Regular Linear Systems Governed by a Boundary Controlled Heat Equation*

C.I. Byrnes[†], D.S. Gilliam[‡],

V.I. Shubov[‡], G. Weiss[§]

Abstract

In this paper we consider a class of distributed parameter systems governed by the heat equation on bounded domains in $\mathbb{R}^n$. We consider two types of boundary inputs (actuators) and two types of boundary outputs (sensors). Allowing for any possible pairing of these, we consider a total of four possible arrangements of our system. The first type of input (control) is through the Neumann boundary condition on a part of the boundary, together with a homogeneous Neumann boundary condition on the remaining part of the boundary. For this type of input, the input space is infinite-dimensional. The second type of input (with a finite-dimensional input space) is obtained by imposing constant normal derivatives on each element of a finite partition of the boundary. The first type of output (observation) is given by evaluation (trace) of the state of the system on a part of the boundary, so that the output space is infinite-dimensional. For the second type of output (with a finite-dimensional output space), we again consider a partition of the boundary of the spatial domain (which may be different from the one considered for the inputs) and each output channel contains the average of the values of the state of the plant on one element of this partition. Our main result is that any possible combination of the aforementioned inputs and outputs provides a regular linear system.

1. Introduction

For most models of infinite-dimensional linear systems the control and observation operators are unbounded with respect to the state space (a Hilbert space). Usual applications include actuators and sensors supported at isolated points or on lower-dimensional hypersurfaces in, or on the boundary of, a spatial domain. As it is well known, such control and

* This work was supported in part by grants from AFOSR and Texas ARP.

† Systems Science and Mathematics, Washington University, St. Louis, MO 63130

‡ Department of Mathematics and Statistics, Texas Tech University, Lubbock, Texas 79409

§ Dept of Electrical and Electronic Engineering, Imperial College of Science, Technology & Medicine, Exhibition Road, London SW7 2BT, UK

observation operators present considerable technical difficulties even at the level of state space formulations of the dynamics. Indeed, there has been and continues to be significant research devoted to formulating analogs of classical feedback control methodologies for such systems. Probably the most general class of distributed parameter systems for which there is a well established theory of representation, transfer functions, feedback, dynamic stabilization and tracking/disturbance rejection is the class of *regular linear systems*.

Regular linear systems form a very general class whose basic properties are rich enough to develop a parallel of the theory of control systems with bounded control and observation operators, as presented in Curtain and Zwart [9]. This class was introduced in Weiss [24, 25] and its theory is under development, see the relatively recent works by Curtain *et al* [8], Logemann *et al* [13, 14, 15], Staffans [20], Weiss and Curtain [26], and their references. As we have mentioned, for regular systems there is now a substantial control theory and, from a practical point of view, construction of specific examples of distributed parameter systems which belong to this class is of considerable importance. In this connection we mention the recent work of Avalos *et al* [1].

A precise description of the systems in question and the statement of main results will be given in Section 2. There, we will also recall some basic facts from the theory of regular linear systems in a form convenient for the present work. Moreover, in this section we suggest a definition of uniformly regular systems. We introduce this definition in order to give a name to a class of systems which appears to be widespread among linear systems governed by partial differential equations (PDEs) (see, e.g., [1] and Theorem 2.9 below). Namely, the transfer function of a uniformly regular system converges to the feedthrough operator in the norm topology.

The well-posedness of our systems will be established in Section 3. To prove well-posedness, we will deal directly with the partial differential equation and use basic a priori estimates and Sobolev embedding theorems (see, e.g., Ladyzhenskaya *et al* [10]).

Section 4 contains a proof of the main regularity result for our class of systems. In this section we deal with an elliptic problem which appears from the original heat problem after applying the Laplace transformation with respect to the time variable. We investigate the behavior of the system's transfer function as the spectral parameter $s \to +\infty$ ($\mathrm{Im}\ s = 0$).

Section 5 is devoted to the proof of the uniform regularity of our main system, with certain choices of input and output spaces. We show that the norm of the transfer function tends to zero as $s \to +\infty$, and therefore, our system is uniformly regular with feedthrough operator zero. We use interpolation theory for Sobolev spaces (see, e.g., Lions and Magenes [12]), in addition to a priori estimates and basic embedding theorems.

In Section 6 we recall an abstract setup for regular linear systems and precisely identify all the spaces and operators involved in this setup. Finally, using the abstract concepts developed in Section 6, in Section 7 we present an alternative operator theoretic proof of the main regularity result, Theorem 2.8.

In conclusion we mention that the results of the present work can be extended to systems governed by general linear parabolic equations on bounded domains in $\mathbb{R}^n$. This extension will be the subject of a forthcoming paper. We also mention that the results of this and forthcoming works provide natural examples for the authors' papers [5, 6] devoted to the problem of output regulation for infinite-dimensional linear systems. We also mention in this connection the works of Byrnes *et al* [3, 4].

2. Problem statement and main results

In this section we present a class of distributed parameter systems with boundary control and boundary observation and we state our main results.

We denote by $x = (x_1, \cdots x_n)$ a point in $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-smooth boundary $\partial\Omega$. More generally, Ω can be a *domain of class* C^2. This means that there exists a covering of Ω by a finite number of open sets $\Omega_1, \cdots \Omega_N$ with the following property: If $\Omega_j \cap \partial\Omega \neq \emptyset$, then there exists a C^2-diffeomorphism mapping Ω_j onto the ball $\mathcal{B} = \{x \in \mathbb{R}^n \, : \, |x| < 1\}$, $\Omega_j \cap \Omega$ onto the "half ball" $\mathcal{B}^+ = \{x \in \mathcal{B} \, : \, x_n > 0\}$, and $\Omega_j \cap \partial\Omega$ onto the "disk" $S = \{x = (x_1, \cdots, x_{n-1}, 0) \in \mathbb{R}^n \, : \, |x| < 1\}$. In other words, we assume that the boundary is "piecewise C^2-smooth."

We assume that our system is governed by the heat equation

$$\frac{\partial z}{\partial t}(x,t) \;=\; \Delta z(x,t), \qquad (x,t) \in \Omega \times [0,\infty). \tag{2.1}$$

For this equation we introduce two types of inputs and two types of outputs.

a) **Type 1 Input:** Assume $\partial\Omega = \mathcal{S}_0 \cup \mathcal{S}$, where $\mathcal{S}_0$ and $\mathcal{S}$ are piecewise (in a sense similar to our earlier definition of a domain of class C^2) C^2-smooth $(n-1)$-hypersurfaces with piecewise C^1-smooth $(n-2)$-dimensional boundaries ($\mathcal{S}$ is assumed to contain the common boundary). We impose the boundary conditions

$$\frac{\partial z}{\partial n}(x,t) = 0, \qquad (x,t) \in \mathcal{S}_0 \times [0,\infty), \tag{2.2}$$

$$\frac{\partial z}{\partial n}(x,t) = u(x,t), \quad (x,t) \in \mathcal{S} \times [0,\infty). \tag{2.3}$$

Here, $\dfrac{\partial z}{\partial n}(x,t) = n(x) \cdot \nabla z(x,t)$ is the derivative of z in the direction of the outward normal $n(x)$ to $\partial\Omega$. It is possible that $\mathcal{S}_0 = \emptyset$ and $\mathcal{S} = \partial\Omega$, in which case the control acts on the entire boundary.

b) **Type 2 Input:** Let $\partial\Omega$ consist of a union of nonoverlapping surfaces,

$$\partial\Omega \;=\; \mathcal{S}_0 \cup \big(\cup_{j=1}^{M} \mathcal{S}_j \big).$$

Each $\mathcal{S}_j$ is a (piecewise) smooth $(n-1)$-dimensional hypersurface with a (piecewise) smooth (C^2) boundary. We assume, in addition, that $|\mathcal{S}_j| > 0$ for $j = 1, \cdots M$, where $|\mathcal{S}_j|$ is the $(n-1)$-dimensional hypersurface measure of $\mathcal{S}_j$. The inputs are implemented through the boundary conditions as

$$\frac{\partial z}{\partial n}(x,t) = 0, \qquad (x,t) \in \mathcal{S}_0 \times [0,\infty) \tag{2.4}$$

$$\frac{\partial z}{\partial n}(x,t) = u_j(t), \qquad (x,t) \in \mathcal{S}_j \times [0,\infty),\ j = 1,2,\cdots M. \tag{2.5}$$

In this case we have a finite-dimensional input space and

$$u(t) = (u_1(t), \cdots u_M(t)) \in \mathbb{R}^M .$$

c) **Type 1 Output:** Let $\partial\Omega$ be represented as a union: $\partial\Omega = \widetilde{\mathcal{S}}_0 \cup \widetilde{\mathcal{S}}$, where $\widetilde{\mathcal{S}}_0$ and $\widetilde{\mathcal{S}}$ have the same properties as $\mathcal{S}_0$ and $\mathcal{S}$ above. They may be the same subsets in the case of co-located actuators and sensors, or they may be different.

The output in this case is the trace of the state function on $\widetilde{\mathcal{S}}$:

$$y(x,t) = z(x,t), \qquad (x,t) \in \widetilde{\mathcal{S}} \times [0,\infty). \tag{2.6}$$

d) **Type 2 Output:** Let

$$\partial\Omega = \widetilde{\mathcal{S}} \cup \big(\cup_{j=1}^{K} \widetilde{\mathcal{S}}_j\big),$$

where the partition has the same properties as in b) above. We define the output vector

$$y(t) = (y_1(t), \cdots y_K(t)) \in \mathbb{R}^K \tag{2.7}$$

by the formula

$$y_m(t) = \frac{1}{\left|\widetilde{\mathcal{S}}_m\right|} \int_{\widetilde{\mathcal{S}}_m} z(x,t)\, d\sigma_x, \tag{2.8}$$

where by $d\sigma_x$ we denote the natural hypersurface measure on $\partial\Omega$.

