \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau u \end{bmatrix}. \tag{4.3}$$

Formula (4.2) is now an immediate consequence of (4.3).

If we denote $\Sigma^a_\tau = \left[\begin{smallmatrix} \mathbb{T}^a_\tau & \Phi^a_\tau \\ \Psi^a_\tau & \mathbb{F}^a_\tau \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ \Psi_\tau & \mathbb{F}_\tau \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathbb{T}_\tau & \Phi_\tau \\ 0 & I \end{smallmatrix}\right]^{-1}$ (this product appears in (4.1) and (4.3)) then, in order to show that $\Sigma^{\mathbf{Я}}$ is a well-posed linear system, it suffices to show that $\Sigma^a = (\mathbb{T}^a, \Phi^a, \Psi^a, \mathbb{F}^a)$ is an anti-causal well-posed linear system. It is well-known that $\mathbb{T}^{\mathbf{Я}}_t = \mathbb{T}_{-t}$ is strongly continuous, and clearly (3.2) is satisfied. Thus, by Proposition 3.3, it suffices to show that $\Sigma^a$ satisfies (3.3). To do this, we proceed as follows. Let $x$ and $y$ be the state trajectory and the output function of $\Sigma$ with inital state $x(0)$ and input function $u$. Using different special cases of (2.3) we find that

$$\begin{aligned}
\begin{bmatrix} x(0) \\ \mathbf{P}_\tau y \\ \mathbf{P}_t \mathbf{S}^*_\tau y \end{bmatrix} &= \begin{bmatrix} I & 0 & 0 \\ \Psi_\tau & \mathbb{F}_\tau & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \\ \mathbf{P}_t \mathbf{S}^*_\tau y \end{bmatrix}, \\
\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau u \\ \mathbf{P}_t \mathbf{S}^*_\tau y \end{bmatrix} &= \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \\ \mathbf{P}_t \mathbf{S}^*_\tau y \end{bmatrix}, \\
\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau u \\ \mathbf{P}_t \mathbf{S}^*_\tau y \end{bmatrix} &= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \Psi_t & 0 & \mathbb{F}_t \end{bmatrix} \begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau u \\ \mathbf{P}_t \mathbf{S}^*_\tau u \end{bmatrix}, \\
\begin{bmatrix} x(\tau+t) \\ \mathbf{P}_\tau u \\ \mathbf{P}_t \mathbf{S}^*_\tau u \end{bmatrix} &= \begin{bmatrix} \mathbb{T}_t & 0 & \Phi_t \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau u \\ \mathbf{P}_t \mathbf{S}^*_\tau u \end{bmatrix}.
\end{aligned} \tag{4.4}$$

We now interpret $\left[\begin{smallmatrix} x(\tau+t) \\ \mathbf{P}_{\tau+t} u \end{smallmatrix}\right]$ as the initial data, from which we express

$$\begin{bmatrix} x(\tau+t) \\ \mathbf{P}_\tau u \\ \mathbf{P}_t \mathbf{S}^*_\tau u \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathbf{P}_\tau \\ 0 & \mathbf{S}^*_\tau \end{bmatrix} \begin{bmatrix} x(\tau+t) \\ \mathbf{P}_{\tau+t} u \end{bmatrix},$$

and then we solve the equations (4.4) successively (working backwards) to obtain

$$\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau u \\ \mathbf{P}_t \mathbf{S}^*_\tau u \end{bmatrix}, \begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau u \\ \mathbf{P}_t \mathbf{S}^*_\tau y \end{bmatrix}, \begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \\ \mathbf{P}_t \mathbf{S}^*_\tau y \end{bmatrix}, \begin{bmatrix} x(0) \\ \mathbf{P}_\tau y \\ \mathbf{P}_t \mathbf{S}^*_\tau y \end{bmatrix}, \begin{bmatrix} x(0) \\ \mathbf{P}_{\tau+t} y \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & \mathbf{P}_\tau & \mathbf{S}_\tau \end{bmatrix} \begin{bmatrix} x(0) \\ \mathbf{P}_\tau y \\ \mathbf{P}_t \mathbf{S}^*_\tau y \end{bmatrix}.$$

Our expression for the last of these vectors, in terms of $\left[\begin{smallmatrix} x(\tau+t) \\ \mathbf{P}_{\tau+t}u \end{smallmatrix}\right]$, shows that we have

$$\begin{bmatrix} \mathbb{T}^a_{\tau+t} & \Phi^a_{\tau+t} \\ \Psi^a_{\tau+t} & \mathbb{F}^a_{\tau+t} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & \mathbf{P}_\tau & \mathbf{S}_\tau \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ \Psi_\tau & \mathbb{F}_\tau & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1}$$
$$\times \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \Psi_t & 0 & \mathbb{F}_t \end{bmatrix} \begin{bmatrix} \mathbb{T}_t & 0 & \Phi_t \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & \mathbf{P}_\tau \\ 0 & \mathbf{S}^*_\tau \end{bmatrix},$$

which reduces to (3.3) after simple matrix multiplications. □

**Remark 4.2.** With $\mathbb{T}^a$, $\Phi^a$, $\Psi^a$ and $\mathbb{F}^a$ defined as in the last proof, (3.3) gives us some additional information. If we change the notation, replacing $\tau$ by $t$ and $\tau+t$ by $\tau$, then we find that for all $0 \le t \le \tau$,

$$\begin{bmatrix} x(t) \\ \mathbf{P}_{\tau-t}\mathbf{Я}_\tau y \end{bmatrix} = \Sigma^{\mathbf{Я}}_{\tau-t} \begin{bmatrix} x(\tau) \\ \mathbf{P}_{\tau-t}\mathbf{Я}_\tau u \end{bmatrix}. \tag{4.5}$$

This formula shows that for all $t \in [0,\tau]$, $x(t)$ is the state of $\Sigma^{\mathbf{Я}}$ at time $\tau - t$ with initial state $x(\tau)$ and input function $\mathbf{Я}_\tau u$. The restriction of the corresponding output function to the interval $[0,\tau]$ is $\mathbf{Я}_\tau y$. In the smooth case described in Theorem 2.5 the output function of $\Sigma^{\mathbf{Я}}$ evaluated at $\tau - t$ is equal to $y(t)$.

**Theorem 4.3.** *The system $\Sigma$ is time-invertible if and only if $sI - A$ has a uniformly bounded inverse on some left half-plane (equivalently, $sI + A$ has a uniformly bounded inverse on some right half-plane). In this case, the growth bound of $\mathbb{T}^{\mathbf{Я}}$ is equal to the infimum of those $\omega \in \mathbb{R}$ for which $(sI + A)^{-1}$ is uniformly bounded on $\mathbb{C}_\omega$.*

*Proof.* Time-invertibility means that $\mathbb{T}$ is invertible, and the first claim in Proposition 4.3 follows from the recent result of Zwart [26] mentioned in the introduction. Indeed, the condition on $sI - A$ used in [26, Theorem 2.2] is even weaker, allowing $\|(sI - A)^{-1}\|$ to grow at a moderate rate, for example, like a polynomial in $|\mathrm{Re}\, s|$. The statement about the growth bound of $\mathbb{T}^{\mathbf{Я}}$ follows from a result in Prüss [11, Proposition 2] (see also Huang Falun [4]). □

We mention that results related to (but weaker than) those of Zwart [26] have been published a little earlier by Liu [7].

Our following theorem describes the system operator and the transfer function of the time-inverted system in terms of the original system operator and transfer function. For transfer functions, we use the convention at the end of Section 2.

**Theorem 4.4.** *With the assumption and the notation of Theorem 4.1, denote the semigroup generator of $\Sigma^{\mathbf{Я}}$ by $A^{\mathbf{Я}}$, its control operator by $B^{\mathbf{Я}}$, its observation operator by $C^{\mathbf{Я}}$ and its combined observation/feedthrough operator by $[C\&D]^{\mathbf{Я}}$. Let $S^{\mathbf{Я}}_\Sigma(s)$ be the system operator of $\Sigma^{\mathbf{Я}}$. Then we have*

$$S^{\mathbf{Я}}_\Sigma(0) = \begin{bmatrix} A^{\mathbf{Я}} & B^{\mathbf{Я}} \\ [C\&D]^{\mathbf{Я}} \end{bmatrix} = \begin{bmatrix} -A & -B \\ C\&D \end{bmatrix}, \tag{4.6}$$

*in particular, $A^{\mathbf{Я}} = -A$, $B^{\mathbf{Я}} = -B$, $C^{\mathbf{Я}} = C$ and $[C\&D]^{\mathbf{Я}} = C\&D$. The transfer function of $\Sigma^{\mathbf{Я}}$ is*

$$\mathbf{G}^{\mathbf{Я}}(s) = \mathbf{G}(-s), \qquad s \in \rho(A^{\mathbf{Я}}). \tag{4.7}$$

*Proof.* From (4.1) we see that

$$\mathbb{T}^{\mathbf{Я}}_\tau = \mathbb{T}_{-\tau}, \qquad \Phi^{\mathbf{Я}}_\tau = -\mathbb{T}_{-\tau}\Phi_\tau.$$

This implies (using (2.8)) that $A^{\mathbf{Я}} = -A$ and $B^{\mathbf{Я}} = -B$. Hence, the space $V$ for the system $\Sigma^{\mathbf{Я}}$ is the same as for the system $\Sigma$ ($V$ for $\Sigma$ was defined in (2.16)).

Let $x_0$, $u$, $x$, and $y$ be as in Theorem 4.1, but this time we require, in addition, that $u \in H^1_{\mathrm{loc}}([0,\infty);U)$ and $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in V$. Then by Theorem 2.5, (2.17) holds. By Remark 4.2, for $t \in [0,\tau]$,

$x(t)$ is the state and $y(t)$ is the output of $\Sigma^{\text{Я}}$ at time $\tau - t$ with initial state $x(\tau)$ and input function $\text{Я}_\tau u$. In particular, by Theorem 2.5, we have that $\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in V$ for all $t \in [0, \tau]$ and

$$\begin{bmatrix} -\dot{x}(t) \\ y(t) \end{bmatrix} = S_\Sigma^{\text{Я}}(0) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \qquad t \in [0, \tau].$$

In particular, taking $t = 0$ we get (4.6). The remaining assertions (concerning $[C\&D]^{\text{Я}}$, $C^{\text{Я}}$ and $\mathbf{G}^{\text{Я}}$) are easy consequences of (4.6). $\square$

If the semigroup $\mathbb{T}$ is invertible, then the spectrum $\sigma(A)$ is contained in a vertical strip $V_\sigma$. Let us denote by $\mathcal{D}^+$ the connected component of the resolvent set $\rho(A)$ which contains a right half-plane, and let us denote by $\mathcal{D}^-$ the connected component of $\rho(A)$ which contains a left half-plane. Thus, $\sigma(A)$ does not separate the half-plane to the right of $V_\sigma$ from the half-plane to the left of $V_\sigma$ if and only if $\mathcal{D}^+ = \mathcal{D}^-$. This is the case, for example, if $\sigma(A)$ is countable. If $\mathcal{D}^+ \neq \mathcal{D}^-$, then we cannot obtain $\mathbf{G}$ on $\mathcal{D}^-$ from $\mathbf{G}$ on $\mathcal{D}^+$ by analytic continuation. In this case, to compute $\mathbf{G}^{\text{Я}}$ on a right half-plane via (4.7) we must first evaluate $\mathbf{G}$ on a left half-plane via (2.18). Such an example is given in Section 8 (Example 8.3).

If $\Sigma$ is regular and time-invertible, it does not follow that $\Sigma^{\text{Я}}$ is regular, see Example 8.5. Even if $\Sigma^{\text{Я}}$ is regular, its feedthrough operator may be different from the feedthrough operator of $\Sigma$, see Example 8.2. Both examples mentioned above have a semigroup for which the resolvent set of $A$ is connected.

## 5. Flow-inversion

The idea behind *flow-inversion* is very similar to the idea behind time-inversion. We still keep the relationship between the data $x(\tau)$, $\mathbf{P}_\tau y$, $x(0)$, and $\mathbf{P}_\tau u$ in (1.4) intact, but this time we interpret $\left[\begin{smallmatrix} x(0) \\ \mathbf{P}_\tau y \end{smallmatrix}\right]$ as the initial data and $\left[\begin{smallmatrix} x(\tau) \\ \mathbf{P}_\tau u \end{smallmatrix}\right]$ as the final data. Taking $x(0) = 0$ in (1.4) we immediately observe that a necessary condition for the flow-inverted system to be well-posed is that $\mathbb{F}_\tau$ is invertible from $L^2([0, \tau]; U)$ to $L^2([0, \tau]; Y)$ for all $\tau > 0$. We shall see that this condition is also sufficient. We use the same standing assumptions and the same notation as in Section 4.

**Theorem 5.1.** *Suppose that $\mathbb{F}_\tau$ is invertible as an operator from $L^2([0, \tau]; U)$ to $L^2([0, \tau]; U)$ for some $\tau > 0$. Then $\mathbb{F}_\tau$ is invertible between these spaces for all $\tau \geq 0$ (note that $\mathbb{F}_0$ is invertible since both its domain and range spaces contain only the zero vector). Define $\Sigma_\tau^\times$ (for all $\tau \geq 0$) by*

$$\Sigma_\tau^\times = \begin{bmatrix} \mathbb{T}_\tau^\times & \Phi_\tau^\times \\ \Psi_\tau^\times & \mathbb{F}_\tau^\times \end{bmatrix} = \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix}^{-1}. \tag{5.1}$$

*We extend $\Phi_\tau^\times$ and $\mathbb{F}_\tau^\times$ to $L^2([0, \infty); U)$ by requiring $\Phi_\tau^\times = \Phi_\tau^\times \mathbf{P}_\tau$ and $\mathbb{F}_\tau^\times = \mathbb{F}_\tau^\times \mathbf{P}_\tau$. Then $\Sigma^\times$ is a well-posed linear system. If $x$ and $y$ are the state trajectory and the output function of $\Sigma$ corresponding to the initial state $x_0 \in X$ and the input function $u \in L^2_{\mathrm{loc}}([0, \infty); U)$ (so that $x(0) = x_0$), then for all $\tau \geq 0$,*

$$\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau u \end{bmatrix} = \Sigma_\tau^\times \begin{bmatrix} x(0) \\ \mathbf{P}_\tau y \end{bmatrix}. \tag{5.2}$$

The system $\Sigma^\times$ introduced above is called the *flow-inverted system* corresponding to $\Sigma$. We see from (5.1) that $\Sigma^\times$ exists if and only if $\mathbb{F}_\tau$ is invertible for all $\tau > 0$, and then $\mathbb{F}_\tau^\times = \mathbb{F}_\tau^{-1}$. It is easy to verify (from (5.2)) that applying time-inversion twice, we get back the original system: $(\Sigma^\times)^\times = \Sigma$.

*Proof.* We begin with the proof of the fact that $\mathbb{F}_\tau$ is invertible for all $\tau \geq 0$ if it is invertible for one $\tau_0 > 0$. Assume that $\mathbb{F}_{\tau_0}$ is invertible from $L^2([0, \tau_0]; U)$ to $L^2([0, \tau_0]; Y)$. First we show that $\mathbb{F}_t$ is invertible for all $t \in [0, \tau_0]$. Clearly $\mathbb{F}_t$ is onto for all such $t$, since $\mathbb{F}_t = \mathbf{P}_t \mathbb{F}_{\tau_0}$. To see that $\mathbb{F}_t$ is also one-to-one, assume that $\mathbb{F}_t u = 0$. Set $\tau = \tau_0 - t$. From (2.6) applied to $\mathbf{S}_\tau u$ we get $\mathbb{F}_{\tau_0} \mathbf{S}_\tau u = \mathbf{S}_\tau \mathbb{F}_t u = 0$, and now the invertibility of $\mathbb{F}_{\tau_0}$ implies that $\mathbf{P}_{\tau_0} \mathbf{S}_\tau u = \mathbf{S}_\tau \mathbf{P}_t u = 0$, which implies that $\mathbf{P}_t u = 0$. Thus, $\mathbb{F}_t$ is one-to-one on $L^2([0,t];U)$ for all $t \in (0,\tau_0]$. Being both one-to-one and onto, by the closed graph theorem, $\mathbb{F}_t$ has a bounded inverse which maps $L^2([0,t];Y)$ onto $L^2([0,t];U)$ (for all $t \in [0,\tau_0]$).

By using (2.6) with both $\tau$ and $t$ replaced by $\tau_0$, we find that $\mathbb{F}_{2\tau_0}$ is invertible (because a block lower triangular operator matrix of size $2\times 2$ is invertible if its two diagonal blocks are invertible). Repeating the same argument we find that $\mathbb{F}_\tau$ is invertible for all $\tau = 2^k\tau_0$, $k = 1,2,3,\ldots$. Combining this with our earlier finding, we conclude that $\mathbb{F}_\tau$ is invertible for all $\tau \geq 0$.

It is easy to see that the initial conditions (2.2) hold.

Next we show that $\Sigma^\times$ satisfies the algebraic conditions (2.3). Let $x$ and $y$ be the state trajectory and the output function of $\Sigma$ corresponding to the initial state $x_0 \in X$ and the input function $u \in L^2_{\mathrm{loc}}([0,\infty);U)$. Then

$$\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau u \end{bmatrix} = \begin{bmatrix} \mathbb{T}^\times_\tau & \Phi^\times_\tau \\ \Psi^\times_\tau & \mathbb{F}^\times_\tau \end{bmatrix} \begin{bmatrix} x(0) \\ \mathbf{P}_\tau y \end{bmatrix} = \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix}^{-1} \begin{bmatrix} x(0) \\ \mathbf{P}_\tau y \end{bmatrix}, \tag{5.3}$$

since $\left[\begin{smallmatrix} x(0) \\ \mathbf{P}_\tau y \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ \Psi_\tau & \mathbb{F}_\tau \end{smallmatrix}\right] \left[\begin{smallmatrix} x(0) \\ \mathbf{P}_\tau u \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} I & 0 \\ \Psi_\tau & \mathbb{F}_\tau \end{smallmatrix}\right]$ is invertible whenever $\mathbb{F}_\tau$ is invertible. We start with $\left[\begin{smallmatrix} x(0) \\ \mathbf{P}_{\tau+t} y \end{smallmatrix}\right]$ and use (4.4) to express successively

$$\begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \\ \mathbf{P}_t \mathbf{S}^*_\tau y \end{bmatrix}, \quad \begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau u \\ \mathbf{P}_t \mathbf{S}^*_\tau y \end{bmatrix}, \quad \begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau u \\ \mathbf{P}_t \mathbf{S}^*_\tau u \end{bmatrix}, \quad \begin{bmatrix} x(\tau+t) \\ \mathbf{P}_\tau u \\ \mathbf{P}_t \mathbf{S}^*_\tau u \end{bmatrix}, \quad \begin{bmatrix} x(\tau+t) \\ \mathbf{P}_{\tau+t} u \end{bmatrix},$$

similarly as in the proof of Theorem 4.1. Thus we get

$$\begin{aligned} \begin{bmatrix} \mathbb{T}^\times_{\tau+t} & \Phi^\times_{\tau+t} \\ \Psi^\times_{\tau+t} & \mathbb{F}^\times_{\tau+t} \end{bmatrix} &= \begin{bmatrix} I & 0 & 0 \\ 0 & \mathbf{P}_\tau & \mathbf{S}_\tau \end{bmatrix} \begin{bmatrix} \mathbb{T}_t & 0 & \Phi_t \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \Psi_t & 0 & \mathbb{F}_t \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ \Psi_\tau & \mathbb{F}_\tau & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & \mathbf{P}_\tau \\ 0 & \mathbf{S}^*_\tau \end{bmatrix} \\ &= \begin{bmatrix} I & 0 & 0 \\ 0 & \mathbf{P}_\tau & \mathbf{S}_\tau \end{bmatrix} \begin{bmatrix} \mathbb{T}^\times_t & 0 & \Phi^\times_t \\ 0 & I & 0 \\ \Psi^\times_t & 0 & \mathbb{F}^\times_t \end{bmatrix} \begin{bmatrix} \mathbb{T}^\times_\tau & \Phi^\times_\tau & 0 \\ \Psi^\times_\tau & \mathbb{F}^\times_\tau & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{P}_\tau \\ 0 & \mathbf{S}^*_\tau \end{bmatrix}, \end{aligned}$$

which is (2.3) applied to the system $\Sigma^\times$.

Let us check that $\mathbb{T}^\times_\tau$ is strongly continuous. Take $x_0 \in X$ and define for all $\tau > 0$,

$$\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau u \end{bmatrix} = \begin{bmatrix} \mathbb{T}^\times_\tau \\ \Psi^\times_\tau \end{bmatrix} x_0 .$$

Then it follows from (5.3) that $x(\tau) = \mathbb{T}_\tau x_0 + \Phi_\tau \mathbf{P}_\tau u$, i.e., $x$ is the state of $\Sigma$ at time $\tau$ with initial state $x_0$ and input $\mathbf{P}_\tau u$ (and the corresponding output of $\Sigma$ is zero). Since the state trajectory is a continuous function of time, we conclude that $x(\tau) \to x_0$ as $\tau \downarrow 0$. This proves that $\mathbb{T}^\times_\tau$ is strongly continuous. $\square$

Now we compute the system operator of the flow-inverted system.

**Theorem 5.2.** *With the assumption and the notation of Theorem 5.1, denote the semigroup generator of $\Sigma^\times$ by $A^\times$, its control operator by $B^\times$, its observation operator by $C^\times$, its combined observation/feedthrough operator by $[C\&D]^\times$ and its system operator by $S^\times_\Sigma(s)$. The operators $[C\&D]^\times$ and $S^\times_\Sigma(s)$ have the same domain $V^\times$, which is the analogue of $V$ from (2.16) for the system $\Sigma^\times$. Then the operator $\left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right]$ maps $V$ continuously onto $V^\times$, with inverse*

$$\begin{bmatrix} I & 0 \\ C\&D \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ [C\&D]^\times \end{bmatrix}, \tag{5.4}$$

*and*

$$S_\Sigma^\times(0) = \begin{bmatrix} A^\times & B^\times \\ [C\&D]^\times \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C\&D \end{bmatrix}^{-1}. \tag{5.5}$$

*In particular,* $A^\times = A + BC^\times$ *on* $X_1^\times$.

Note that, since $(\Sigma^\times)^\times = \Sigma$, it follows from this theorem that $A = A^\times + B^\times C$ on $X_1$ and that

$$S_\Sigma(0) = \begin{bmatrix} A & B \\ C\&D \end{bmatrix} = \begin{bmatrix} A^\times & B^\times \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ [C\&D]^\times \end{bmatrix}^{-1}. \tag{5.6}$$

*Proof.* Let $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in V$, i.e., $x_0 \in X$, $u_0 \in U$, and $Ax_0 + Bu_0 \in X$. Define

$$y_0 = C\&D \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}.$$

Choose an arbitrary $u \in H^1_{\mathrm{loc}}([0,\infty); U)$ with $u(0) = u_0$. Let $x$ and $y$ be the state trajectory and the output function of $\Sigma$ with initial state $x_0$ and input function $u$. Then, by Theorem 2.5(i), $x \in C^1([0,\infty); X)$, $y \in H^1_{\mathrm{loc}}([0,\infty); Y)$ and for all $t \geq 0$,

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S_\Sigma(0) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \qquad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ C\&D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \tag{5.7}$$

In particular, $y(0) = y_0$. On the other hand, we can also consider the system $\Sigma^\times$ with initial state $x_0$ and input function $y$. By Theorem 5.1, the state trajectory and the output function of this system are $x$ and $u$, where $x$ and $u$ are the same functions as above. The fact that $x$ is continuously differentiable implies, in particular, that $\dot{x}(0) = A^\times x_0 + B^\times y_0 \in X$, therefore $\left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right] \in V^\times$ and for all $t \geq 0$,

$$\begin{bmatrix} \dot{x}(t) \\ u(t) \end{bmatrix} = S_\Sigma^\times(0) \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ [C\&D]^\times \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \tag{5.8}$$

In particular, taking $t = 0$ in (5.7) and (5.8) we find that $\left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right]$ maps $V$ into $V^\times$, that it has a left inverse $\left[\begin{smallmatrix} I & 0 \\ [C\&D]^\times \end{smallmatrix}\right]$, and that

$$S_\Sigma(0) \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} \dot{x}(0) \\ y_0 \end{bmatrix} = \begin{bmatrix} A^\times & B^\times \\ 0 & I \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} A^\times & B^\times \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C\&D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}.$$

Thus, $S_\Sigma(0) = \left[\begin{smallmatrix} A^\times & B^\times \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right]$. By interchanging the roles of $\Sigma$ and $\Sigma^\times$ we find that $\left[\begin{smallmatrix} I & 0 \\ [C\&D]^\times \end{smallmatrix}\right]$ is also a right inverse of $\left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right]$ and that $S_\Sigma^\times(0) = \left[\begin{smallmatrix} A & B \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} I & 0 \\ [C\&D]^\times \end{smallmatrix}\right]$. This implies both (5.4) and (5.5).

Finally, since $X_1^\times \times \{0\}$ is included in $V^\times$ and since $[C\&D]^\times \left[\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right] = C^\times z$ for all $z \in X_1^\times$, the formulas (5.4) and (5.5) imply that $A^\times z = Az + BC^\times z$. $\square$

From Theorem 5.2 it is easy to derive explicit formulas for the system operator and the transfer function of the flow-inverted system.