Remark 2.1. The output (2.7), (2.8) is physically natural. Indeed, measuring the "temperature" $z(x,t)$ at the points $x \in \partial\Omega$ would produce the trace considered in (2.6). However, in reality a measuring device could only measure the average value of the "temperature" over a small subregion $\widetilde{\mathcal{S}}_m \subset \partial\Omega$. Therefore, instead of the function $y(x,t)$ defined on part of the boundary, it produces a discrete set of values as given in (2.8).

Combining the heat equation (2.1) with any of the above pairs of inputs and outputs we obtain four possible types of distributed parameter boundary control systems. (A rigorous definition of a "boundary control system" within the framework of abstract functional analysis can be found in [17]. Here we use this term just to indicate that our systems are governed by partial differential equations and the controls are implemented through the boundary conditions.) In order to identify a pairing of type 1 or type 2 input with a type 1 or type 2 output we will use the notation (i,j), where $i = 1,2$ refers to the type of input and $j = 1,2$ refers to the type of output. Thus we have four possible systems which we call type $(1,1)$, $(1,2)$, $(2,1)$ and $(2,2)$, corresponding to the types of inputs and outputs.

The main result of the present work, which will be stated precisely at the end of this section (in Theorems 2.6 and 2.8), is the following.

With appropriate choices of input and output spaces U and Y all of the above systems are regular linear systems.

To clarify the above statement, we need to recall some facts from the theory of regular linear systems, but we will not present their theory here. Instead, we organize our results in a series of five statements, which imply that our systems are regular (in particular, well-posed), according to the theory in [7], [24], [25]. In fact, the first four statements are a reformulation, in our specific framework, of the four conditions equivalent to well-posedness listed in [7], while the fifth statement, which is called Theorem 2.8, implies the main equivalent formulation of the regularity condition derived in [24, Theorem 1.3].

We turn now to the formulation of the aforementioned statements. In this formulation we need the input and output spaces U and Y. It is convenient for us to postpone the precise definitions of these spaces to the statement of our main results: Theorems 2.6 and 2.9. At this point it is only important that U and Y are Hilbert spaces (infinite or finite dimensional depending on the type of inputs or outputs).

Statement 1. *Define the state space of our system to be $X = L^2(\Omega)$. The uncontrolled system described by (2.1) with Neumann boundary condition*

$$\frac{\partial z}{\partial n}(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,\infty) \tag{2.9}$$

generates a strongly continuous semigroup $(\mathbb{T}_t)_{t\geq 0}$ on the space X. Moreover, $\mathbb{T}_t$ is a self-adjoint contraction for every $t \geq 0$.

This is a well known fact (see, e.g., [10, 12, 16]) which does not require a proof.

Statement 2. *Denote the input space by U both in the case of the* ***type 1 input*** *and* ***type 2 input****. Consider the initial-boundary value problem defined by (2.1) with the initial condition*

$$z(x,0) = 0, \tag{2.10}$$

and either the boundary conditions (2.2), (2.3) or the boundary conditions (2.4), (2.5). Assume that the controls in both problems are continuous functions of their arguments. Then each of the above problems has a classical solution $z(x,t)$. For each $\tau > 0$ define the map $u(\cdot) \to z(\cdot,\tau)$. The statement is that for any $\tau > 0$ this map can be extended to a bounded linear operator (called an input map)

$$\Phi_\tau \,:\, L^2([0,\tau],U) \to X. \tag{2.11}$$

We remark that, since the solvability theory for the above problems is well known, all that is necessary to prove the above statement is to establish (for each of the problems) the following a priori estimate:

$$\|z(\tau)\|^2 \leq M_1(\tau) \int_0^\tau \|u(t)\|_U^2 \, d\tau, \quad \tau > 0\,, \tag{2.12}$$

where $M_1(\tau) > 0$ is independent of u and is defined for all $\tau > 0$ (in fact, it will appear as a continuous function of $\tau \in [0,\infty)$). Here and later, $\|\cdot\|$ denotes the norm in X.

Statement 3. *Denote the output space by Y both in the case of **type 1 output** and in the case of **type 2 output**. Consider the uncontrolled initial-boundary problem defined by* (2.1)*, the homogeneous Neumann boundary condition* (2.9)*, and the initial condition*

$$z(x,0) = z_0(x), \quad z_0 \in X. \tag{2.13}$$

As is well known, this problem has a unique weak solution (which is also a classical solution) $z(x,t)$. Using this solution, we can evaluate either the output (2.6) *or the output* (2.7), (2.8)*. For every $\tau \geq 0$, define the map $z_0 \to y(\cdot)$. The statement is that for every $\tau > 0$, this map can be extended to a bounded linear operator (called the output map)*

$$\Psi_\tau : X \to L^2([0,\tau], Y).$$

To prove this statement, it is sufficient to establish (for each of the above two outputs) the following a priori estimate:

$$\int_0^\tau \|y(t)\|_Y^2 \, dt \leq M_2(\tau)\|z_0\|^2, \tag{2.14}$$

where $M_2(\tau) > 0$ is independent of z_0 and is defined for all $\tau > 0$.

Statement 4. *Consider again the initial-boundary problems from Statement* 2 *with continuous controls. For each of their classical solutions we calculate either the **type 1 output** or the **type 2 output*** ((2.6) *or* (2.7), (2.8))*. So, in each of the four cases, and on any finite time interval $[0,\tau]$, we can define the **input-output map** $u(\cdot) \to y(\cdot)$. The statement is that for any $\tau > 0$, this map can be extended to a bounded linear operator*

$$\mathbb{F}_\tau : L^2([0,\tau], U) \to L^2([0,\tau], Y).$$

To prove this statement, we have to establish in each case the a priori estimate:

$$\int_0^\tau \|y(t)\|_Y^2 \, dt \leq M_3(\tau) \int_0^\tau \|u(t)\|_U^2 \, dt, \tag{2.15}$$

where $M_3(\tau) > 0$ is independent of u and is defined for all $\tau > 0$.

We regard $L^2([0,\tau], U)$ as a subspace of $L^2_{\text{loc}}([0,\infty), U)$, and similarly for Y in place of U. We denote by $\mathbf{P}_\tau$ the operator of truncation to $[0,\tau]$, so that $(\mathbf{P}_\tau v)(t) = v(t)$ for $t \in [0,\tau]$ and $(\mathbf{P}_\tau v)(t) = 0$ for $t > \tau$. It is easy to show that the various operators $\mathbb{F}_\tau$ are compatible with each other, in the sense that for $t > \tau$ we have $\mathbf{P}_\tau \mathbb{F}_t = \mathbb{F}_\tau$. This implies that there exists a unique operator $\mathbb{F} : L^2_{\text{loc}}([0,\infty), U) \to L^2_{\text{loc}}([0,\infty), Y)$ such that $\mathbb{F}_\tau = \mathbf{P}_\tau \mathbb{F} = \mathbf{P}_\tau \mathbb{F} \mathbf{P}_\tau$, for all $\tau \geq 0$. This $\mathbb{F}$ is called the *extended input-output map* of the system. It is easy to see that the operator $\mathbb{F}$ is *shift-invariant*, which means that

$$\mathbb{F}\mathbf{S}_\tau = \mathbf{S}_\tau \mathbb{F} \qquad \forall\, \tau \geq 0,$$

where $\mathbf{S}_\tau$ is the operator of right shift by τ on $L^2_{\text{loc}}([0,\infty), U)$ and also on $L^2_{\text{loc}}([0,\infty), Y)$.

For any $\alpha \in \mathbb{R}$ and Hilbert space W define the weighted space

$$L^2_\alpha([0,\infty), W) = \left\{ f \in L^2_{\text{loc}}([0,\infty), W) : \int_0^\infty \|f(t)\|_W^2 \, e^{-2\alpha t} \, dt < \infty \right\}$$

with the norm being the square root of the integral appearing above.

Proposition 2.2. *For any $\beta > 0$, the extended input-output map $\mathbb{F}$ can be restricted to a bounded linear operator*

$$\mathbb{F} \,:\, L^2_\beta([0,\infty),U) \to L^2_\beta([0,\infty),Y)\,.$$

The above proposition is equivalent to the a priori estimate

$$\int_0^\infty \|y(t)\|_Y^2\, e^{-2\beta t}\, dt \leq M_0(\beta) \int_0^\infty \|u(t)\|_U^2\, e^{-2\beta t}\, dt \tag{2.16}$$

for all $\beta > 0$ (here, $M_0(\beta) > 0$ is a constant independent of u). Proposition 2.2 follows from Statement 4 and the fact that the growth bound of the semigroup $\mathbb{T}$ is zero, see [24, Propositions 3.2 and 4.1]. We will also give an independent short proof of Proposition 2.2 using directly the specific form of our systems.

We recall some facts about transfer functions. Let W be a Banach space, $\alpha \in \mathbb{R}$, and $g \in L^2_\alpha([0,\infty),W)$. The Laplace transform $\widehat{g}$ of g is defined by the formula

$$\widehat{g}(s) \;=\; \int_0^\infty e^{-st} g(t)\, dt\,.$$

The above integral is convergent and defines $\widehat{g}(s) \in W$ for any $s \in \mathbb{C}$ such that $\mathrm{Re}\,(s) > \alpha$. In fact, $\widehat{g}(s)$ is an analytic W-valued function on the right half-plane

$$\mathbb{C}_\alpha \;=\; \{s \in \mathbb{C} \,:\, \mathrm{Re}\,(s) > \alpha\}\,.$$

(We note that $\widehat{g}$ might have analytic extensions to larger half-planes.) We denote by $H^\infty_\alpha(W)$ the space of all bounded and analytic W-valued functions on $\mathbb{C}_\alpha$.

Denote by $\mathcal{L}(W_1,W_2)$ the Banach space of all bounded linear operators from a Hilbert space W_1 to a Hilbert space W_2. We need the following proposition, which is a corollary of [24, Theorem 3.6 and Proposition 3.2].

Proposition 2.3. *Let $\mathbb{F}$ be a shift-invariant bounded linear operator from $L^2_\beta([0,\infty),U)$ to $L^2_\beta([0,\infty),Y)$, for some $\beta \in \mathbb{R}$. Then there exists a unique $\mathbf{G} \in H^\infty_\beta(\mathcal{L}(U,Y))$ such that $y = \mathbb{F}u$ if and only if*

$$\widehat{y}(s) \;=\; \mathbf{G}(s)\widehat{u}(s) \quad \forall\; s \in \mathbb{C}_\beta\,.$$

This $\mathbf{G}$ is called the *transfer function* of $\mathbb{F}$. Of course, $\mathbf{G}$ may have analytic extensions to larger domains in $\mathbb{C}$. An operator-valued analytic function defined on a domain in $\mathbb{C}$ is called *well-posed* if its domain includes a half-plane $\mathbb{C}_\alpha$ such that the function is bounded on $\mathbb{C}_\alpha$. Thus, any transfer function occurring as in the last proposition is well-posed. It follows from the last two propositions that the extended input-output map of our heat equation system (with any of the two possible inputs and the two possible outputs) has a well-posed transfer function $\mathbf{G}$. More precisely, $\mathbf{G}$ is defined on $\mathbb{C}_0$ and it is bounded on $\mathbb{C}_\beta$ for every $\beta > 0$. This $\mathbf{G}$ is also called the transfer function of the system.

We recall from [24] the following equivalent characterization of regular transfer functions (since it is equivalent to the original definition of regularity, which we do not recall here, we state this characterization as a definition).

Definition 2.4. Let $\mathbf{G}$ be a well-posed $\mathcal{L}(U, Y)$-valued function. $\mathbf{G}$ is called *regular* if the strong limit

$$\lim_{\substack{s \to +\infty \\ s \in \mathbb{R}}} \mathbf{G}(s) \;=\; D \in \mathcal{L}(U, Y) \tag{2.17}$$

exists, i.e., for any $v \in U$, $\|\mathbf{G}(s)v - Dv\|_Y \to 0$ as $s \to +\infty$, $s \in \mathbb{R}$. The operator D defined in (2.17) is then called the *feedthrough operator* of $\mathbf{G}$.

If $\mathbf{G}$ is the transfer function of a well-posed system Σ and $\mathbf{G}$ is regular, then the system is called regular and D from (2.17) is called the feedthrough operator of Σ.

It happens sometimes for systems governed by PDEs that a condition stronger than (2.17) is satisfied, namely, the limit (2.17) exists in the norm topology, see for example [1] or our Theorem 2.9. In view of this fact, we suggest the following definition.

Definition 2.5. Let $\mathbf{G}$ be a well-posed $\mathcal{L}(U, Y)$-valued function. $\mathbf{G}$ is called *uniformly regular* if there exists $D \in \mathcal{L}(U, Y)$ such that

$$\lim_{\substack{s \to +\infty \\ s \in \mathbb{R}}} \|\mathbf{G}(s) - D\| \;=\; 0\,,$$

i.e., $\mathbf{G}(s)$ converges to the feedthrough operator D in the operator norm topology.

If the transfer function of a well-posed system Σ is uniformly regular, we say that Σ is uniformly regular. Clearly, a uniformly regular system is regular.

Now we are in a position to give precise statements of the main results of this work.

Theorem 2.6. *With the choices $U = H^{-1/2}(\mathcal{S})$ and $Y = H^{1/2}(\widetilde{\mathcal{S}})$ the system of type $(1,1)$ is well-posed, i.e., Statements $1-4$ hold.*

Corollary 2.7. *The systems of type $(1,2)$, $(2,1)$, and $(2,2)$ are all well-posed also with the following choices of U and Y:*

$U = H^{-1/2}(\mathcal{S})$, $\quad Y = \mathbb{R}^K$ *for systems of type* $(1,2)$;

$U = \mathbb{R}^M$, $\quad Y = H^{1/2}(\widetilde{\mathcal{S}})$ *for systems of type* $(2,1)$;

$U = \mathbb{R}^M$, $\quad Y = \mathbb{R}^K$ *for systems of type* $(2,2)$.

Here, it is of course assumed that $\mathbb{R}^M$ and $\mathbb{R}^K$ are equipped with the Euclidean norms.

Theorem 2.8. *With the choices $U = H^{-1/2}(\mathcal{S})$ and $Y = H^{1/2}(\widetilde{\mathcal{S}})$ the system of type $(1,1)$ is regular with feedthrough operator $D = 0$, i.e., in addition to Statements $1-4$ the condition (2.17) holds as well.*

We give two proofs for this theorem: one with a PDE flavour in Section 4 and one with an operator theory flavour in Section 7.

Theorem 2.9. *The system of type* $(1,1)$ *is uniformly regular, with feedthrough operator* $D=0$*, with either of the following two choices of input and output spaces:*

$$(i) \qquad U = H^{-1/2}(\mathcal{S}), \quad Y = H^{1/2-\varepsilon}(\widetilde{\mathcal{S}}), \qquad \text{where} \qquad \varepsilon > 0\,; \tag{2.18}$$

$$(ii) \qquad U = H^{-1/2+\varepsilon}(\mathcal{S}), \quad Y = H^{1/2}(\widetilde{\mathcal{S}}), \qquad \text{where} \qquad \varepsilon > 0\,. \tag{2.19}$$

Moreover, the following estimates hold for $\|\mathbf{G}(s)\|$ *for real* s*:*

$$\text{In case } (i) \qquad \|\mathbf{G}(s)\| \;\leq\; M_1(s) \;=\; C_1(\varepsilon)\;\; (2s-1)^{-\frac{\varepsilon}{2}}, \qquad \text{for} \qquad s \geq 1; \tag{2.20}$$

$$\text{In case } (ii) \quad \|\mathbf{G}(s)\| \;\leq\; M_2(s) \;=\; C_2(\varepsilon)\;\; (2s-1)^{-\frac{\varepsilon}{2(1+\varepsilon)}}, \qquad \text{for} \qquad s \geq 1. \tag{2.21}$$

Here, $C_1(\varepsilon), C_2(\varepsilon) \geq 0$ *depend only on* ε *and on the domain* Ω*.*

Remark 2.10. a) Each of the estimates (2.20) and (2.21) is sufficient for the system $(1,1)$ to be uniformly regular with $D=0$. Indeed, it follows from these estimates that $\lim \|\mathbf{G}(s)\| = 0$ (for $s \in \mathbb{R}$, $s \to +\infty$), so that (2.17) holds and $D=0$.

b) In all of the above three theorems our assumptions about the input and output spaces are on the borderline: if we take $U = H^{-1/2-\beta}(\mathcal{S})$ and $Y = H^{1/2+\alpha}(\widetilde{\mathcal{S}})$ with $\alpha \geq 0$, $\beta \geq 0$, $\alpha+\beta > 0$, then the resulting system is no longer well-posed.

c) If for the system of type $(1,1)$ we define the input and output spaces by $U = L^2(\mathcal{S})$ and $Y = L^2(\widetilde{\mathcal{S}})$, then the system is well-posed and regular with $D=0$. Moreover, the system is uniformly regular. This follows easily from Theorems 2.8 and 2.9, using the continuous embeddings

$$L^2(\mathcal{S}) \hookrightarrow H^{-1/2}(\mathcal{S}) \quad \text{and} \quad H^{1/2}(\widetilde{\mathcal{S}}) \hookrightarrow L^2(\widetilde{\mathcal{S}}).$$

Corollary 2.11. *The systems of type* $(1,2)$*,* $(2,1)$ *and* $(2,2)$ *are uniformly regular with* $D=0$*, if we make the same choices of* U *and* Y *as in Corollary* 2.7*.*

Remark 2.12. In the next three sections we prove Theorems 2.6, 2.8 and 2.9. Corollaries 2.7 and 2.11 are obvious, because the second output is obtained from the first via a bounded operator (defined in (2.8)). This operator maps $H^{1/2}(\widetilde{\mathcal{S}})$ onto $\mathbb{R}^M$. The dual statement holds for the controls, i.e., the second input space $\mathbb{R}^K$ is embedded into the first input space $H^{-1/2}(\mathcal{S})$ via the bounded operator which maps $(u_1, \cdots u_K) \in \mathbb{R}^K$ into the function $\sum_{j=1}^K u_j \chi_{\mathcal{S}_j}$, where $\chi_{\mathcal{S}_j}$ is the characteristic function of $\mathcal{S}_j$.

The above paragraph shows that the system of type $(1,1)$ is the main system. The results for the systems of type $(1,2)$, $(2,1)$ and $(2,2)$ are easy consequences of the main case. We have introduced all four types of systems because they are all meaningful from the point of view of applications.

3. Proof of well-posedness

In this section we prove Theorem 2.6, i.e., we show that Statements 1-4 hold for the system defined by (2.1), (2.2), (2.3) and (2.6). For this, we derive all the necessary a priori estimates (2.12), (2.14), (2.15) and (2.16). We recall that the solvability theory for parabolic equations is well known (see, e.g., [10, 12]) and, as was indicated in Section 2, Statement 1 does not require any proof, since it is well known.