**Corollary 5.3.** *Suppose that* $\Sigma$ *is flow-invertible. We use the notation from Theorem* 5.2 *and we denote by* $\mathbf{G}^\times$ *the transfer function of* $\Sigma^\times$. *Then for all* $s \in \rho(A^\times)$, $S_\Sigma(s)$ *is boundedly invertible and*

$$S_\Sigma^{-1}(s) = \begin{bmatrix} -(sI - A^\times)^{-1} & (sI - A^\times)^{-1} B^\times \\ -C^\times (sI - A^\times)^{-1} & \mathbf{G}^\times(s) \end{bmatrix}. \tag{5.9}$$

*If, in addition,* $s \in \rho(A)$, *then also* $\mathbf{G}(s)$ *is invertible and*

$$S_\Sigma^{-1}(s) = \begin{bmatrix} -(sI - A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} (sI - A)^{-1} B \\ I \end{bmatrix} \mathbf{G}^{-1}(s) \begin{bmatrix} C(sI - A)^{-1} & I \end{bmatrix}. \tag{5.10}$$

*In particular,* $U$ *and* $Y$ *are isomorphic and*

$$\mathbf{G}^\times(s) = \mathbf{G}^{-1}(s), \qquad s \in \rho(A) \cap \rho(A^\times). \tag{5.11}$$

*Moreover, for all $s \in \rho(A) \cap \rho(A^\times)$, the operator $(sI - A^\times)^{-1}(sI - A)$ which maps $\mathcal{D}(A)$ onto $\mathcal{D}(A^\times)$ satisfies*

$$\begin{aligned}(sI - A^\times)^{-1}(sI - A) &= I - (sI - A)^{-1}B\mathbf{G}^{-1}(s)C \\ &= I - (sI - A^\times)^{-1}B^\times C,\end{aligned}$$

*and its inverse $(sI - A)^{-1}(sI - A^\times)$ satisfies*

$$\begin{aligned}(sI - A)^{-1}(sI - A^\times) &= I - (sI - A^\times)^{-1}B^\times \mathbf{G}(s)C^\times \\ &= I - (sI - A)^{-1}BC^\times.\end{aligned}$$

*Proof.* It follows from (5.6) that for all $s \in \mathbb{C}$,

$$S_\Sigma(s) = \begin{bmatrix} A^\times - sI & B^\times \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ [C\&D]^\times \end{bmatrix}^{-1}.$$

For $s \in \rho(A^\times)$, the right-hand side is invertible as an operator from $V$ to $X \times Y$, hence so is the left-hand side, and inverting both sides we get (5.9). Formula (5.10) can be derived from (5.9) combined with the factorization

$$S_\Sigma^{-1}(s) = \begin{bmatrix} I & (sI - A)^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} -(sI - A)^{-1} & 0 \\ 0 & \mathbf{G}^{-1}(s) \end{bmatrix} \begin{bmatrix} I & 0 \\ C(sI - A)^{-1} & I \end{bmatrix},$$

valid for $s \in \rho(A)$, which follows from Proposition II.3.6. In the next three formulas we have simply written out some of the identities that we get by combining (5.9) and (5.10). The fact that $U$ and $Y$ must be isomorphic follows from the fact that for certain $s$, $\mathbf{G}(s)$ is a boundedly invertible mapping of $U$ onto $Y$. □

From Corollary 5.3 we get the following interesting criterion for flow-invertibility and an expression for the growth bound of the semigroup of the flow-inverted system.

**Theorem 5.4.** *The system $\Sigma$ is flow-invertible if and only if $S_\Sigma$ has a uniformly bounded inverse on some right half-plane, or equivalently, if and only if $\mathbf{G}$ has a uniformly bounded inverse on some right half-plane. In this case, the growth bound of $\mathbb{T}^\times$ is the infimum of those $\omega \in \mathbb{R}$ for which $S_\Sigma^{-1}$ is uniformly bounded on $\mathbb{C}_\omega$.*

*Proof.* First we prove that if $\Sigma$ is flow-invertible, then $S_\Sigma^{-1}$ is uniformly bounded on $\mathbb{C}_\omega$ for any $\omega > \omega_{\mathbb{T}}^\times$, the growth bound of $\mathbb{T}^\times$. Indeed, by the standard Hille–Yoshida estimates, $\|(sI - A^\times)^{-1}\|$ is uniformly bounded on $\mathbb{C}_\omega$. By [21, Proposition 2.3] (or [18, Proposition 4.2.8]) with $\Sigma$ replaced by $\Sigma^\times$, $\|(sI - A^\times)^{-1}B^\times\|$ in uniformly bounded on $\mathbb{C}_\omega$, and by applying this result to the dual system we find that also $\|C^\times(sI - A^\times)^{-1}\|$ is uniformly bounded on $\mathbb{C}_\omega$. Finally, by Proposition I.4.1 (or Theorem II.2.7(3)) applied to $\Sigma^\times$, $\mathbf{G}^\times(s)$ is uniformly bounded on $\mathbb{C}_\omega$. Thus, by (5.9), $S_\Sigma^{-1}$ is uniformly bounded on $\mathbb{C}_\omega$ for any $\omega > \omega_{\mathbb{T}}^\times$.

Now we prove the equality stated in the last sentence of the theorem. What we have proved so far means that $\omega_0 \le \omega_{\mathbb{T}}^\times$, where

$$\omega_0 = \inf\{\omega \in \mathbb{R} \mid S_\Sigma^{-1} \text{ is uniformly bounded on } \mathbb{C}_\omega\}.$$

On the other hand, by the left upper corner of (5.9) and by the well-known result of Prüss [11, Proposition 2] (see also Huang [4]), $\omega_{\mathbb{T}}^\times \le \omega_0$, so that $\omega_{\mathbb{T}}^\times = \omega_0$.

Next we prove that if $S_\Sigma^{-1}$ is uniformly bounded on $\mathbb{C}_\omega$ for some $\omega \in \mathbb{R}$, then $\mathbf{G}^{-1}$ exists and it is uniformly bounded on some right half-plane. Without loss of generality, take $\omega > \omega_{\mathbb{T}}$. Then it follows from the first factorization in Proposition II.3.6 that $\mathbf{G}^{-1}$ exists and it is uniformly bounded on $\mathbb{C}_\omega$.

Finally, we show that if $\mathbf{G}^{-1}$ exists and it is uniformly bounded on $\mathbb{C}_\omega$ for some $\omega \in \mathbb{R}$, then $\Sigma$ is flow-invertible. By the converse part of Theorem I.3.6, $\mathbf{G}^{-1}$ determines a shift-invariant and continuous operator $\mathbb{F}_\infty^\times$ from $L^2_{\text{loc}}([0,\infty);Y)$ to $L^2_{\text{loc}}([0,\infty);U)$, and of course $\mathbb{F}_\infty^\times = (\mathbb{F}_\infty)^{-1}$. Since shift-invariant operators are causal, it follows that for every $\tau > 0$, $\mathbb{F}_\tau^\times = \mathbf{P}_\tau \mathbb{F}_\infty^\times$ is the inverse of $\mathbb{F}_\tau$ as an operator from $L^2([0,\tau];U)$ to $L^2([0,\tau];Y)$. Thus, $\Sigma$ is flow-invertible. □

**Remark 5.5.** It is important to observe that the concept of flow-inversion can be reduced to the concept of static output feedback and conversely. (For a detailed treatment of output feedback we refer the reader to [23].) To see that flow-inversion can be regarded as a special case of output feedback, we argue as follows. By Corollary 5.3, a necessary condition for flow-invertibility is that there exists a boundedly invertible operator $E$ mapping $Y$ onto $U$. Assume that this is the case. We then extend the original system $\Sigma$ by adding another input signal $z \in L^2_{\mathrm{loc}}([0,\infty);Y)$ so that the new input signal is $\left[\begin{smallmatrix} z \\ u \end{smallmatrix}\right]$, and we also replace the output signal by $\left[\begin{smallmatrix} z-y \\ -Ez+u \end{smallmatrix}\right]$, where $y = \Psi_\infty x_0 + \mathbb{F}_\infty u$ is the original output signal of $\Sigma$. Thus, the extended system $\Sigma_E$ has equal input and output spaces, namely $Y \times U$. The state space remains the same, the new control operator is $\begin{bmatrix} 0 & B \end{bmatrix}$, the new observation operator is $\left[\begin{smallmatrix} -C \\ 0 \end{smallmatrix}\right]$, the new transfer function is $\left[\begin{smallmatrix} I & -\mathbf{G} \\ -E & I \end{smallmatrix}\right]$, and the state trajectories of $\Sigma_E$ are the same as those for $\Sigma$. By some simple algebra, $\Sigma$ is flow-invertible if and only if $\left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right]$ is an admissible feedback operator for $\Sigma_E$, and we get the flow-inverted system $\Sigma^\times$ from the closed-loop version of $\Sigma_E$ by dropping its second input and first output.

To see that output feedback can be regarded as a special case of flow-inversion, we argue in a similar way. Let $K \in \mathcal{L}(Y;U)$. We extend the original system $\Sigma$ by adding the same input as earlier, and replacing the output signal by $\left[\begin{smallmatrix} z+y \\ -Kz-u \end{smallmatrix}\right]$, where $y = \Psi_\infty x_0 + \mathbb{F}_\infty u$ is the original output signal of $\Sigma$. Again the extended system $\Sigma_K$ has equal input and output spaces, namely $Y \times U$. The state space remains the same, the new control operator is $\begin{bmatrix} 0 & B \end{bmatrix}$, the new observation operator is $\left[\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right]$, the new transfer function is $\left[\begin{smallmatrix} I & \mathbf{G} \\ -K & -I \end{smallmatrix}\right]$, and the state trajectories of $\Sigma_K$ are the same as for $\Sigma$. By some simple algebra, $K$ is an admissible feedback operator for $\Sigma$ if and only if $\Sigma_K$ is flow-invertible, and we get the closed-loop system $\Sigma^K$ from the flow-inverted system $\Sigma_K^\times$ by dropping the first input and the second output.

Thus, it is possible to rederive many of the results given in [23] (those which do not use regularity) from the results presented in this section. (The proofs presented here are significantly shorter than those in [23].) Conversely, many of the results presented here (with the exception of Theorems 5.1 and 5.2) could have been obtained from the corresponding feedback results in [23].

By using the last remark, we can reduce the flow-inversion of regular linear systems to a certain kind of output feedback for such systems.

**Proposition 5.6.** *With the assumptions and the notation of Corollary* 5.3*, assume that $\Sigma$ is regular and let $D$ be its feedthrough operator (see* (2.13)*). Then $\Sigma^\times$ is regular if and only if $D$ is invertible. If this is the case then, denoting the feedthrough operator of $\Sigma^\times$ by $D^\times$, we have for all $x \in \mathcal{D}(A^\times)$*

$$A^\times x = Ax - BD^{-1}C_\Lambda x, \qquad C^\times x = -D^{-1}C_\Lambda x,$$

*and for all $v \in Y$*

$$B^\times v = BD^{-1}v, \qquad D^\times v = D^{-1}v.$$

*Proof.* Consider the system $\Sigma_E$ obtained from $\Sigma$ by adding another input and output signal, as described in the first part of Remark 5.5. If $\Sigma$ is regular and its generating operators are $A, B, C$ and $D$, then $\Sigma_E$ is also regular, with generating operators $A_E = A$, $B_E = \begin{bmatrix} 0 & B \end{bmatrix}$, $C_E = \left[\begin{smallmatrix} -C \\ 0 \end{smallmatrix}\right]$, and $D_E = \left[\begin{smallmatrix} I & -D \\ -E & I \end{smallmatrix}\right]$. As noted in Remark 5.5, the flow-invertibility of $\Sigma$ implies that $K = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right]$ is an admissible feedback operator for $\Sigma_E$. According to [23, Theorem 4.7], $\Sigma_E^K$ is regular if and only if $\left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] - D_E$ is invertible, or equivalently, if and only if $D$ is invertible. If this is the case, then according to the theory in [23, Section 7], the generating operators of $\Sigma_E^K$ are

$$A_E^K = A + \begin{bmatrix} 0 & B \end{bmatrix}\left[\begin{smallmatrix} 0 & D \\ E & 0 \end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix} -C_\Lambda \\ 0 \end{smallmatrix}\right], \qquad B_E^K = \begin{bmatrix} 0 & B \end{bmatrix}\left[\begin{smallmatrix} 0 & D \\ E & 0 \end{smallmatrix}\right]^{-1},$$

$$C_E^K = \left[\begin{smallmatrix} 0 & D \\ E & 0 \end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix} -C_\Lambda \\ 0 \end{smallmatrix}\right], \qquad D_E^K = \left[\begin{smallmatrix} I & -D \\ -E & I \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & D \\ E & 0 \end{smallmatrix}\right]^{-1}.$$

From here we obtain the generating operators of $\Sigma^\times$ as stated in the proposition by dropping the second input and the first output. □

**Remark 5.7.** We refer the reader to [23] for an explanation of how the identity $B^\times = BD^{-1}$ in Proposition 5.6 should be interpreted. (In principle, the range of $B$ is a subspace of $X_{-1}$ whereas the range of $B^\times$ is a subspace of $X_{-1}^\times$, but there is a subspace of $X_{-1} \cap X_{-1}^\times$ which contains both of these ranges.) The formula $A^\times x = Ax - BD^{-1}C_\Lambda x$ in the last proposition actually holds for all $x$ in a much larger space than $\mathcal{D}(A^\times)$, as a consequence of [23, Proposition 7.10]. This larger space includes a space known as $\mathcal{D}(C_L)$ (defined in Parts I and II), which is such that $\mathcal{D}(A) \subset \mathcal{D}(C_L) \subset \mathcal{D}(C_\Lambda)$. Another noteworthy fact is that we have $C_\Lambda^\times x = -D^{-1}C_\Lambda x$ for all $x \in \mathcal{D}(C_\Lambda)$, as a consequence of [23, Proposition 7.1].

We do not know any example of a flow-invertible regular linear system whose flow-inverse is not regular. Maybe such a system does not exist.

## 6. Time-flow-inversion

As explained in the introduction, in the time-flow-inverted system we still let the relationship between $x(0)$, $\mathbf{P}_\tau u$, $x(\tau)$, and $\mathbf{P}_\tau y$ be the same as in (1.4), but this time we interpret $\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \end{bmatrix}$ as the initial data and $\begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \end{bmatrix}$ as the final data. We use the standing assumptions and the notation from Section 4.

**Theorem 6.1.** *Suppose that $\Sigma_\tau$ is invertible as an operator from $X \times L^2([0,\tau];U)$ to $X \times L^2([0,\tau];Y)$ for some $\tau > 0$. Then $\Sigma_\tau$ is invertible between these spaces for all $\tau \geq 0$ (note that $\Sigma_0$ is the identity on $X \times \{0\}$.) Define $\Sigma_\tau^\leftarrow$ (for all $\tau \geq 0$) by*

$$\Sigma_\tau^\leftarrow = \begin{bmatrix} \mathbb{T}_\tau^\leftarrow & \Phi_\tau^\leftarrow \\ \Psi_\tau^\leftarrow & \mathbb{F}_\tau^\leftarrow \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \boldsymbol{\mathcal{A}}_\tau \end{bmatrix} \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & \boldsymbol{\mathcal{A}}_\tau \end{bmatrix}. \tag{6.1}$$

*Then $\Sigma^\leftarrow$ is a well-posed linear system. If $x$ and $y$ are the state trajectory and the output function of $\Sigma$ corresponding to the initial state $x_0 \in X$ and the input function $u \in L^2_{\mathrm{loc}}([0,\infty);U)$, then for all $\tau \geq 0$,*

$$\begin{bmatrix} x(0) \\ \boldsymbol{\mathcal{A}}_\tau u \end{bmatrix} = \Sigma_\tau^\leftarrow \begin{bmatrix} x(\tau) \\ \boldsymbol{\mathcal{A}}_\tau y \end{bmatrix}. \tag{6.2}$$

The system $\Sigma^\leftarrow$ introduced above is called the *time-flow-inverted system* corresponding to $\Sigma$. Clearly $\Sigma^\leftarrow$ exists if and only if $\Sigma_\tau$ is invertible for some (hence, by the preceding theorem, for all) $\tau > 0$, and then $(\Sigma^\leftarrow)^\leftarrow = \Sigma$.

*Proof.* We begin with the proof of the fact that $\Sigma_\tau$ is invertible for all $\tau \geq 0$ if it is invertible for one $\tau_0 > 0$. Assume that $\Sigma_{\tau_0}$ is invertible from $X \times L^2([0,\tau_0];U)$ to $X \times L^2([0,\tau_0];Y)$. Take some nonnegative $\tau$ and $t$ such that $\tau + t = \tau_0$. Then it follows from (2.3) that $\Sigma_t$ is onto and that $\Sigma_\tau$ is one-to-one. This being true for all $\tau$ and $t$ with $\tau + t = \tau_0$, we find that $\Sigma_\tau$ is invertible for all $\tau \leq \tau_0$. To remove the condition $\tau \leq \tau_0$, it suffices to observe that by (2.3) with $\tau = t = \tau_0$, $\Sigma_{2\tau_0}$ is invertible, hence $\Sigma_{4\tau_0}$ is invertible, etc. Clearly (6.2) follows from (1.4) and (6.1).

Now let us show that $\Sigma^\leftarrow$ is a well-posed linear system. To do this it suffices to show that the system $\Sigma^a$ determined by $\Sigma_\tau^a = \Sigma_\tau^{-1}$ for all $\tau \geq 0$ is an anti-causal well-posed linear system, as defined in Section 3. This follows from (2.3), because by inverting the right-hand side of (2.3) we get (3.3), with $\mathbb{T}^a$, $\Phi^a$, $\Psi^a$ and $\mathbb{F}^a$ being the components of $\Sigma^a$. Formula (2.3) (with $\tau$ replaced by $t$ and $\tau + t$ replaced by $\tau$) and Remark 2.3 imply that for all $0 \leq t \leq \tau$,

$$\begin{bmatrix} x(t) \\ \mathbf{P}_{\tau-t}\boldsymbol{\mathcal{A}}_\tau u \end{bmatrix} = \Sigma_{\tau-t}^\leftarrow \begin{bmatrix} x(\tau) \\ \mathbf{P}_{\tau-t}\boldsymbol{\mathcal{A}}_\tau y \end{bmatrix}, \qquad t \in [0,\tau]. \tag{6.3}$$

This formula shows that for all $t \in [0,\tau]$, $x(t)$ is the state of $\Sigma^{\boldsymbol{\mathcal{A}}}$ at time $\tau - t$ with initial state $x(\tau)$ and input function $\boldsymbol{\mathcal{A}}_\tau y$.

Now we check that $\mathbb{T}_\tau^\leftarrow$ is strongly continuous. Take $x_1 \in X$, let

$$\begin{bmatrix} x(0) \\ \boldsymbol{\mathcal{A}}_1 u \end{bmatrix} = \begin{bmatrix} \mathbb{T}_1^\leftarrow \\ \Psi_1^\leftarrow \end{bmatrix} x_1$$

and, for all $t \in [0,1]$, let $x(t)$ be the state of $\Sigma$ at time $t$ with initial state $x(0)$ and input $u$ (the corresponding output $y$ is then zero). Then $x(t)$ is a continuous function of $t$. On the other hand, by (6.3), $x(t) = \mathbb{T}^{\leftarrow}_{1-t}x_1$. Thus $\mathbb{T}^{\leftarrow}_{1-t}x_1 \to x_1$ as $t \to 1$, and this shows that $\mathbb{T}^{\leftarrow}$ is strongly continuous. $\square$

We have seen in Section 3 that the duality transformation has a simple interpretation in terms of the Lax–Phillips semigroup induced by the system. Now we show that a similar interpretation holds for time-flow-inversion.

**Proposition 6.2.** *We use the assumptions and the notation of Theorem* 6.1. *Let* $\omega \in \mathbb{R}$, *and let* $\mathfrak{T}$ *be the Lax–Phillips semigroup of index* $\omega$ *induced by* $\Sigma$. *Then* $\Sigma$ *is time-flow-invertible if and only if* $\mathfrak{T}$ *is invertible, in which case the Lax–Phillips semigroup of index* $-\omega$ *induced by* $\Sigma^{\leftarrow}$ *is given by*

$$\mathfrak{T}^{\leftarrow}_\tau = \begin{bmatrix} 0 & 0 & \boldsymbol{\mathfrak{R}} \\ 0 & I & 0 \\ \boldsymbol{\mathfrak{R}} & 0 & 0 \end{bmatrix} \mathfrak{T}^{-1}_\tau \begin{bmatrix} 0 & 0 & \boldsymbol{\mathfrak{R}} \\ 0 & I & 0 \\ \boldsymbol{\mathfrak{R}} & 0 & 0 \end{bmatrix}, \qquad \tau \geq 0. \tag{6.4}$$

*Proof.* Since for any real $\alpha$, $\boldsymbol{\mathfrak{R}}$ maps $L^2_\alpha((-\infty,0];Y)$ onto $L^2_{-\alpha}([0,\infty);Y)$ and also $L^2_\alpha([0,\infty);U)$ onto $L^2_{-\alpha}((-\infty,0];U)$, the operator $\mathfrak{T}^{\leftarrow}_\tau$ defined in (6.4) is bounded on the space $L^2_{-\omega}((-\infty,0];U) \times X \times L^2_{-\omega}((-\infty,0];Y)$.

Let $\left[\begin{smallmatrix} y_0 \\ x_0 \\ u_0 \end{smallmatrix}\right] \in L^2_\omega((-\infty,0];Y) \times X \times L^2_\omega([0,\infty);U)$, let $\tau > 0$ and let $\left[\begin{smallmatrix} y_\tau \\ x_\tau \\ u_\tau \end{smallmatrix}\right] = \mathfrak{T}_\tau \left[\begin{smallmatrix} y_0 \\ x_0 \\ u_0 \end{smallmatrix}\right]$. This means explicitly (see (II.6.1)) that

$$\begin{aligned} \mathbf{P}_{(-\infty,-\tau]}y_\tau &= \mathcal{S}_{-\tau}y_0, \\ \begin{bmatrix} x_\tau \\ \mathbf{P}_\tau \mathcal{S}_\tau y_\tau \end{bmatrix} &= \Sigma_\tau \begin{bmatrix} x_0 \\ \mathbf{P}_\tau u_0 \end{bmatrix}, \\ u_\tau &= \mathcal{S}_{-\tau}\mathbf{P}_{[\tau,\infty)}u_0. \end{aligned}$$

The first and last of the three equations above define invertible mappings from $L^2((-\infty,0];Y)$ onto $L^2((-\infty,-\tau];Y)$ and from $L^2([\tau,\infty);U)$ onto $L^2([0,\infty);U)$, so that $\mathfrak{T}$ is invertible if and only if $\Sigma$ is time-flow-invertible. In this case we get

$$\begin{aligned} y_0 &= \mathcal{S}_\tau \mathbf{P}_{(-\infty,-\tau]}y_\tau, \\ \begin{bmatrix} x_0 \\ \boldsymbol{\mathfrak{R}}_\tau \mathbf{P}_\tau u_0 \end{bmatrix} &= \Sigma^{\leftarrow}_\tau \begin{bmatrix} x_\tau \\ \boldsymbol{\mathfrak{R}}_\tau \mathbf{P}_\tau \mathcal{S}_\tau y_\tau \end{bmatrix}, \\ \mathbf{P}_{[\tau,\infty)}u_0 &= \mathcal{S}_\tau u_\tau, \end{aligned}$$

or equivalently,

$$\begin{aligned} \mathcal{S}_{-\tau}\boldsymbol{\mathfrak{R}}u_\tau &= \mathbf{P}_{(-\infty,-\tau]}\boldsymbol{\mathfrak{R}}u_0, \\ \Sigma^{\leftarrow}_\tau \begin{bmatrix} x_\tau \\ \mathbf{P}_\tau \boldsymbol{\mathfrak{R}} y_\tau \end{bmatrix} &= \begin{bmatrix} x_0 \\ \mathcal{S}_\tau \mathbf{P}_{[-\tau,0]}\boldsymbol{\mathfrak{R}}u_0 \end{bmatrix}, \\ \mathcal{S}_{-\tau}\mathbf{P}_{[\tau,\infty)}\boldsymbol{\mathfrak{R}}y_\tau &= \boldsymbol{\mathfrak{R}}y_0. \end{aligned}$$

This is the same as what we get by applying both sides of (6.4) to $\left[\begin{smallmatrix} \boldsymbol{\mathfrak{R}}u_\tau \\ x_\tau \\ \boldsymbol{\mathfrak{R}}y_\tau \end{smallmatrix}\right]$. $\square$

Next we investigate the system operator of the time-flow-inverted system.

**Theorem 6.3.** *With the assumption and the notation of Theorem* 6.1, *denote the semigroup generator of* $\Sigma^{\leftarrow}$ *by* $A^{\leftarrow}$, *its control operator by* $B^{\leftarrow}$, *its observation operator by* $C^{\leftarrow}$, *its combined observation/feedthrough operator by* $[C\&D]^{\leftarrow}$ *and its system operator by* $S^{\leftarrow}_\Sigma(s)$. *The operators* $[C\&D]^{\leftarrow}$ *and* $S^{\leftarrow}_\Sigma(s)$ *have the same domain* $V^{\leftarrow}$, *which is the analogue of* $V$ *from* (2.16) *for the system* $\Sigma^{\leftarrow}$. *Then the operator* $\left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right]$ *maps* $V$ *continuously onto* $V^{\leftarrow}$, *with inverse*

$$\begin{bmatrix} I & 0 \\ \multicolumn{2}{c}{C\&D} \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ \multicolumn{2}{c}{[C\&D]^{\leftarrow}} \end{bmatrix}, \tag{6.5}$$

*and*

$$S^{\leftarrow}_\Sigma(0) = \begin{bmatrix} A^{\leftarrow} & B^{\leftarrow} \\ \multicolumn{2}{c}{[C\&D]^{\leftarrow}} \end{bmatrix} = \begin{bmatrix} -A & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \multicolumn{2}{c}{C\&D} \end{bmatrix}^{-1}. \tag{6.6}$$

*In particular,* $A^{\leftarrow} = -A - BC^{\leftarrow}$ *on* $X^{\leftarrow}_1$.