***Proof of Statement* 2**. We consider the initial-boundary problem (2.1), (2.2), (2.3) with the zero initial condition (2.10). Taking the X-inner product of both sides of (2.1) with $z(x,t)$, integrating by parts, and using the boundary conditions, we obtain

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2 + \|\nabla z(t)\|^2 \;=\; \int_{\mathcal{S}} u(x,t)z(x,t)\,d\sigma_x\,. \tag{3.1}$$

To estimate the right-hand side of (3.1), we recall that the Dirichlet trace operator

$$\gamma_0(\mathcal{S})\;:\;H^1(\Omega)\to H^{1/2}(\mathcal{S})$$

is bounded, so that

$$\|\gamma_0(\mathcal{S})z(t)\|_{H^{1/2}(\mathcal{S})} \le m_0\|z(t)\|_{H^1(\Omega)}, \tag{3.2}$$

where $m_0 > 0$ is a constant depending on Ω and $\mathcal{S}$. We denote the integral on the right-hand side of (3.1) by $\langle u(t), z(t)\rangle$ and treat it as the duality pairing between $H^{-1/2}(\mathcal{S})$ and $H^{1/2}(\mathcal{S})$. We have, using (3.2),

$$\begin{aligned}\big\langle u(t), z(t)\big\rangle &\le \|u(t)\|_{H^{-1/2}(\mathcal{S})}\|z(t)\|_{H^{1/2}(\mathcal{S})}\\ &\le \varepsilon\|z(t)\|^2_{H^{1/2}(\mathcal{S})} + \frac{1}{4\varepsilon}\|u(t)\|^2_{H^{-1/2}(\mathcal{S})}\\ &\le \varepsilon m_0^2\|z(t)\|^2_{H^1(\Omega)} + \frac{1}{4\varepsilon}\|u(t)\|^2_{H^{-1/2}(\mathcal{S})}\,.\end{aligned}$$

Substituting this into (3.1) and using the fact that

$$\|z\|^2_{H^1(\Omega)} \;=\; \|z\|^2 + \|\nabla z\|^2,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2 + (1-\varepsilon m_0^2)\|\nabla z(t)\|^2 \le \varepsilon m_0^2\|z(t)\|^2 + \frac{1}{4\varepsilon}\|u(t)\|^2_{H^{-1/2}(\mathcal{S})}. \tag{3.3}$$

Now we select ε so that $\varepsilon m_0^2 = \dfrac{1}{2}$, so that (3.3) becomes

$$\frac{d}{dt}\|z(t)\|^2 + \|\nabla z(t)\|^2 \;\le\; \|z(t)\|^2 + m_0^2\|u(t)\|^2_{H^{-1/2}(\mathcal{S})}. \tag{3.4}$$

Omitting the second term on the left, taking into account that $z(0) = 0$, and applying Gronwall's lemma, we obtain

$$\|z(\tau)\|^2 \;\le\; m_0^2\int_0^\tau e^{(\tau-t)}\|u(t)\|^2_{H^{-1/2}(\mathcal{S})}\,dt\,. \tag{3.5}$$

This implies (2.12) with $M_1(\tau) = m_0^2 e^\tau$. □

Remark 3.1. The formula (3.1) implies that, choosing $\mathcal{S} = \widetilde{\mathcal{S}}$ and U, Y as in Theorem 2.6, we have

$$\frac{d}{dt}\|z(t)\|^2 \;\le\; 2\langle u(t), y(t)\rangle_{U,Y}\,.$$

This means that the system from Theorem 2.6 is *passive* in the sense of van der Schaft [18]. For a detailed treatment of passive infinite-dimensional systems see Staffans [21].

***Proof of Statement* 3**. We consider the "uncontrolled" initial boundary value problem (2.1), (2.9), (2.13). For this problem the right-hand side of the relation (3.1) is zero and, therefore, integrating over the interval $[0, \tau]$ and taking into account the initial condition (2.13), we obtain

$$\|z(\tau)\|^2 + 2\int_0^\tau \|\nabla z(t)\|^2\,dt \;=\; \|z_0\|^2.$$

From here, using also that $\mathbb{T}$ is a contraction semigroup, we see that

$$\int_0^\tau \|\nabla z(t)\|^2\,dt \;\le\; \frac{1}{2}\|z_0\|^2 \quad \text{and} \quad \int_0^\tau \|z(t)\|^2\,dt \le \tau\,\|z_0\|^2\,.$$

Therefore,

$$\int_0^\tau \|z(t)\|^2_{H^1(\Omega)}\,dt \le \left(\frac{1}{2}+\tau\right)\|z_0\|^2. \tag{3.6}$$

Combining (3.6) with (3.2) with $\mathcal{S}$ replaced by $\widetilde{\mathcal{S}}$ and recalling that for the output (2.7) we have

$$\|y(t)\|_Y = \|z(t)\|_{H^{1/2}(\widetilde{\mathcal{S}})},$$

we immediately obtain the desired estimate (2.14) with $M_2(\tau) = m_0^2\big(1/2+\tau\big)$. □

***Proof of Statement* 4**. We again consider the system (2.1), (2.2), (2.3), (2.6) with zero initial condition (2.10): $z_0 = 0$. To prove the statement we have to derive the inequality (2.15) as an a priori estimate.

For the above system we have inequality (3.4) obtained in the proof of Statement 2. Multiplying both sides of (3.4) by e^{-t}, integrating from 0 to τ, and taking into account that $z(0) = 0$, we obtain

$$\|z(\tau)\|^2 e^{-\tau} + \int_0^\tau \|\nabla z(t)\|^2\,e^{-t}\,dt \le m_0^2\int_0^\tau \|u(t)\|^2_{H^{-1/2}(\mathcal{S})}\,e^{-t}\,dt. \tag{3.7}$$

Multiplying both sides of (3.7) by e^τ, omitting the first term on the left and taking into account that $1 \le e^{(\tau-t)} \le e^\tau$, we obtain

$$\int_0^\tau \|\nabla z(t)\|^2\,dt \le m_0^2\,e^\tau\int_0^\tau \|u(t)\|^2_{H^{-1/2}(\mathcal{S})}\,dt\,. \tag{3.8}$$

Now recall the inequality (3.5). Replacing in (3.5) τ by t, t by ξ, and integrating with respect to the new variable t from 0 to τ, we obtain

$$\int_0^\tau \|z(t)\|^2\,dt \le m_0^2\int_0^\tau\int_0^t e^{(t-\xi)}\,\|u(\xi)\|^2_{H^{-1/2}}\,d\xi\,dt \tag{3.9}$$

$$= m_0^2 \int_0^\tau \left(e^{(\tau-\xi)} - 1\right) \|u(\xi)\|^2_{H^{-1/2}}\, d\xi \;\le\; m_0^2\,(e^\tau - 1) \int_0^\tau \|u(t)\|^2_{H^{-1/2}}\, dt.$$

Here on the second step we changed the order of integration and performed an elementary integration with respect to t. It remains to add the inequalities (3.8) and (3.9), take into account (3.2), and recall that

$$\|y(t)\|_Y \;=\; \|\gamma_0(\mathcal{S}) z(t)\|_{H^{1/2}(\mathcal{S})}\,.$$

As a result we arrive at the desired estimate (2.15) with $M_3(\tau) = m_0^4\,(2e^\tau - 1)$. □

As has already been mentioned in Section 2, [24, Proposition 3.2] implies that Statement 4 and Proposition 2.2 are equivalent. However, we will show that in our specific context, (2.16) can be easily derived as an a priori estimate and, moreover, that it holds for any $\beta > 0$ with a constant $M_0(\beta)$ such that $M_0(\beta) \to \infty$ as $\beta \to 0$.

***Proof of Proposition* 2.2.** We return to the inequality (3.3) from the proof of Statement 2. Let us select any $\delta > 0$ and pick ε in (3.3) so that $\varepsilon m_0^2 = \dfrac{\delta}{2}$. In this case (3.3) takes the form

$$\frac{d}{dt}\|z(t)\|^2 + (2-\delta)\|\nabla z(t)\|^2 \le \delta\|z(t)\|^2 + \frac{m_0^2}{\delta}\|u(t)\|^2_{H^{-1/2}(\mathcal{S})}. \tag{3.10}$$

Multiplying both sides of (3.10) by $e^{-\delta t}$ we can rewrite it in the form

$$\frac{d}{dt}\left(\|z(t)\|^2 e^{-\delta t}\right) + (2-\delta)\|\nabla z(t)\|^2 e^{-2\delta t} \le \frac{m_0^2}{\delta}\|u(t)\|^2_{H^{-1/2}(\mathcal{S})} e^{-\delta t}\,. \tag{3.11}$$

Integrating (3.11) from 0 to τ and keeping in mind that $z_0 = 0$, we obtain

$$\|z(\tau)\|^2 e^{-\delta\tau} + (2-\delta)\int_0^\tau \|\nabla z(t)\|^2 e^{-2\delta t}\, dt \le \frac{m_0^2}{\delta}\int_0^\tau \|u(t)\|^2_{H^{-1/2}(\mathcal{S})} e^{-\delta t}\, dt\,.$$

We multiply both sides above by $e^{-\delta\tau}$ and integrate from 0 to ∞. Taking into account the formula (f is any function such that the integrals are convergent)

$$\int_0^\infty e^{-\delta\tau}\left(\int_0^\tau f(t)\, e^{-\delta t}\, dt\right) d\tau \;=\; \frac{1}{\delta}\int_0^\infty f(t) e^{-2\delta t}\, dt\,,$$

we obtain

$$\int_0^\infty \|z(t)\|^2\, e^{-2\delta t}\, dt + \left(\frac{2}{\delta} - 1\right)\int_0^\infty \|\nabla z(t)\|^2\, e^{-2\delta t}\, dt \le \frac{m_0^2}{\delta^2}\int_0^\infty \|u(t)\|^2_{H^{-1/2}(\mathcal{S})} e^{-2\delta t}\, dt\,.$$

This immediately implies that

$$\int_0^\infty \|z(t)\|^2_{H^1(\Omega)}\, e^{-2\delta t}\, dt \le \widetilde{M}_0(\delta)\int_0^\infty \|u(t)\|^2_{H^{-1/2}(\mathcal{S})} e^{-2\delta t}\, dt \tag{3.12}$$

where

$$\widetilde{M}_0(\delta) \;=\; \left[\min\,\left(1; \frac{2}{\delta} - 1\right)\right]^{-1} \frac{m_0^2}{\delta^2}\,.$$

Combining (3.12) with (3.2) we obtain the desired estimate (2.16) with $\beta = 2\delta$ and

$$M_0 \;=\; M_0(\beta) \;=\; m_0^2 \widetilde{M}_0(\beta/2)\,.$$ □

4. Proof of regularity

In this section we prove regularity, i.e., Theorem 2.8. Consider the control system (2.1), (2.2), (2.3), (2.6) and assume that the initial state is $z_0 = 0$. Denote by $\widehat{z}(x,s)$, $\widehat{u}(x,s)$, and $\widehat{y}(x,s)$ the Laplace transforms of the state, input, and output respectively. As usual, for a fixed $s \in \mathbb{C}$ we will use the notation $\widehat{z}(s) = \widehat{z}(\cdot, s) \in X$, $\widehat{u}(s) = \widehat{u}(\cdot, s) \in U$, and $\widehat{y}(s) = \widehat{y}(\cdot, s) \in Y$ for $\widehat{z}$, $\widehat{u}$ and $\widehat{y}$ considered as functions of x only.