Note that, since $(\Sigma^{\leftarrow})^{\leftarrow} = \Sigma$, it follows from this theorem that $A = -A^{\leftarrow} - B^{\leftarrow}C$ on $X_1$ and that

$$S_\Sigma(0) = \begin{bmatrix} A & B \\ C\&D \end{bmatrix} = \begin{bmatrix} -A^{\leftarrow} & -B^{\leftarrow} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ [C\&D]^{\leftarrow} \end{bmatrix}^{-1}. \tag{6.7}$$

*Proof.* We begin the proof exactly in the same way as the proof of Theorem 5.2, up to formula (5.7). We then fix some $\tau > 0$. By (6.3), on the interval $[0, \tau]$, the function $t \mapsto x(\tau - t)$ is the state and $\boldsymbol{Я}_\tau u$ is the output function of $\Sigma^{\leftarrow}$ with initial state $x(\tau)$ and input function $\boldsymbol{Я}_\tau y$. The fact that $x$ is continuously differentiable implies (as in the proof of Theorem 5.2) that $\left[\begin{smallmatrix} x(\tau) \\ y(\tau) \end{smallmatrix}\right] \in V^{\leftarrow}$ and that for all $t \in [0, \tau]$,

$$\begin{bmatrix} -\dot{x}(t) \\ u(t) \end{bmatrix} = S_\Sigma^{\leftarrow}(0) \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ [C\&D]^{\leftarrow} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \tag{6.8}$$

In particular, taking $t = 0$ in (5.7) and (6.8) we find that $\left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right]$ maps $V$ into $V^{\leftarrow}$, that it has a left inverse $\left[\begin{smallmatrix} I & 0 \\ [C\&D]^{\leftarrow} \end{smallmatrix}\right]$, and that

$$S_\Sigma(0) \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} \dot{x}(0) \\ y_0 \end{bmatrix} = \begin{bmatrix} -A^{\leftarrow} & -B^{\leftarrow} \\ 0 & I \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -A^{\leftarrow} & -B^{\leftarrow} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C\&D \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

Thus, $S_\Sigma(0) = \left[\begin{smallmatrix} -A^{\leftarrow} & -B^{\leftarrow} \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right]$. By interchanging the roles of $\Sigma$ and $\Sigma^{\leftarrow}$ we find that $\left[\begin{smallmatrix} I & 0 \\ [C\&D]^{\leftarrow} \end{smallmatrix}\right]$ is also a right inverse of $\left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right]$ and that $S_\Sigma^{\leftarrow}(0) = \left[\begin{smallmatrix} -A & -B \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} I & 0 \\ [C\&D]^{\leftarrow} \end{smallmatrix}\right]$. This implies both (6.5) and (6.6).

Finally, since $X_1^{\leftarrow} \times \{0\}$ is included in $V^{\leftarrow}$ and since $[C\&D]^{\leftarrow} \left[\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right] = C^{\leftarrow} z$ for all $z \in X_1^{\leftarrow}$, the formulas (6.5) and (6.6) imply that $A^{\leftarrow} z = -Az - BC^{\leftarrow} z$. ☐

**Corollary 6.4.** *Suppose that $\Sigma$ is time-flow-invertible. We use the notation from Theorem 6.1. Then for all $s \in \rho(A^{\leftarrow})$, $S_\Sigma(-s) = \left[\begin{smallmatrix} sI+A & B \\ C\&D \end{smallmatrix}\right]$ is invertible and*

$$S_\Sigma^{-1}(-s) = \begin{bmatrix} (sI - A^{\leftarrow})^{-1} & (sI - A^{\leftarrow})^{-1} B^{\leftarrow} \\ C^{\leftarrow}(sI - A^{\leftarrow})^{-1} & \mathbf{G}^{\leftarrow}(s) \end{bmatrix}. \tag{6.9}$$

*If, in addition, $s \in \rho(-A)$, then $\mathbf{G}(-s)$ is invertible and*

$$S_\Sigma^{-1}(-s) = \begin{bmatrix} (sI + A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -(sI + A)^{-1} B \\ I \end{bmatrix} \mathbf{G}^{-1}(-s) \begin{bmatrix} -C(sI + A)^{-1} & I \end{bmatrix}.$$

*In particular, $\mathbf{G}^{\leftarrow}(s) = \mathbf{G}^{-1}(-s)$ for all $s \in \rho(A^{\leftarrow}) \cap \rho(-A)$. Thus, $U$ and $Y$ are isomorphic (i.e., they have the same dimension) if $\rho(A^{\leftarrow}) \cap \rho(-A)$ is nonempty.*

*Proof.* It follows from (6.7) that for all $s \in \mathbb{C}$

$$S_\Sigma(-s) = \begin{bmatrix} sI + A & B \\ C\&D \end{bmatrix} = \begin{bmatrix} sI - A^{\leftarrow} & -B^{\leftarrow} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ [C\&D]^{\leftarrow} \end{bmatrix}^{-1}.$$

For $s \in \rho(A^{\leftarrow})$, the right-hand side is invertible as an operator from $V$ to $X \times Y$, hence so is the left-hand side, and inverting both sides we get (6.9). The verification of the next formula in the corollary is a straightforward algebraic manipulation which uses the same factorization as the proof of Corollary 5.3. ☐

**Remark 6.5.** The set $\rho(A^{\leftarrow}) \cap \rho(-A)$ can be empty, as we show in Example 8.7. In the first part of that example we construct two time-flow-invertible systems for which $U$ and $Y$ have different dimensions (1 and 0). In the second part of the example we construct a time-flow-invertible system $\Sigma^c$ with input and output spaces $U^c = Y^c = \mathbb{C}$ and transfer function $\mathbf{G}^c(s) = 0$ for all $s \in \mathbb{C}_0$. For each of these three systems, $\sigma(A)$ and $\sigma(A^{\leftarrow})$ is the whole left half-plane.

We now state an expanded version of Theorem 1.1.

**Theorem 6.6.** *The system $\Sigma$ is time-flow-invertible if and only if $S_\Sigma(s)$ has a uniformly bounded inverse for all $s$ in some left half-plane (equivalently, $S_\Sigma(-s)$ has a uniformly bounded inverse on some right half-plane). In this case the growth bound of $\mathbb{T}^{\leftarrow}$, denoted by $\omega_{\mathbb{T}}^{\leftarrow}$, is equal to the infimum of those $\omega \in \mathbb{R}$ for which $S_\Sigma^{-1}(-s)$ is uniformly bounded on $\mathbb{C}_\omega$.*

*Proof.* If $\Sigma$ is time-flow-invertible, then it follows from (6.9) that $S_\Sigma^{-1}(-s)$ is uniformly bounded on $\mathbb{C}_\omega$ for any $\omega > \omega_{\mathbb{T}}^{\leftarrow}$. The details of this are as in the proof of Theorem 5.4. Moreover, as in that proof we find that $\omega_0 \le \omega_{\mathbb{T}}^{\leftarrow}$, where

$$\omega_0 \;=\; \inf\{\omega\in\mathbb{R} \mid S_\Sigma^{-1}(-s) \text{ is uniformly bounded on } \mathbb{C}_\omega\}.$$

Conversely, suppose that $S_\Sigma^{-1}(-s)$ is uniformly bounded on some half-plane $\mathbb{C}_\omega$. This means that $\omega_0$ defined above is $< +\infty$, and $\omega \ge \omega_0$. Let $\mathfrak{A}$ be the generator of $\mathfrak{T}$, the Lax–Phillips semigroup of index $\omega_0$ induced by $\Sigma$. Then, by Theorem II.6.3(iii), the resolvent set of $\mathfrak{A}$ contains the open left half-plane bounded by $-\omega$. It follows from Proposition II.6.4(ii) (and some trivial estimates) that $(sI-\mathfrak{A})^{-1}$ is uniformly bounded on the open left half-plane bounded by $-\omega$. By Theorem 4.3, $\mathfrak{T}$ is invertible and the growth bound of $\mathfrak{T}^{-1}$ is at most $\omega$. Since this argument is valid for any $\omega \ge \omega_0$, we obtain that the growth bound of $\mathfrak{T}^{-1}$ is $\le \omega_0$. By Proposition 6.2, the existence of $\mathfrak{T}^{-1}$ implies that $\Sigma$ is time-flow invertible. It is easy to see (from Propositions II.6.2 and 6.2) that the growth bound of $\mathbb{T}^{\leftarrow}$ is dominated by the growth bound of $\mathfrak{T}^{-1}$, so that $\omega_{\mathbb{T}}^{\leftarrow} \le \omega_0$. Thus, $\omega_{\mathbb{T}}^{\leftarrow} = \omega_0$ whenever $\Sigma$ is time-flow-invertible. $\square$

We end this section with some comments on how the different inversions interact with each other and with the duality transformation.

**Proposition 6.7.** *The dual $\Sigma^d$ of a system $\Sigma$ is time-invertible, flow-invertible, or time-flow-invertible if and only if $\Sigma$ has the same property. In this case the appropriate inverse of the dual system is related to the dual of the inverted system as follows (for all $\tau \ge 0$):*

$$\begin{aligned}(\Sigma^{\mathfrak{R}})^d_\tau &= \left[\begin{smallmatrix}-I & 0\\ 0 & I\end{smallmatrix}\right](\Sigma^d)^{\mathfrak{R}}_\tau\left[\begin{smallmatrix}-I & 0\\ 0 & I\end{smallmatrix}\right],\\ (\Sigma^\times)^d_\tau &= \left[\begin{smallmatrix}-I & 0\\ 0 & I\end{smallmatrix}\right](\Sigma^d)^\times_\tau\left[\begin{smallmatrix}-I & 0\\ 0 & I\end{smallmatrix}\right],\\ (\Sigma^{\leftarrow})^d_\tau &= (\Sigma^d)^{\leftarrow}_\tau.\end{aligned}\tag{6.10}$$

We leave the easy algebraic proof of this proposition to the reader.

It is easy to find examples of systems $\Sigma$ where none, or any one but not the other two, or any two but not the third one, or all three of the "inverted" systems $\Sigma^{\mathfrak{R}}$, $\Sigma^\times$, and $\Sigma^{\leftarrow}$ exist. Indeed, all combinations are possible, as can be seen by comparing the different conditions for the existence of the different inverses. By inspecting (4.1), (5.1), and (6.1) we can draw some additional conclusions. For example, if $\Sigma$ is both time-invertible and flow-invertible, then both the time-inverted and flow-inverted systems are time-flow-invertible, and they are time-flow-inverses of each other. Similar statements are true when $\Sigma$ is both time-invertible and time-flow-invertible, or both flow-invertible and time-flow-invertible. Finally, if all the three inverses $\Sigma^{\mathfrak{R}}$, $\Sigma^\times$, and $\Sigma^{\leftarrow}$ exist, then they are all time-invertible, flow-invertible, and time-flow-invertible, and a combination of any two of the inversions gives the third.

## 7. Conservative Systems

Time-flow-inversion and duality are naturally linked to the theory of conservative linear systems (see [1], [5], [8], [25]).

**Definition 7.1.** The well-posed system $\Sigma$ is called *isometric*, or *co-isometric*, or *conservative* if for all $\tau \ge 0$, the operator $\Sigma_\tau$ from (1.4) is isometric, or co-isometric, or unitary, respectively (from $X\times L^2([0,\tau];U)$ to $X\times L^2([0,\tau];Y)$).

There are many different ways of characterizing unitary operators, and this gives us several of the equivalences in the following theorem.

**Theorem 7.2.** *The following conditions are equivalent:*

(i) $\Sigma$ *is conservative;*
(ii) $\Sigma$ *is time-flow-invertible and isometric;*

(iii) $\Sigma$ *is time-flow-invertible and co-isometric;*
(iv) $\Sigma$ *is time-flow-invertible, and* $\Sigma^{\leftarrow} = \Sigma^d$*;*
(v) *the Lax–Phillips semigroup of index zero induced by* $\Sigma$ *is unitary;*
(vi) $\Sigma$ *is time-flow-invertible,* $\left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right]$ *maps* $V$ *continuously onto* $V^d$*, and*

$$\begin{bmatrix} A & B \\ C\&D \end{bmatrix}^* \begin{bmatrix} I & 0 \\ C\&D \end{bmatrix} = \begin{bmatrix} -A & -B \\ 0 & I \end{bmatrix}.$$

Additional equivalent characterizations of conservativity are given in [8].

*Proof.* (i) $\Leftrightarrow$ (ii): This holds because the operator $\Sigma_\tau$ is unitary if and only if it is invertible and isometric.

(i) $\Leftrightarrow$ (iii): The operator $\Sigma_\tau$ is unitary iff it is invertible and co-isometric.

(i) $\Leftrightarrow$ (iv): The operator $\Sigma_\tau$ is unitary if and only if it is invertible and its inverse is equal to its adjoint $\Sigma_\tau^*$.

(iv) $\Leftrightarrow$ (v): This follows Propositions 3.9 and 6.2 (and the fact that $\mathfrak{T}_\tau$ is unitary if and only if $\mathfrak{T}_\tau$ is invertible and the inverse is equal to $\mathfrak{T}_\tau^*$).

(iv) $\Leftrightarrow$ (vi): Two systems are equal if they have the same system operator. $\square$

**Corollary 7.3.** *If* $\Sigma$ *is conservative, and if for some* $\omega \in \mathbb{R}$*, both the limits* $\lim_{\varepsilon\to 0+} \mathbf{G}(\varepsilon + i\omega)$ *and* $\lim_{\varepsilon\to 0+} \mathbf{G}(-\varepsilon + i\omega)$ *exist in the strong sense and are equal (in particular, this requires that* $-\varepsilon + i\omega \in \rho(A)$ *for all sufficiently small* $\varepsilon > 0$ *so that* $\mathbf{G}(-\varepsilon + i\omega)$ *is defined), then this limit is a unitary operator. In particular,* $\mathbf{G}(s)$ *is unitary at each pure imaginary point* $s \in \rho(A)$ *(such points need not exist).*

See Example 8.7 for a case where this corollary does *not* apply.

*Proof.* By Theorem 7.2(iv), $\mathbf{G}^d(s) = \mathbf{G}^{\leftarrow}(s)$ for all $s \in \rho(A^{\leftarrow}) = \rho(A^d)$. In particular, taking $s = \varepsilon + i\omega$ with $\varepsilon > 0$ and $\omega \in \mathbb{R}$, and assuming that $-\varepsilon + i\omega \in \rho(A)$ we get from Corollary 6.4,

$$\mathbf{G}^*(\varepsilon + i\omega) = \mathbf{G}^d(\epsilon - i\omega) = \mathbf{G}^{\leftarrow}(\varepsilon - i\omega) = \mathbf{G}^{-1}(-\varepsilon + i\omega).$$

Thus, $\mathbf{G}^*(\varepsilon + i\omega)\mathbf{G}(-\varepsilon + i\omega) = I$. Denote the common limit in the first sentence of the theorem by $\mathbf{G}(i\omega)$, and take the limit as $\varepsilon \to 0+$ in the formula above to see that $\mathbf{G}^*(i\omega)\mathbf{G}(i\omega) = I$. A similar argument shows that also $\mathbf{G}(i\omega)\mathbf{G}^*(i\omega) = I$. Thus $\mathbf{G}(i\omega)$ is unitary, and in particular this is true if $i\omega \in \rho(A)$. $\square$

## 8. Examples

**Example 8.1.** Here we construct an example of a regular linear system $\Sigma$ whose dual $\Sigma^d$ is not regular (although $\Sigma^d$ must be weakly regular, by Proposition 3.7).

Let $U = l^2$ (square-summable sequences) and $Y = \mathbb{C}$. We define $g : \mathbb{C}_0 \to \mathbb{C}$ by

$$g(s) = \frac{s}{(1+s)^2}, \qquad s \in \mathbb{C}_0.$$

Clearly

$$|g(s)| \le \min\{|s|^{-1}, |s|\}, \qquad s \in \mathbb{C}_0, \tag{8.1}$$

since $|(1+s)| \ge \max\{|s|, 1\}$ for all $s \in \mathbb{C}_0$. For some $a > 1$ we define the transfer function $\mathbf{G} : \mathbb{C}_0 \to \mathcal{L}(U, Y)$ by

$$\mathbf{G}(s) = \begin{bmatrix} g(s) & g(s/a) & g(s/a^2) & g(s/a^3) & \cdots \end{bmatrix}.$$

We claim that this function is bounded on $\mathbb{C}_0$. By (8.1), for all $s \in \mathbb{C}_0$,

$$\begin{aligned}\|\mathbf{G}(s)\|^2 &= \sum_{k=0}^{\infty} |g(a^{-k}s)|^2 \le \sum_{k=0}^{\infty} \min\{a^{2k}|s|^{-2}, a^{-2k}|s|^2\} \\ &\le \sum_{k=-\infty}^{\infty} \min\{a^{2k}|s|^{-2}, a^{-2k}|s|^2\} \\ &= \sum_{k=-\infty}^{\infty} \min\{e^{2(k\log a - \log|s|)}, e^{-2(k\log a-\log|s|)}\} \\ &= \sum_{k=-\infty}^{\infty} e^{-2\left|\log|s|-k\log a\right|}.\end{aligned}$$

The function $f(\alpha) = \sum_{k=-\infty}^{\infty} e^{-2|\alpha - k\log a|}$ is periodic in $\alpha$ with period $\log a$. Elementary computations show that its maximum value is $f_{max} = (1+a^{-2})/(1-a^{-2})$, at $\alpha = k\log a$, and its minimum value is $f_{min} = 2a^{-1}/(1-a^{-2})$, at $\alpha = (k+1/2)\log a$, where $k \in \mathbb{Z}$. In particular, this shows that $\mathbf{G}$ is bounded on $\mathbb{C}_0$.

Let us show that $\mathbf{G}$ is analytic. For any $v = (v_k) \in l^2$ we have

$$\mathbf{G}(s)v = \sum_{k=1}^{\infty} g(a^{-k}s)v_k.$$

Since $|g(s)| \le |s|$ (see (8.1)), the partial sums of the above series converge uniformly on compact subsets of $\mathbb{C}_0$. The terms of the series are analytic, so that $\mathbf{G}(s)v$ is analytic for any $v \in l^2$. Thus, $\mathbf{G}$ is strongly analytic, and hence analytic.

We show that $\mathbf{G}$ is regular. Let $v \in l^2$ and $\varepsilon > 0$. We can find $n \in \mathbb{N}$ such that

$$\left(\sum_{k=n}^{\infty} |v_k|^2\right)^{\frac{1}{2}} \le \frac{\varepsilon}{(f_{max})^{\frac{1}{2}}}.$$

Then, for $\lambda \in (0,\infty)$, by (8.1) and the Cauchy-Schwarz inequality,

$$\begin{aligned}|\mathbf{G}(\lambda)v| &\le \sum_{k=0}^{\infty} |g(a^{-k}\lambda)v_k| \\ &\le \sum_{k=0}^{n-1} |g(a^{-k}\lambda)v_k| + \Big(\sum_{k=n}^{\infty} |g(a^{-k}\lambda)|^2\Big)^{\frac{1}{2}} \Big(\sum_{k=n}^{\infty} |v_k|^2\Big)^{\frac{1}{2}} \\ &\le \sum_{k=0}^{n-1} a^k\lambda^{-1}|v_k| + (f_{max})^{\frac{1}{2}} \Big(\sum_{k=n}^{\infty}|v_k|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

For sufficiently large $\lambda$, the first term on the last right-hand side above is $\le \varepsilon$, while the second term is $\le \varepsilon$ according to the choice of $n$. Hence, $|\mathbf{G}(\lambda)v| \le 2\varepsilon$ for large $\lambda$, which shows that $\mathbf{G}(\lambda)v \to 0$ as $\lambda \to \infty$. Thus, $\mathbf{G}$ is (strongly) regular, and its feedthrough operator is zero.

For every bounded and analytic $\mathcal{L}(U,Y)$-valued function $\mathbf{G}$ defined on a right half-plane there exists a well-posed linear system $\Sigma$ with input space $U$ and output space $Y$, such that the transfer function of $\Sigma$ is $\mathbf{G}$, see Salamon [14] or Staffans [17]. Thus, we can construct a system $\Sigma$ with the transfer function $\mathbf{G}$ defined earlier.

Let us show that the transfer function of the dual system, $\mathbf{G}^d(s) = \mathbf{G}^*(\overline{s})$, is not regular. The input space of the dual system is $\mathbb{C}$, so regularity of $\mathbf{G}^d$ would mean that $\lim_{\lambda\to+\infty} \mathbf{G}^*(\lambda) = 0$ in $U$, since the feedthrough operator of $\mathbf{G}$ is zero. But this is not true, since for all positive integers $k$, $\|\mathbf{G}^*(a^k)\| > |g(1)| = 1/4$.

We can also show that for certain $a$, $\lim_{\lambda\to+\infty} \|\mathbf{G}(\lambda)\|$ does not exist. In fact, this is true for any $a > 1$, but it is easiest to prove it for large $a$. Indeed, for $\log\lambda = (k+1/2)\log a$ with $k \in \mathbb{Z}$, we have $\|\mathbf{G}(\lambda)\| \leq (f_{min})^{\frac{1}{2}}$, and from the expression of $f_{min}$ we see that it can be made arbitrarily small for large $a$.

Using the same example we can show that the cascade connection of two weakly regular systems is not necessarily weakly regular (see, e.g., [18, Section 7.2] for the definition of a cascade connection). Consider the cascade of $\Sigma$ and $\Sigma^d$, i.e., the output of $\Sigma^d$ becomes the input of $\Sigma$. Each of these systems is weakly regular. However, the transfer function of the cascade is the scalar function $\mathbf{G}(s)\mathbf{G}^*(\overline{s})$, which for real values of $s$ is equal to $\|\mathbf{G}(s)\|^2$. This transfer function is not regular since, as we observed earlier, $\lim_{\lambda\to+\infty}\|\mathbf{G}(\lambda)\|^2$ does not exist.

Since a cascade connection is a special case of a feedback connection (see, e.g., [24] or [18, Section 7.2]), and a feedback connection may be regarded as a special case of a flow-inversion (see Remark 5.5), the same example shows that in general, weak regularity is not preserved under feedback and under flow-inversion.

**Example 8.2.** This is our first example meant to illustrate the difficulties with time-inversion. We construct a regular linear system $\Sigma$ which is time-invertible, the time-inverted system $\Sigma^{\text{Я}}$ is also regular but its feedthrough operator $D^{\text{Я}}$ is different from the feedthrough operator $D$ of $\Sigma$. (This could not be the case for a finite-dimensional system.) Our system is SISO (single-input single-output, i.e., $U = Y = \mathbb{C}$), so that there is no difference between regularity and weak regularity. The spectrum of the semigroup generator $A$ does not separate the complex plane.

Consider $X = L^2[0,\tau]$, where $\tau > 0$, and let $\mathbb{T}$ be the semigroup of periodic left shifts on $X$:

$$(\mathbb{T}_t x)(\xi) = x(\xi \dot{+} t),$$

where $\xi \dot{+} t$ denotes addition modulo $\tau$. Clearly, $\mathbb{T}$ extends to a unitary group on $X$, and its generator is $A = \frac{\mathrm{d}}{\mathrm{d}\xi}$, with the domain

$$\mathcal{D}(A) = \{x \in H^1[0,\tau] \mid x(0) = x(\tau)\}.$$

Note that $\mathcal{D}(A^*) = \mathcal{D}(A)$ and $A^* = -A$. Consider $Y = \mathbb{C}$ and $C \in \mathcal{L}(X_1; Y)$ defined by $Cx = x(0)$, which is easily seen to determine a continuous operator from $X$ to $L^2_{\mathrm{loc}}[0,\infty)$ via (2.10). Using the terminology of [20], Parts I, II and other papers, $B = C^*$ is an admissible control operator for the dual semigroup $\mathbb{T}^*$. Since $\mathbb{T}^*_t = \mathbb{T}_{-t}$, it follows that $B$ is an admissible control operator for $\mathbb{T}$. Actually, for $t \leq \tau$,

$$(\Phi_t u)(\xi) = \begin{cases} 0 & \text{for } \ \xi \in [0, \tau - t), \\ u(\xi - \tau + t) & \text{for } \ \xi \in [\tau - t, \tau]. \end{cases}$$

It is not difficult to check that $(sI - A)^{-1}B \in H^1[0,\tau] \subset \mathcal{D}(C_\Lambda)$ and

$$C_\Lambda(sI - A)^{-1}B = \frac{e^{-\tau s}}{1 - e^{-\tau s}},$$

which is bounded on $\mathbb{C}_\alpha$, for any $\alpha > 0$. By the main result of Curtain and Weiss [3] (see also Proposition 2.1 in [24]), for any $D \in \mathcal{L}(U; Y)$, there exists a regular linear system $\Sigma$ whose generating operators are $A, B, C$ and $D$. We choose $D = 1$ and then (by (2.15)) the transfer function of $\Sigma$ is

$$\mathbf{G}(s) = \frac{1}{1 - e^{-\tau s}}.$$

This transfer function (which is often encountered in an engineering area called repetitive control) has poles at $s = ik\nu$, where $k \in \mathbb{Z}$ and $\nu = 2\pi/\tau$. These poles are exactly at the spectrum of $A$. $\mathbf{G}$ has an analytic continuation to the whole complex plane except the poles mentioned earlier.