Applying the Laplace transformation to (2.1), (2.2), (2.3), (2.6) and taking into account that $z(x,0) = 0$, we arrive at the following elliptic boundary value problem:

$$s\widehat{z}(x,s) = \Delta \widehat{z}(x,s), \tag{4.1}$$

$$\frac{\partial \widehat{z}}{\partial n}(x,s) = 0, \quad x \in \mathcal{S}_0, \tag{4.2}$$

$$\frac{\partial \widehat{z}}{\partial n}(x,s) = \widehat{u}(x,s), \quad x \in \mathcal{S}, \tag{4.3}$$

$$\widehat{y}(x,s) = \widehat{z}(x,t), \quad x \in \widetilde{\mathcal{S}}. \tag{4.4}$$

Let us replace in (4.3) the Laplace transform $\widehat{u}(x,s)$ of the control $u(x,t)$ by the function $v(x)$ independent of s. In other words, consider instead of (4.3) the boundary condition

$$\frac{\partial \widehat{z}}{\partial n}(x,s) = v(x), \quad x \in \mathcal{S}. \tag{4.5}$$

We consider the boundary value problem (4.1), (4.2), (4.4), (4.5). To prove Theorem 2.8, we only have to show that the following holds for the solutions of this problem:

$$\lim_{\substack{s \to +\infty \\ s \in \mathbb{R}}} \|\widehat{y}(s)\|_Y = \lim_{\substack{s \to +\infty \\ s \in \mathbb{R}}} \|\widehat{z}(s)\|_{H^{1/2}(\widetilde{\mathcal{S}})} = 0 \quad \forall\ v \in U = H^{-1/2}(\mathcal{S}). \tag{4.6}$$

Indeed, the well-posedness of our system has already been established. So, it remains to prove that (2.17) holds with $D = 0$. But, (4.6) is equivalent to

$$\lim_{\substack{s \to +\infty \\ s \in \mathbb{R}}} \|\mathbf{G}(s)v\|_Y = 0 \quad \forall\ v \in U,$$

i.e., it is equivalent to (2.17) with $D = 0$.

We claim now that (4.6) is a corollary of the following two facts about the solutions of the problem (4.1), (4.2), (4.4) and (4.5).

Fact 1. There exists an $M_0 > 0$ such that

$$\|\widehat{z}(s)\|_{H^{1/2}(\widetilde{\mathcal{S}})} \le M_0 \|v\|_{H^{-1/2}(\mathcal{S})}, \quad \text{for} \quad s \ge 1 \quad \text{and for any} \quad v \in H^{-1/2}(\mathcal{S}). \tag{4.7}$$

Fact 2. (4.6) holds for any $v \in L^2(\mathcal{S})$, i.e.,

$$\lim_{\substack{s \to +\infty \\ s \in \mathbb{R}}} \|\widehat{z}(s)\|_{H^{1/2}(\mathcal{S})} = 0 \quad \forall\ v \in L^2(\mathcal{S}). \tag{4.8}$$

To explain the above claim, we notice that (4.7) is equivalent to the estimate $\|\mathbf{G}(s)\| \le M_0$ for all real $s \ge 1$ and hence, (4.8) implies (4.6), because $L^2(\mathcal{S})$ is dense in $H^{-1/2}(\mathcal{S})$.

***Proof of* (4.7).** Calculating the $L^2(\Omega)$ inner product of both sides of (4.1) with $\widehat{z}(\cdot, s)$, integrating by parts, and taking into account the boundary conditions (4.2), (4.5), we obtain the relation

$$s\,\|\widehat{z}(s)\|^2 + \|\nabla\widehat{z}(s)\|^2 \;=\; \int_{\mathcal{S}} v(x)\widehat{z}(x,s)\,d\sigma_x\,. \tag{4.9}$$

Treating the right-hand side of (4.9) as the duality pairing $\langle v(s), \widehat{z}(s)\rangle$ between $H^{-1/2}(\mathcal{S})$ and $H^{1/2}(\mathcal{S})$, we obtain

$$\begin{aligned}\left|\int_{\mathcal{S}} v(x)\widehat{z}(x,s)\,d\sigma_x\right| &\le \|\widehat{z}(s)\|_{H^{1/2}(\mathcal{S})}\|v\|_{H^{-1/2}(\mathcal{S})}\\ &\le \delta\|\widehat{z}(s)\|^2_{H^{1/2}(\mathcal{S})} + \frac{1}{4\delta}\|v\|^2_{H^{-1/2}(\mathcal{S})}\\ &\le \delta m_0^2\|\widehat{z}(s)\|^2_{H^1(\Omega)} + \frac{1}{4\delta}\|v\|^2_{H^{-1/2}(\mathcal{S})}\,. \end{aligned}\tag{4.10}$$

In the third step above we have used the boundedness of the Dirichlet trace operator γ_0, see the estimate (3.2). Combining (4.9), (4.10) and using that $\|\widehat{z}(s)\|^2_{H^1(\Omega)} = \|\widehat{z}(s)\|^2 + \|\nabla\widehat{z}(s)\|^2$, we obtain the inequality

$$(s-\delta m_0^2)\|\widehat{z}(s)\|^2 + (1-\delta m_0^2)\|\nabla\widehat{z}(s)\|^2 \le \frac{1}{4\delta}\|v\|^2_{H^{-1/2}(\mathcal{S})}\,.$$

Choosing $\delta m_0^2 = \frac{1}{2}$, we rewrite the above inequality as

$$(2s-1)\|\widehat{z}(s)\|^2 + \|\nabla\widehat{z}(s)\|^2 \le m_0^2\|v\|^2_{H^{-1/2}(\mathcal{S})}\,. \tag{4.11}$$

Assuming $s \ge 1$ and replacing the coefficient $(2s-1)$ in the first term in (4.11) by 1, we obtain

$$\|\widehat{z}(s)\|_{H^1(\Omega)} \le m_0\|v\|_{H^{-1/2}(\mathcal{S})}, \quad s \ge 1\,. \tag{4.12}$$

Combining (4.12) with the inequality (3.2), in which $\mathcal{S}$ is replaced by $\widetilde{\mathcal{S}}$, we immediately obtain (4.7) with $M_0 = m_0^2$. $\square$

***Proof of* (4.8).** In this proof we will use a well known interpolation-type (or multiplicative) inequality (see, e.g., [10, Chapter 2, (2.21)] and Remark 4.2 below).

Proposition 4.1. *The following inequality holds for any $\varphi \in H^1(\Omega)$:*

$$\|\varphi\|_{L^2(\mathcal{S})} \le \gamma\|\varphi\|^{1/2}_{H^1(\Omega)}\|\varphi\|^{1/2}, \tag{4.13}$$

where $\gamma = \gamma(\Omega, \mathcal{S}) > 0$ is a constant depending only on Ω and $\mathcal{S}$.

We will derive (4.8) as a corollary of a priori estimates for the problem (4.1), (4.2), (4.4), (4.5) with $v \in L^2(\mathcal{S})$ in the boundary condition (4.5).

We return to the relation (4.9) and estimate the right-hand side using the Cauchy-Schwarz inequality in $L^2(\mathcal{S})$. We obtain

$$s\,\|\widehat{z}(s)\|^2 + \|\nabla\widehat{z}(s)\|^2 \le \|\widehat{z}(s)\|_{L^2(\mathcal{S})}\|v\|_{L^2(\mathcal{S})}\,.$$

Replacing the coefficient s in the first term above by 1, we obtain

$$\|\widehat{z}(s)\|^2_{H^1(\Omega)} \le \|\widehat{z}(s)\|_{L^2(\mathcal{S})}\|v\|_{L^2(\mathcal{S})}, \quad s \ge 1. \tag{4.14}$$

Now we return to the inequality (4.11). Combining (4.11) with the bounded embedding $L^2(\mathcal{S}) \hookrightarrow H^{-1/2}(\mathcal{S})$, i.e., with the inequality $\|v\|_{H^{-1/2}(\mathcal{S})} \le c_0\|v\|_{L^2(\mathcal{S})}$, we obtain

$$(2s-1)\|\widehat{z}(s)\|^2 + \|\nabla\widehat{z}(s)\|^2 \le m_0^2c_0^2\|v\|^2_{L^2(\mathcal{S})}\,.$$

Omitting here the second term on the left, we obtain

$$\|\widehat{z}(s)\| \le m_0c_0(2s-1)^{-\frac{1}{2}}\|v\|_{L^2(\mathcal{S})}, \quad s > \frac{1}{2}\,. \tag{4.15}$$