Since $\mathbb{T}$ extends to a group, the system $\Sigma$ is time-invertible and the transfer function of $\Sigma^{\text{Я}}$ can be obtained from the analytic continuation of $\mathbf{G}$ to the left half-plane:

$$\mathbf{G}^{\text{Я}}(s) = \mathbf{G}(-s) = \frac{1}{1 - e^{\tau s}} = \frac{-e^{-\tau s}}{1 - e^{-\tau s}}.$$

We see from here (by taking a limit) that $\Sigma^{\text{Я}}$ is regular and its feedthrough operator is $D^{\text{Я}} = 0$, which is different from $D$.

**Example 8.3.** This is again an example of a time-invertible system $\Sigma$, but here $\sigma(A)$ separates $\mathbb{C}$ and the restrictions of $\mathbf{G}$ and $\mathbf{G}^{\mathbf{Я}}$ to the right half-plane cannot be obtained from each other (see the comments at the end of Section 4). Actually, these restrictions could be any two functions in $H^\infty(\mathbb{C}_0)$. More precisely, we let $\mathbf{G}_+$ be an arbitrary function in $H^\infty(\mathbb{C}_0)$, and let $\mathbf{G}_-$ be an arbitrary function which is analytic and bounded on the left half-plane. We shall construct a time-invertible SISO system $\Sigma$ whose transfer function $\mathbf{G}$ is defined on $\mathbb{C} \setminus i\mathbb{R}$, its restriction to $\mathbb{C}_0$ is equal to $\mathbf{G}_+$, and its restriction to the left half-plane is equal to $\mathbf{G}_-$. Thus, $\mathbf{G}^{\mathbf{Я}}(s) = \mathbf{G}_-(-s)$ for $s \in \mathbb{C}_0$, and $\mathbf{G}^{\mathbf{Я}}(s) = \mathbf{G}_+(-s)$ for $\operatorname{Re} s < 0$.

Our construction of the realization of the transfer function described above is based on Remark II.5.4, i.e., we define the system $\Sigma$ which realizes $\mathbf{G}$ by defining its semigroup $\mathbb{T}$, its extended input map $\widetilde{\mathbf{\Phi}}_0$, its extended output map $\Psi_\infty$, and its bilaterally shift-invariant input-output map $\mathcal{F}$. (The realization described below is a slight modification of the bilateral input shift realization described in [18, Section 2.6]; another possibility would have been to modify the bilateral output shift realization described in [18, Section 2.6] in a similar way.)

The state space of $\Sigma$ is $X = L^2(\mathbb{R})$ and we take the semigroup $\mathbb{T}$ of $\Sigma$ to be the bilateral left shift semigroup on $X$: $\mathbb{T}_t = \mathcal{S}_{-t}$ (this $\mathbb{T}$ extends to a group, of course). The generator $A$ of $\mathbb{T}$ is the usual differentiation operator $A = \frac{\mathrm{d}}{\mathrm{d}\xi}$, with domain $H^1(\mathbb{R})$. We define the extended input map $\widetilde{\mathbf{\Phi}}_0$ of the system $\Sigma$ by

$$\widetilde{\mathbf{\Phi}}_0 u = \mathbf{P}_- u, \qquad u \in L^2(\mathbb{R}).$$

This operator satisfies the functional equation

$$\mathbb{T}_t \widetilde{\mathbf{\Phi}}_0 = \widetilde{\mathbf{\Phi}}_0 \mathcal{S}_{-t} \mathbf{P}_-, \qquad t \geq 0,$$

as required in Remark II.5.4. It is not difficult to see that the control operator $B$ corresponding to this extended input map is the unit pulse $B = \delta_0$ (which is a distribution in $H^{-1}(\mathbb{R}) = X_{-1}$, by definition $\langle \delta_0, \varphi \rangle = \varphi(0)$ for all $\varphi \in H^1(\mathbb{R})$).

We define two bilaterally shift-invariant operators $\mathcal{F}$ and $\mathcal{F}_-$ acting on $L^2(\mathbb{R})$ as follows: For almost all $\omega \in \mathbb{R}$ we define $\mathbf{G}_+(i\omega)$ by a nontangential limit of $\mathbf{G}_+$ from the right, and we define $\mathbf{G}_-(i\omega)$ by a nontangential limit of $\mathbf{G}_-$ from the left. Both limit functions are in $L^\infty(i\mathbb{R})$. For all $u \in L^2(\mathbb{R})$, we define $\mathcal{F}u$ and $\mathcal{F}_- u$ through their bilateral Laplace transforms for almost all $\omega \in \mathbb{R}$:

$$\widehat{\mathcal{F}u}(i\omega) = \mathbf{G}_+(i\omega)\hat{u}(i\omega), \qquad \widehat{\mathcal{F}_- u}(i\omega) = \mathbf{G}_-(i\omega)\hat{u}(i\omega). \tag{8.2}$$

Both $\mathcal{F}$ and $\mathcal{F}_-$ commute with $\mathcal{S}_t$ for all $t \in \mathbb{R}$, $\mathcal{F}$ is causal and $\mathcal{F}_-$ is anticausal:

$$\mathbf{P}_- \mathcal{F} \mathbf{P}_+ = 0, \qquad \mathbf{P}_+ \mathcal{F}_- \mathbf{P}_- = 0.$$

Moreover, $\mathcal{D}(A) = H^1(\mathbb{R})$ is invariant under both $\mathcal{F}$ and $\mathcal{F}_-$ (since both of these operators commute with $\mathbb{T}_t$). We define the bilaterally shift-invariant input-output map of $\Sigma$ to be $\mathcal{F}$. Finally, we define the extended output map of $\Sigma$ by

$$\Psi_\infty = \mathbf{P}_+(\mathcal{F} - \mathcal{F}_-). \tag{8.3}$$

Then $\Psi_\infty$ satisfies the functional equation

$$\Psi_\infty \mathbb{T}_t = \mathbf{S}_t^* \Psi_\infty, \qquad t \geq 0,$$

as required in Remark II.5.4. The corresponding observation operator $C$ is

$$Cx = ((\mathcal{F} - \mathcal{F}_-)x)(0), \qquad x \in H^1(\mathbb{R}).$$

We have now verified most of the identities listed in Remark II.5.4. The only remaining one is also easily verified:

$$\mathbf{P}_+ \mathcal{F} \mathbf{P}_- = \mathbf{P}_+(\mathcal{F} - \mathcal{F}_-)\mathbf{P}_- = \Psi_\infty \widetilde{\mathbf{\Phi}}_0.$$

By Remark II.5.4, if we define the families $\Phi$, $\Psi$ and $\mathbb{F}$ by

$$\Phi_\tau = \widetilde{\mathbf{\Phi}}_0 \mathcal{S}_{-\tau} \mathbf{P}_+ = \mathcal{S}_{-\tau} \mathbf{P}_\tau, \qquad \Psi_\tau = \mathbf{P}_\tau \Psi_\infty, \qquad \mathbb{F}_\tau = \mathbf{P}_\tau \mathcal{F} \mathbf{P}_\tau,$$

for all $\tau \geq 0$, then $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ is a well-posed linear system. In particular, $\mathbb{F}_\infty u = \mathcal{F}u$ for all $u \in L^2[0, \infty)$, and this together with the definition of $\mathcal{F}$ implies that the transfer function $\mathbf{G}$ of $\Sigma$ satisfies $\mathbf{G}(s) = \mathbf{G}_+(s)$ for all $s \in \mathbb{C}_0$.

The system $\Sigma$ is time-invertible, since $\mathbb{T}$ is a group. We compute the time-inverted system $\Sigma^{\mathbf{Я}}$. Using (4.1), we get (as for any time-inverted system)

$$\begin{bmatrix} \mathbb{T}_\tau^{\mathbf{Я}} & \Phi_\tau^{\mathbf{Я}} \\ \Psi_\tau^{\mathbf{Я}} & \mathbb{F}_\tau^{\mathbf{Я}} \end{bmatrix} = \begin{bmatrix} \mathbb{T}_{-\tau} & -\mathbb{T}_{-\tau}\Phi_\tau \mathbf{Я}_\tau \\ \mathbf{Я}_\tau \Psi_\tau \mathbb{T}_{-\tau} & \mathbf{Я}_\tau(\mathbb{F}_\tau - \Psi_\tau \mathbb{T}_{-\tau}\Phi_\tau)\mathbf{Я}_\tau \end{bmatrix}.$$

In particular, we get (after a short computation) that $\mathbb{F}_\tau^{\mathbf{Я}} = \mathbf{Я}_\tau \mathcal{F}_- \mathbf{Я}_\tau$. We claim that the extended input-output map of $\Sigma^{\mathbf{Я}}$ is

$$\mathbb{F}_\infty^{\mathbf{Я}} u = \mathbf{Я}\mathcal{F}_-\mathbf{Я}u, \qquad u \in L^2[0,\infty). \tag{8.4}$$

Recall that $\mathbb{F}_\infty^{\mathbf{Я}}$ is uniquely determined by the fact that $\mathbf{P}_\tau \mathbb{F}_\infty^{\mathbf{Я}} = \mathbb{F}_\tau^{\mathbf{Я}}$, for all $\tau \geq 0$. To prove (8.4), we use the elementary identities $\mathbf{P}_\tau \mathbf{Я} = \mathbf{Я}_\tau \mathcal{S}_\tau$, $\mathbf{Я}_\tau = \mathbf{Я}_\tau \mathbf{P}_+$ and $\mathbf{P}_+ \mathcal{S}_\tau \mathbf{Я} u = \mathbf{Я}_\tau u$ for all $u \in L^2[0,\infty)$. These identities, together with $\mathbf{P}_+ \mathcal{F}_- = \mathbf{P}_+ \mathcal{F}_- \mathbf{P}_+$ (which is just another way of writing that $\mathcal{F}_-$ is anticausal) enable us to make the following reasoning: for every $u \in L^2[0,\infty)$,

$$\begin{aligned} \mathbf{P}_\tau \mathbf{Я}\mathcal{F}_-\mathbf{Я}u &= \mathbf{Я}_\tau \mathcal{S}_\tau \mathcal{F}_- \mathbf{Я}u = \mathbf{Я}_\tau \mathbf{P}_+ \mathcal{F}_- \mathcal{S}_\tau \mathbf{Я}u \\ &= \mathbf{Я}_\tau \mathcal{F}_- \mathbf{P}_+ \mathcal{S}_\tau \mathbf{Я}u = \mathbf{Я}_\tau \mathcal{F}_- \mathbf{Я}_\tau u = \mathbb{F}_\tau^{\mathbf{Я}} u, \qquad \tau \geq 0, \end{aligned}$$

which proves (8.4). Now (8.4) together with the definition of $\mathcal{F}_-$ in (8.2) implies that the transfer function $\mathbf{G}^{\mathbf{Я}}$ of $\Sigma^{\mathbf{Я}}$ satisfies (by looking at its boundary values on $i\mathbb{R}$) $\mathbf{G}^{\mathbf{Я}}(i\omega) = \mathbf{G}_-(-i\omega)$, for almost every $\omega \in \mathbb{R}$. This implies that $\mathbf{G}^{\mathbf{Я}}(s) = \mathbf{G}_-(-s)$ for all $s \in \mathbb{C}_0$. Therefore, by (4.7), the restriction of $\mathbf{G}$ to the left half-plane is $\mathbf{G}_-$. Thus, $\Sigma$ has all the properties mentioned at the beginning of this example.

We remark that the extended input operator $\widetilde{\Phi}_0^{\mathbf{Я}}$, the extended output operator $\Psi_\infty^{\mathbf{Я}}$, and the bilaterally shift-invariant input-output map $\mathcal{F}^{\mathbf{Я}}$ of $\Sigma^{\mathbf{Я}}$ are given by

$$\widetilde{\Phi}_0^{\mathbf{Я}} = -\mathbf{Я}\mathbf{P}_-, \qquad \Psi_\infty^{\mathbf{Я}} = \mathbf{P}_+\mathbf{Я}(\mathcal{F} - \mathcal{F}_-), \qquad \mathcal{F}^{\mathbf{Я}} = \mathbf{Я}\mathcal{F}_-\mathbf{Я}.$$

**Example 8.4.** Here we construct a time-invertible SISO system $\Sigma$ such that neither $\Sigma$ nor $\Sigma^{\mathbf{Я}}$ is regular. The spectrum of $A$ is contained in the imaginary axis $i\mathbb{R}$ and it does not separate the open right half-plane from the open left half-plane. It is based on Example 8.3, with specific choices of $\mathbf{G}_-$ and $\mathbf{G}_+$, but using a smaller state space, after factoring out an unnecessary subspace of $L^2(\mathbb{R})$.

Let $E = (-\infty,-1] \cup [1,\infty)$ and put $\Omega = \mathbb{C} \setminus iE$. Thus, $\Omega$ contains the open left and right half-planes and also a connecting bridge between them. For $s \in \Omega$, $s^2+1$ is not a real number in $(-\infty, 0]$. Since the function log can be defined to be analytic on $\mathbb{C} \setminus (-\infty, 0]$ and such that $\log z$ is real for $z > 0$, we can define

$$\mathbf{G}(s) = \cos \log(s^2+1), \qquad \text{for} \quad s \in \Omega. \tag{8.5}$$

Then $\mathbf{G}$ is a bounded analytic function on $\Omega$ (its nontangential limits on $iE$ are different, depending if we come from the right or from the left). Moreover, $\mathbf{G}$ does not have limits as $s \to +\infty$ or $s \to -\infty$ along the real axis. The non-regular SISO transfer function $\cos \log s$ was proposed by K. Morris in [9], and we have arrived at our $\mathbf{G}$ above by modifying her example.