Now we are prepared to estimate the norm of the output

$$\|\widehat{y}(s)\|_Y = \|\widehat{z}(s)\|_{H^{1/2}(\widetilde{\mathcal{S}})}$$

under the assumption $v \in L^2(\mathcal{S})$. First, we use (4.13) with $\varphi = \widehat{z}(s)$ to estimate $\|\widehat{z}(s)\|_{L^2(\mathcal{S})}$ in the right-hand side of (4.14). We have

$$\|\widehat{z}(s)\|^2_{H^1(\Omega)} \le \gamma\|\widehat{z}(s)\|^{1/2}_{H^1(\Omega)}\|\widehat{z}(s)\|^{1/2}\|v\|_{L^2(\mathcal{S})}, \quad s \ge 1\,,$$

and therefore,

$$\|\widehat{z}(s)\|_{H^1(\Omega)} \le \gamma^{2/3}\|\widehat{z}(s)\|^{1/3}\|v\|^{2/3}_{L^2(\mathcal{S})}, \quad s \ge 1\,.$$

Using (4.15) to estimate the right-hand side above, we obtain

$$\|\widehat{z}(s)\|_{H^1(\Omega)} \le \gamma^{2/3}(m_0c_0)^{1/3}(2s-1)^{-1/6}\|v\|_{L^2(\mathcal{S})}, \quad s \ge 1\,.$$

Finally, we recall the inequality (3.2) with $\mathcal{S}$ replaced by $\widetilde{\mathcal{S}}$ and we combine it with the above inequality to obtain

$$\|\widehat{z}(s)\|_{H^{1/2}(\widetilde{\mathcal{S}})} \le C_1(2s-1)^{-1/6}\|v\|_{L^2(\mathcal{S})}, \quad s \ge 1\,, \tag{4.16}$$

where $C_1 = \gamma^{2/3}m_0^{4/3}c_0^{1/3}$. Now notice that (4.8) follows immediately from (4.16). $\square$

Remark 4.2. We have referred to [10, Chapter 2] as a source for Proposition 4.1. In fact, in the above reference an inequality slightly different from (4.13) is used:

$$\|\psi\|_{L^2(\mathcal{S})} \le \gamma_0\|\nabla\psi\|^{\frac{1}{2}}\|\psi\|^{\frac{1}{2}}, \tag{4.17}$$

for any $\psi \in H^1(\Omega)$ such that $\int_\Omega \psi(x)\,dx = 0$. (4.13) can be easily derived from (4.17). Indeed, for any $\varphi \in H^1(\Omega)$ define its average value by

$$\langle\varphi\rangle = \frac{1}{|\Omega|}\int_\Omega \varphi(x)\,dx,$$

where $|\Omega|$ is the volume of Ω. Define $\psi = \varphi - \langle\varphi\rangle$. Clearly, $\int_\Omega \psi(x)\,dx = 0$ and, therefore, (4.17) holds for ψ. Substituting $\psi = \varphi - \langle\varphi\rangle$ into (4.17) and using the triangle inequality and the obvious inequality $\big|\langle\varphi\rangle\big| \le |\Omega|^{-1/2}\|\varphi\|$, we can obtain (4.13) after a short calculation.

5. Proof of uniform regularity

In this section we prove Theorem 2.9. Consider again the elliptic boundary value problem (4.1), (4.2), (4.4), (4.5), with $\widehat{u}(x,s) = v(x)$. All we need to do is to show that the following inequality holds for the solution of this problem:

$$\|\widehat{y}(s)\|_Y \le M(s)\|v\|_U, \quad s \ge 1, \tag{5.1}$$

where $M(s) = M_1(s)$ from (2.20) if Y and U are defined by (2.18) (*Case (i)*) or $M(s) = M_2(s)$ from (2.19) if Y and U are defined by (2.19) (*Case (ii)*). For both cases we have to derive (5.1) as an a priori estimate. We treat these cases separately.

Proposition 5.1. *The Dirichlet trace operator*

$$\gamma_0(\mathcal{S}) \,:\, H^\alpha(\Omega) \to H^{\alpha-\frac{1}{2}}(\mathcal{S}), \quad \alpha > \frac{1}{2}, \tag{5.2}$$

is bounded and, therefore, for any $\varphi \in H^{1-\varepsilon}(\Omega)$ with $\varepsilon \in [0, \frac{1}{2})$ we have

$$\|\varphi\|_{H^{1/2-\varepsilon}(\mathcal{S})} \le m_\varepsilon \|\varphi\|_{H^{1-\varepsilon}(\Omega)}, \tag{5.3}$$

where $m_\varepsilon = m_\varepsilon(\Omega, \mathcal{S}) > 0$ is a constant depending on Ω, $\mathcal{S}$ and ε.

Note that (5.3) is a generalization of (3.2). For the proof of this and also of the next proposition, we refer to [12].

Proposition 5.2. *Let $0 \le s_1 \le s_2$ and $0 \le \theta \le 1$. The following interpolation inequality holds for any $\varphi \in H^{s_2}(\Omega)$:*

$$\|\varphi\|_{H^{\theta s_1 + (1-\theta)s_2}(\Omega)} \le C\|\varphi\|^\theta_{H^{s_1}(\Omega)} \|\varphi\|^{1-\theta}_{H^{s_2}(\Omega)}, \tag{5.4}$$

where $C = C_\Omega(\theta, s_1, s_2) > 0$ is a constant depending only on Ω, θ, s_1 and s_2.

In fact, we will need a particular case of (5.4) with $s_1 = 0$, $s_2 = 1$. Namely,

$$\|\varphi\|_{H^{1-\theta}(\Omega)} \le C_\theta \|\varphi\|^\theta \|\varphi\|^{1-\theta}_{H^1(\Omega)}, \tag{5.5}$$

where $C_\theta = C_\Omega(\theta, 0, 1)$. Now we turn to the derivation of (5.1).

Case (i): $U = H^{-1/2}(\mathcal{S})$, $Y = H^{1/2-\varepsilon}(\widetilde{\mathcal{S}})$, $0 < \varepsilon < 1/2$.

Omitting in (4.11) the second term on the left we obtain

$$\|\widehat{z}(s)\| \le m_0 (2s-1)^{-1/2} \|v\|_{H^{-1/2}(\mathcal{S})}, \quad s > \frac{1}{2}. \tag{5.6}$$

We also have to recall the inequality (4.12) which follows from (4.11).

Now we are ready to estimate the norm of the output:

$$\|\widehat{y}(s)\|_Y = \|\widehat{z}(s)\|_{H^{1/2-\varepsilon}(\mathcal{S})} \le m_\varepsilon \|\widehat{z}(s)\|_{H^{1-\varepsilon}(\Omega)} \le m_\varepsilon C_\varepsilon \|\widehat{z}(s)\|^\varepsilon \|\widehat{z}(s)\|^{1-\varepsilon}_{H^1(\Omega)}. \tag{5.7}$$

Here, in the first step we have used (5.3) for $\varphi = \widehat{z}(s)$ and in the second step we have used (5.5) with $\varphi = \widehat{z}(s)$, $\theta = \varepsilon$.

It remains to substitute (5.6) and (4.12) into the right-hand side of (5.7). As a result, we obtain for all $s \geq 1$ the inequality

$$\|\widehat{y}(s)\|_Y = \|\widehat{z}(s)\|_{H^{1/2-\varepsilon}(\mathcal{S})} \leq M_1(s)\|v\|_{H^{-1/2}(\mathcal{S})} = M_1(s)\|v\|_U,$$

where $M_1(s) = C_1(\varepsilon)(2s-1)^{-\varepsilon/2}$ and $C_1(\varepsilon) = m_0 m_\varepsilon C_\varepsilon$. This inequality is exactly the desired estimate (5.1) with $M(s) = M_1(s)$, which in turn is equivalent to (2.20).

Case (ii): $U = H^{-1/2+\varepsilon}(\mathcal{S})$, $Y = H^{1/2}(\widetilde{\mathcal{S}})$, $0 < \varepsilon < 1/2$.

Consider again (4.9) and let us treat the right hand side as the duality pairing $\langle v, \widehat{z}(s)\rangle$ between $H^{-1/2+\varepsilon}(\mathcal{S})$ and $H^{1/2-\varepsilon}(\mathcal{S})$. We obtain the following estimates:

$$\left|\int_{\mathcal{S}} \widehat{u}(x,s)\widehat{z}(x,s)\,d\sigma_x\right| \leq \|\widehat{z}(s)\|_{H^{1/2-\varepsilon}(\mathcal{S})}\|v\|_{H^{-1/2+\varepsilon}(\mathcal{S})}$$

$$\leq m_\varepsilon\|\widehat{z}(s)\|_{H^{1-\varepsilon}(\Omega)}\|v\|_{H^{-1/2+\varepsilon}(\mathcal{S})} \leq \widetilde{m}_\varepsilon\|\widehat{z}(s)\|_{H^1(\Omega)}\|v\|_{H^{-1/2+\varepsilon}(\mathcal{S})}$$

$$\leq \varepsilon\widetilde{m}_\varepsilon^2\|\widehat{z}(s)\|^2_{H^1(\Omega)} + \frac{1}{4\varepsilon}\|v\|^2_{H^{-1/2+\varepsilon}(\mathcal{S})}. \tag{5.8}$$

Here, in the second step we have used (5.3) with $\theta = \varepsilon$ and $\varphi = \widehat{z}(s)$. In the third step we have used the bounded embedding $H^1(\Omega) \hookrightarrow H^{1-\varepsilon}(\Omega)$.