We now use Example 8.3 to construct a time-invertible realization $\widetilde{\Sigma}$ of $\mathbf{G}$ (which is not yet the final realization). We simply take $\mathbf{G}_+$ to be the restriction of $\mathbf{G}$ to $\mathbb{C}_0$, and $\mathbf{G}_-$ to be the restriction of $\mathbf{G}$ to the open left half-plane. According to Example 8.3 we get a realization $\widetilde{\Sigma}$ whose state space is $\widetilde{X} = L^2(\mathbb{R})$, whose semigroup is $\widetilde{\mathbb{T}}_t = \mathcal{S}_{-t}$, with generator $\widetilde{A} = \frac{\mathrm{d}}{\mathrm{d}\xi}$, $\mathcal{D}(\widetilde{A}) = H^1(\mathbb{R})$, and whose control operator is $\widetilde{B} = \delta_0$. The extended output map of $\widetilde{\Sigma}$ is given, according to (8.2) and (8.3), as follows: $y = \widetilde{\Psi}_\infty x$ if and only if $y = \mathbf{P}_+ z$, where $\hat{z} = (\mathbf{G}_+ - \mathbf{G}_-)\hat{x}$.

Since $\mathbf{G}_+(i\omega) - \mathbf{G}_-(i\omega) = 0$ if (and only if) $\omega \in [-1,1]$, the space $X_0$ of all band-limited functions in $L^2(\mathbb{R})$ whose spectrum is confined to $[-1,1]$ (i.e., their bilateral Laplace transforms vanish on $iE$) is an unobservable subspace for $\widetilde{\Sigma}$ (this means that $X_0$ is invariant for $\widetilde{\mathbb{T}}$ and $X_0 \subset \operatorname{Ker} \widetilde{\Psi}_\infty$). Moreover, $X_0$ is invariant also for the adjoint semigroup $\widetilde{\mathbb{T}}^*$. We factor out $X_0$, obtaining a reduced system $\Sigma$ whose state space is the orthogonal complement of $X_0$. Thus, the state space of $\Sigma$ is

$$X = \left\{ x \in L^2(\mathbb{R}) \mid \hat{x} \in L^2(iE) \right\}.$$

Thus, the functions in $X$ contain "only high frequencies". The semigroup $\mathbb{T}_t$ of the reduced system $\Sigma$ is the restriction of $\mathcal{S}_{-t}$ to $X$. The spectrum of the generator $A$ of this semigroup is $iE$, and so $\rho(A) = \Omega$ is connected. Denoting the orthogonal projection of $L^2(\mathbb{R})$ onto $X$ by $\mathbf{P}_X$, we have $\Phi_\tau = \mathbf{P}_X\widetilde{\Phi}_\tau = \mathbf{P}_X\mathcal{S}_{-\tau}\mathbf{P}_\tau$. $\Psi_\tau$ is simply the restriction of $\widetilde{\Psi}_\tau$ to $X$. Thus, for every $x \in X$, we have $y = \Psi_\tau x$ if and only if $y = \mathbf{P}_\tau z$, where $\hat{z} = (\mathbf{G}_+ - \mathbf{G}_-)\hat{x}$. The extended input-output map of the reduced system $\Sigma$ coincides with that of $\widetilde{\Sigma}$, and therefore the restriction of the transfer function $\mathbf{G}$ of $\Sigma$ to $\mathbb{C}_0$ is $\mathbf{G}_+$. As $\Omega$ is connected, the transfer function of $\Sigma$ must be equal to the (analytic) function $\mathbf{G}$ defined in (8.5) on all of $\Omega$.

Finally, we make an interesting observation. In this example $\mathbf{G}_+(s) = \mathbf{G}_-(-s)$ for all $s \in \mathbb{C}_0$, and $\Omega$ is invarant under a 180° rotation of the complex plane. In particular, $\mathcal{F}_- = \boldsymbol{\mathcal{A}}\mathcal{F}\boldsymbol{\mathcal{A}}$. This implies that the realization $\widetilde{\Sigma}$ constructed as in Example 8.3 has the property that $\widetilde{\Sigma}^{\boldsymbol{\mathcal{A}}}$ is unitarily similar to $\widetilde{\Sigma}$, with similarity operator $-\boldsymbol{\mathcal{A}}$. Since $\boldsymbol{\mathcal{A}}$ commutes with $\mathbf{P}_X$, the reduced system $\Sigma$ has the same property: $\Sigma^{\boldsymbol{\mathcal{A}}}$ is unitarily similar to $\Sigma$, with similarity operator $-\boldsymbol{\mathcal{A}}$.

**Example 8.5.** Taking the cascade connection of $\Sigma^{\boldsymbol{\mathcal{A}}}$ from Example 8.2 and $\Sigma$ from Example 8.4, we get a time-invertible system whose transfer function is the product of the two transfer functions, i.e., for all $s \in \Omega$ with $e^{-\tau s} \neq 1$,

$$\mathbf{G}(s) = \frac{-e^{-\tau s}}{1 - e^{-\tau s}} \cos\, \log(s^2 + 1).$$

We once more refer the reader to [18, Section 7.2] for a closer description of the cascade connection; in particular, the semigroup of the cascade connection is invertible whenever the two factors are time-invertible. It is easy to see that this system is regular (with feedthrough operator zero) since $\mathbf{G}(s) \to 0$ as $s \to +\infty$, but the time-inverted system is not regular, since $\lim_{s\to-\infty} \mathbf{G}(s)$ does not exist.

**Example 8.6.** The construction presented in Example 8.4 can be extended into a general procedure to construct time-invertible systems with $\rho(A)$ connected, whose transfer functions behave in a specified way at $+\infty$ and $-\infty$. In all the cases we may choose the group $\mathbb{T}$ to be the same left-shift as in Example 8.4, and we only vary the transfer function $\mathbf{G}$. We start with an arbitrary $H^\infty$ function $\varphi$ in the unit disk $\mathbb{D}$. Then we use a conformal map $\eta$ to map the unit disk onto the region $\Omega$ in Example 8.4 in such a way that 1 is mapped onto $+\infty$ and $-1$ is mapped onto $-\infty$. After that we define $\mathbf{G}(s) = \varphi(\eta^{-1}(s))$ and realize $\mathbf{G}$ in the same way as we did in Example 8.4. By an appropriate choice of $\varphi$ we can adjust the behavior of $\mathbf{G}$ at $\pm\infty$ (and also at any point of $iE$). For example, if $\varphi$ is bounded away from zero at $\pm 1$ but does not have limits (taken along the real axis) at these points, then the system that we get is time-invertible, flow-invertible, and time-flow-invertible, but neither the system itself nor any of the inverted systems is regular.

**Example 8.7.** Here we give the construction promised in Remark 6.5. Consider the state space $X = L^2(-\infty, 0]$. On $X$ we define the semigroup $\mathbb{T}$ by restricting the bilateral left-shift to $X$, i.e., $\mathbb{T}_t = \mathcal{S}_{-t}$. The generator of $\mathbb{T}$ is $A = \frac{\mathrm{d}}{\mathrm{d}\xi}$ on its domain $H^1_0(-\infty, 0] = \{u \in H^1(-\infty, 0] \mid u(0) = 0\}$. We take $U = \mathbb{C}$, and define the extended input map (defined in Section II.5) to be $\widetilde{\Phi}_0 = I$. The corresponding control operator is the unit pulse at zero: $B = \delta_0$. We take $Y = \{0\}$. This forces us to take $C = 0$ and $\Psi_\infty = 0$. Using Remark II.5.4 we can check that the operators

$$\Sigma_\tau = \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \end{bmatrix} = \begin{bmatrix} \mathcal{S}_{-\tau} & \mathcal{S}_{-\tau}\mathbf{P}_\tau \end{bmatrix}$$

determine a well-posed linear system $\Sigma$ (with no output). Clearly, $\Sigma_\tau$ maps $X \times L^2[0, \tau]$ one-to-one onto $X = L^2(-\infty, 0]$, and it is even norm-preserving (i.e., unitary). Therefore, $\Sigma$ *is conservative,* hence time-flow-invertible. The time-flow-inverted system $\Sigma^{\leftarrow}$ concides with the dual $\Sigma^d$, and it is given by

$$\Sigma^{\leftarrow}_\tau = \Sigma^d_\tau = \begin{bmatrix} \mathbf{P}_-\mathcal{S}_\tau \\ \boldsymbol{\mathcal{A}}_\tau\mathcal{S}_\tau \end{bmatrix}.$$

The input space of this system is $\{0\}$, and its output space is $\mathbb{C}$ (the observation operator is point observation at zero). We get a slightly different version of $\Sigma^{\leftarrow}$ if we use a unitary similarity tranformation of its state space: we use $\boldsymbol{\mathcal{A}}$ to map $L^2(-\infty, 0]$ onto $L^2[0, \infty)$. This means that we construct the conservative system $\Sigma^1$ as follows: we replace $\mathbb{T}_\tau^{\leftarrow} = \mathbf{P}_-\mathcal{S}_\tau$ by the semigroup $\mathbb{T}_\tau^1 = \boldsymbol{\mathcal{R}}\mathbb{T}_\tau^{\leftarrow}\boldsymbol{\mathcal{R}}$ acting on $X^1 = L^2[0,\infty)$. It turns out that $\mathbb{T}_\tau^1 = \mathbf{S}_\tau^*$ (the unilateral left-shift). Similarly, we replace $\Psi_\tau^{\leftarrow} = \boldsymbol{\mathcal{R}}_\tau\mathcal{S}_\tau$ by $\Psi_\tau^{\leftarrow}\boldsymbol{\mathcal{R}}$. It turns out that $\Psi_\tau^1 = \mathbf{P}_\tau$. Thus,

$$\Sigma_\tau^1 = \begin{bmatrix} \mathbf{S}_\tau^* \\ \mathbf{P}_\tau \end{bmatrix}$$

is another example of a conservative system with no input.

$\Sigma$ and $\Sigma^1$ constructed above are examples of a time-flow-invertible (even conservative) systems where the input and output spaces have different dimensions. In both cases the whole left half-plane belongs to the spectrum of the semigroup generator (as it must always do when $U$ and $Y$ are not isomorphic, see Corollary 6.4).

By combining the systems $\Sigma$ and $\Sigma^1$, we can construct a time-flow-invertible system $\Sigma^c$ whose input and output spaces are the same, but whose transfer function is zero. This time we take the state space to be $X^c = L^2(\mathbb{R})$, and the semigroup $\mathbb{T}^c$ to be a modified bilateral left-shift which does not permit any information to pass through the origin:

$$\mathbb{T}_\tau^c = \mathcal{S}_{-\tau}(I - \mathbf{P}_\tau).$$

Using the orthogonal splitting $X = L^2(-\infty,0] \oplus L^2[0,\infty)$, which corresponds to the state spaces of the two systems which we are combining, we can write

$$\mathbb{T}_\tau^c = \begin{bmatrix} \mathcal{S}_{-\tau} & 0 \\ 0 & \mathbf{S}_\tau^* \end{bmatrix}.$$

The generator of $\mathbb{T}^c$ is $A^c = \frac{\mathrm{d}}{\mathrm{d}\xi}$, defined on $\mathcal{D}(A^c) = H_0^1(-\infty,0] \oplus H^1[0,\infty)$. We take $U^c = Y^c = \mathbb{C}$, $\widetilde{\Phi}_0^c = \left[\begin{smallmatrix} \mathbf{P}_- \\ 0 \end{smallmatrix}\right]$ and $\Psi_\infty^c = \mathbf{P}_+ = \begin{bmatrix} 0 & I \end{bmatrix}$. The control operator of $\Sigma^c$ is $B^c = \left[\begin{smallmatrix} \delta_0 \\ 0 \end{smallmatrix}\right]$, and its observation operator is given by $C^c \left[\begin{smallmatrix} x_- \\ x_+ \end{smallmatrix}\right] = x_+(0)$, for all $x = \left[\begin{smallmatrix} x_- \\ x_+ \end{smallmatrix}\right] \in \mathcal{D}(A^c)$. By the formulas in Remark II.5.4, we get

$$\Phi_\tau^c = \mathcal{S}_{-\tau}\mathbf{P}_\tau, \qquad \Psi_\tau^c = \mathbf{P}_\tau.$$

This system is easily seen to be conservative, hence time-flow-invertible.

Finally, we remark that Corollary 7.3 *does not apply to any of the three conservative systems in this example,* due to the fact that $\mathbf{G}$ is not defined on the left half-plane (which belongs to the spectrum of the semigroup generator).

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