Let us choose $\varepsilon\widetilde{m}_\varepsilon^2 = 1/2$ and use (5.8) to estimate the right-hand side of (4.9). Arguing in the same way as in the derivation of (5.6) we obtain the estimate

$$\|\widehat{z}(s)\| \leq \widetilde{m}_\varepsilon(2s-1)^{-1/2}\,\|v\|_{H^{-1/2+\varepsilon}(\mathcal{S})}, \quad s > \frac{1}{2}. \tag{5.9}$$

Let us now again come back to (4.9) and estimate the right-hand side using the inequality obtained on the second step in (5.8). We have

$$s\,\|\widehat{z}(s)\|^2 + \|\nabla\widehat{z}(s)\|^2 \leq m_\varepsilon\|\widehat{z}(s)\|_{H^{1-\varepsilon}(\Omega)}\|v\|_{H^{-1/2+\varepsilon}(\mathcal{S})}. \tag{5.10}$$

Using (5.5) with $\theta = \varepsilon$ and $\varphi = \widehat{z}(s)$ we obtain

$$\|\widehat{z}(s)\|_{H^{1-\varepsilon}(\Omega)} \leq C_\varepsilon\|\widehat{z}(s)\|^\varepsilon\|\widehat{z}(s)\|^{1-\varepsilon}_{H^1(\Omega)}. \tag{5.11}$$

Substituting (5.11) into (5.10), assuming that $s \geq 1$, and replacing the coefficient s in the first term on the left in (5.11) by 1, we obtain the inequality

$$\|\widehat{z}(s)\|^2_{H^1(\Omega)} \leq m_\varepsilon C_\varepsilon\|\widehat{z}(s)\|^\varepsilon\|\widehat{z}(s)\|^{1-\varepsilon}_{H^1(\Omega)}\|v\|_{H^{-1/2+\varepsilon}(\mathcal{S})}.$$

It follows immediately from here that, denoting $\widetilde{C}_\varepsilon = (m_\varepsilon C_\varepsilon)^{1/(1+\varepsilon)}$,

$$\|\widehat{z}(s)\|_{H^1(\Omega)} \leq \widetilde{C}_\varepsilon\|\widehat{z}(s)\|^{\varepsilon/(1+\varepsilon)}\|v\|^{1/(1+\varepsilon)}_{H^{-1/2+\varepsilon}(\mathcal{S})}.$$

Substituting (5.9) into the right-hand side above, we obtain

$$\|\widehat{z}(s)\|_{H^1(\Omega)} \leq \widehat{C}_\varepsilon (2s-1)^{-\varepsilon/(2(1+\varepsilon))} \|v\|_{H^{-1/2+\varepsilon}(\mathcal{S})}, \quad s \geq 1, \tag{5.12}$$

where $\widehat{C}_\varepsilon = \widetilde{C}_\varepsilon (\widetilde{m})^{\varepsilon/(1+\varepsilon)}$. We are now ready to estimate the norm of the output:

$$\begin{aligned}\|\widehat{y}(s)\|_Y = \|\widehat{z}(s)\|_{H^{1/2}(\widetilde{\mathcal{S}})} &\leq m_0 \|\widehat{z}(s)\|_{H^1(\Omega)} \\ &\leq m_0 \widehat{C}_\varepsilon (2s-1)^{-\varepsilon/(2(1+\varepsilon))} \|v\|_{H^{-1/2+\varepsilon}(\widetilde{\mathcal{S}})} \\ &= M_2(s)\, \|v\|_U, \quad s \geq 1. \end{aligned} \tag{5.13}$$

Here, first we have used (5.3) with $\varepsilon = 0$, $\mathcal{S}$ replaced with $\widetilde{\mathcal{S}}$, and $\varphi = \widehat{z}(s)$. In the second step we have just substituted (5.12). Inequality (5.13) is exactly the desired estimate (5.1) with $M(s) = M_2(s)$, which in turn is equivalent to (2.21) (with $C_2(\varepsilon) = m_0 \widetilde{C}_\varepsilon$). □

6. Abstract setup

The fact that each of the systems considered in the preceding sections is regular implies (see [25]) that each of these systems has the following state space representation, with the state space $X = L^2(\Omega)$: for almost every $t \geq 0$,

$$\dot{z}(t) = Az(t) + Bu(t)\,, \tag{6.1a}$$
$$y(t) = C_\Lambda z(t) + Du(t)\,. \tag{6.1b}$$

Here A is the semigroup generator of the corresponding semigroup $\mathbb{T}$, B is the control operator, C_Λ is the Λ-extension of the observation operator C (see [24, 25]), and D is the feedthrough operator. The aim of this section is to precisely identify all the above operators and the spaces involved in their definitions.

We begin with A, which is just the Laplace operator: $A = \Delta$, defined on the domain

$$\mathcal{D}(A) = \left\{ \varphi \in H^2(\Omega) \, : \, \frac{\partial \varphi}{\partial n}\bigg|_{\partial\Omega} = 0 \right\}.$$

It is well known that A is self-adjoint and dissipative, so that it is the infinitesimal generator of a strongly continuous semigroup of contractions $\mathbb{T}$ on X (see [10, 12, 16]).

Define two new Hilbert spaces X_1 and X_{-1} as follows: $X_1 = \mathcal{D}(A)$ with the norm $\|z\|_1 = \|(sI - A)z\|$, where $s \in \rho(A)$, the resolvent set of A. Note that in our case the norm $\|\cdot\|_1$ is equivalent to the standard $H^2(\Omega)$-norm due to the well known elliptic regularity estimate for the operator A (see, e.g., [10, 12]). The space X_{-1} is the completion of X with respect to the norm

$$\|z\|_{-1} = \|(sI - A)^{-1} z\|, \quad \text{where} \quad s \in \rho(A).$$

These spaces are independent of the choice of $s \in \rho(A)$. We have

$$X_1 \hookrightarrow X \hookrightarrow X_{-1},$$

densely and with continuous embeddings. Since A is self-adjoint, the space X_{-1} *can be identified with the dual of* X_1 *with respect to the pivot space* X.

Now we give a more explicit description of the space X_{-1}. Recall (see [12]) that

$$H_0^2(\Omega) \;=\; \left\{ \varphi \in H^2(\Omega) \;:\; u\big|_{\partial\Omega} = \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} = 0 \right\}.$$

The space of test functions $\mathcal{D}(\Omega)$ is dense in $H_0^2(\Omega)$ and, therefore, the dual of $H_0^2(\Omega)$ can be identified with a subspace of the space of distributions $\mathcal{D}(\Omega)^*$:

$$H^{-2}(\Omega) \;=\; \left[H_0^2(\Omega)\right]^*.$$

Thus, we have a Gelfand triple $H_0^2(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-2}(\Omega)$. $H^{-2}(\Omega)$ consists of all those distributions from $\mathcal{D}(\Omega)^*$ which can be extended to bounded linear functionals on $H_0^2(\Omega)$. Recall also that for any $\alpha > 0$, the space $H^{-\alpha}(\partial\Omega)$ is defined as the dual of $H^{\alpha}(\partial\Omega)$ with respect to the pivot space $L^2(\partial\Omega)$:

$$H^{\alpha}(\partial\Omega) \hookrightarrow L^2(\partial\Omega) \hookrightarrow H^{-\alpha}(\partial\Omega).$$

Proposition 6.1. *The following linear isomorphism holds:*

$$X_{-1} \cong H^{-2}(\Omega) \oplus H^{-3/2}(\partial\Omega) \tag{6.2}$$

where $\oplus$ *denotes the direct sum.*

Proof. Consider the Dirichlet trace operator from (5.2) with $\alpha = 2$. This operator is onto, i.e., its range is $H^{3/2}(\partial\Omega)$. Moreover, the restriction of this operator to the subspace $X_1 \subset H^2(\Omega)$ is still onto:

$$\gamma_0(\partial\Omega) \;:\; X_1 \to H^{3/2}(\partial\Omega), \quad \gamma_0(\partial\Omega)X_1 = H^{3/2}(\partial\Omega). \tag{6.3}$$

Indeed, (6.3) is equivalent to the following well-known result.

Fact. *For every* $\varphi \in H^{3/2}(\partial\Omega)$ *there exists a* $u \in H^2(\Omega))$ *such that*

$$u\big|_{\partial\Omega} = \varphi \quad \text{and} \quad \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} = 0.$$

Moreover, for each $\varphi \in H^{3/2}(\partial\Omega)$ *the corresponding* $u \in H^2(\Omega)$ *can be chosen in such a way that the extension mapping* E*, defined by* $E\varphi = u$*, is a bounded linear operator*

$$E \;:\; H^{3/2}(\partial\Omega) \to X_1 \subset H^2(\Omega).$$

The extension operator E is a right inverse of the trace operator:

$$\gamma_0(\partial\Omega) \circ E \;=\; I_{H^{3/2}(\partial\Omega)}.$$

Obviously, the null-space of the operator in (6.3) is

$$\mathrm{Ker}(\gamma_0(\partial\Omega)) = H_0^2(\Omega).$$

It follows from this and (6.3) that $\gamma_0(\partial\Omega)$ is an isomorphism from $X_1/H_0^2(\Omega)$ to $H^{3/2}(\partial\Omega)$. Hence, we have the following isomorphism from $H_0^2(\Omega) \oplus H^{3/2}(\partial\Omega)$ to X_1:

$$\Xi(v, \varphi) = v + E\varphi .$$

It is easy to see that for any $u \in X_1$ we have

$$\Xi^{-1}u = \Big(u - E\gamma_0(\partial\Omega)u, \gamma_0(\partial\Omega)u\Big) .$$

Finally, Ξ^* is the corresponding isomorphism of the dual spaces, as claimed in (6.2). □

Now our aim is to identify the control operator B corresponding to the **type 1 input** (see (2.2) and (2.3)) and to the **type 2 input** (see (2.4) and (2.5)). According to the theory of well-posed linear systems (which include the regular linear systems), see [17, 24, 25, 15], $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$ and for every $\tau \geq 0$, the operator Φ_τ from (2.11) is given by

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau - t} Bu(t)\, dt . \tag{6.4}$$

We recall the definition of a "simple layer" distribution (see, e.g., [22]). Let, for a moment, $\mathcal{S}$ be a C^1-piecewise smooth (see Section 2) $(n-1)$-dimensional bounded hypersurface in $\mathbb{R}^n$. The *simple layer distribution* $\delta_{\mathcal{S}} \in \mathcal{D}(\mathbb{R}^n)^*$ (or δ-function concentrated on $\mathcal{S}$) is defined by

$$\langle \delta_{\mathcal{S}}, \varphi \rangle = \int_{\mathcal{S}} \varphi(x)\, d\sigma_x \qquad \forall\ \varphi \in \mathcal{D}(\mathbb{R}^n) . \tag{6.5}$$

The functional $\delta_{\mathcal{S}}$ defined in (6.5) can be extended from $\mathcal{D}(\mathbb{R}^n)$ to the space $H^{1/2+\varepsilon}(\mathbb{R}^n)$ for any $\varepsilon > 0$. This follows from the boundedness of the trace operator

$$\gamma_0(\mathcal{S})\ :\ H^{1/2+\varepsilon}(\mathbb{R}^n) \to H^\varepsilon(\mathcal{S}) \hookrightarrow L^2(\mathcal{S}).$$

Indeed, the right-hand side of (6.5) defines a bounded linear functional on $L^2(\mathcal{S})$. Denoting the extension by the same symbol $\delta_{\mathcal{S}}$, we obtain that $\delta_{\mathcal{S}} \in H^{-1/2-\varepsilon}(\mathbb{R}^n)$ for any $\varepsilon > 0$. If $\mathcal{S} \subset \Omega$ then we have $\delta_{\mathcal{S}} \in H^{-1/2-\varepsilon}(\Omega)$.

If $\mathcal{S} \subset \partial\Omega$ then $\delta_{\mathcal{S}} \in \widetilde{H}^{-1/2-\varepsilon}(\Omega)$, where $\widetilde{H}^{-\alpha}(\Omega)$ denotes the dual of $H^\alpha(\Omega)$, $\alpha > 0$ (see, e.g., [19]). $\widetilde{H}^{-\alpha}(\Omega)$ can be identified with a subspace of the space of distributions $H^{-\alpha}(\mathbb{R}^n) = [H^\alpha(\mathbb{R}^n)]^* \subset \mathcal{D}(\mathbb{R}^n)^*$ (recall that $H_0^\alpha(\mathbb{R}^n) = H^\alpha(\mathbb{R}^n)$ for all $\alpha > 0$):

$$\widetilde{H}^{-\alpha}(\Omega) = \left\{ f \in H^{-\alpha}(\mathbb{R}^n)\ :\ \mathrm{supp}(f) \subseteq \overline{\Omega} \right\} .$$

We also recall that if $0 \leq \alpha \leq 1/2$, then the space of test functions $\mathcal{D}(\Omega)$ is dense in $H^\alpha(\Omega)$. Therefore, in this case

$$H_0^\alpha(\Omega) = H^\alpha(\Omega) \qquad \text{and} \qquad \widetilde{H}^{-\alpha}(\Omega) = H^{-\alpha}(\Omega)$$

(see, e.g., [12, Chapter I, Section 11]). It follows that if $\mathcal{S} \subset \partial\Omega$, then $\delta_{\mathcal{S}}$ can be considered as an element of the space X_{-1}. Indeed, formula (6.5) defines a bounded linear functional on the space X_1, since $\gamma_0(\mathcal{S})X_1 \subset H^{3/2}(\mathcal{S})$ (see (5.2)).

Proposition 6.2. *Let $\mathcal{S}$ and $\mathcal{S}_j$ be the hypersurfaces defined in Section 2.*

a) In the case of the ***type 1 input*** *defined in (2.2), (2.3), the corresponding control operator B is given by the formula*

$$Bu = u\delta_{\mathcal{S}} \qquad \text{for any} \quad u \in U = H^{-\frac{1}{2}}(\mathcal{S}),$$

or equivalently, by

$$\langle Bu, \varphi\rangle = \langle u, \gamma_0(\mathcal{S})\varphi\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}}, \tag{6.6}$$

for any $\varphi \in H^1(\Omega)$ (in particular, for any $\varphi \in X_1$). Thus, we have that $Bu \in \widetilde{H}^{-1}(\Omega) \subset X_{-1}$, for all $u \in U$.

b) In the case of the ***type 2 input*** *(2.4), (2.5), the operator B is given by*

$$Bu = \sum_{j=1}^{M} u_j\, \delta_{\mathcal{S}_j}, \tag{6.7}$$

for every $u = (u_1, \cdots u_M) \in U = \mathbb{R}^M$, which means that for all $\varphi \in H^1(\Omega)$,

$$\langle Bu, \varphi\rangle = \sum_{j=1}^{M} u_j \int_{\mathcal{S}_j} \varphi(x)\, d\sigma_x .$$

Proof. The proposition follows from the comparison of the variational formulations of the problems (2.1), (2.2), (2.3) or (2.1), (2.4), (2.5) and the equation (6.1a). To see this, consider the problem (2.1), (2.2), (2.3) with $u \in C([0,\infty), H^{-1/2}(\mathcal{S}))$ (for the other problem the argument is similar). The weak solution of this problem (see, e.g., [11], [12]) can be defined as a function z which satisfies

$$z \in C((0,\infty), H^1(\Omega)) \cap C^1((0,\infty), \widetilde{H}^{-1}(\Omega)),$$

$$\int_\Omega z_t(x,t)\varphi(x)\, dx = -\int_\Omega \nabla z(x,t)\cdot\nabla\varphi(x)\, dx + \int_{\mathcal{S}} u(x,t)\varphi(x)\, d\sigma_x \tag{6.8}$$

for any $\varphi \in H^1(\Omega)$. If we impose the initial condition $z(x,0) = z_0(x)$ with $z_0 \in X$, then the solution exists and is unique. On the other hand, (6.1a) is equivalent to

$$\langle \dot z(t), \varphi\rangle = \langle Az(t), \varphi\rangle + \langle Bu(t), \varphi\rangle \quad \text{for all} \;\; \varphi \in H^1(\Omega).$$

Clearly, this coincides with (6.8) if B is defined by (6.6).

The fact that $Bu = u\delta_{\mathcal{S}}$ is well defined as a distribution and $Bu \in \widetilde{H}^{-1}(\Omega)$ if $u \in H^{-1/2}(\mathcal{S})$ is clear from (6.6). Indeed, if $\varphi \in H^1(\Omega)$ then $\gamma_0(\mathcal{S})\varphi \in H^{1/2}(\mathcal{S})$. Therefore, the right-hand side of (6.6) defines a bounded linear functional on $H^1(\Omega)$. Since $X_1 \subset H^1(\Omega)$, we have in particular $Bu \in X_{-1}$. □

The observation operators C do not require any special discussion. Indeed, they are defined by (2.6) or (2.8) and, obviously, in both cases $C \in \mathcal{L}(X_1, Y)$. In the case of (2.6) the operator C is just the trace operator $\gamma_0(\widetilde{\mathcal{S}})$ restricted to X_1.

The operator C_Λ is defined in the abstract theory (see [24, 25]) by

$$C_\Lambda z_0 = \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} z_0 \tag{6.9}$$

on the domain $\mathcal{D}(C_\Lambda) \subset X$, which contains all the elements $z_0 \in X$ such that the limit (6.9) exists. In particular, every state $x \in X = L^2(\Omega)$ which in an open neighborhood of $\mathcal{S}$ is of class $H^{1/2+\varepsilon}$, with $\varepsilon > 0$, belongs to $\mathcal{D}(C_\Lambda)$.

As was shown in Theorem 2.8, for each of our systems the feedthrough operator $\mathcal{D} = 0$.

7. Proof of regularity using operators

In this section we present an alternative operator theoretic proof of the main regularity result, Theorem 2.8. In this development we use the spaces and operators discussed in Section 6. We begin the proof with two remarks.

1. Notice that $H^1(\Omega) = \mathcal{D}((-A)^{1/2})$ and the inner product in $H^1(\Omega)$, which by definition is

$$\big(\psi, \varphi\big)_{H^1} = \big(\psi, \varphi\big) + \big(\nabla\psi, \nabla\varphi\big) \qquad \text{for any} \quad \phi, \varphi \in H^1(\Omega)\,, \tag{7.1}$$

can be represented in the form

$$\big(\psi, \varphi\big)_{H^1} = \big((I-A)^{1/2}\psi, (I-A)^{1/2}\varphi\big) \quad \text{for any} \quad \psi, \varphi \in \mathcal{D}((-A)^{1/2}). \tag{7.2}$$

(Recall that $(\cdot,\cdot)$ and $\|\cdot\|$ denote the $L^2(\Omega)$-inner product and norm respectively.) Indeed, if $\psi \in X_1 = \mathcal{D}(A)$, then $A\psi = \Delta\psi$ and hence (by Green's formula) we get

$$\big(\psi, \varphi\big)_{H^1} = \big((I-\Delta)\psi, \varphi\big) = \big(\psi, \varphi\big) + \big(\nabla\psi, \nabla\varphi\big)\,.$$

Since X_1 is dense in $H^1(\Omega)$, we obtain (7.2) extending by continuity.

2. For any open subset Q of the boundary $\partial\Omega$, we denote by $\gamma_0(Q)$ the Dirichlet trace operator corresponding to Q (i.e., the Dirichlet trace restricted to Q). It is well known that

$$\gamma_0(Q) \in \mathcal{L}(H^1(\Omega), H^{1/2}(Q))\,.$$

Similarly, we denote by $\gamma_1(Q)$ the Neumann trace operator corresponding to Q (i.e., the Neumann trace restricted to Q). It is known that

$$\gamma_1(Q) \in \mathcal{L}(H^2(\Omega), H^{1/2}(Q)\,.$$

Proposition 7.1. *Define* $Z = \{f \in H^1(\Omega) : \Delta f \in L^2(\Omega)\}$, *which is a Hilbert space with the norm given by* $\|f\|_Z^2 = \|f\|_{H^1}^2 + \|\Delta f\|$. *The Neumann trace operator* $\gamma_1(Q)$ *can be extended so that*

$$\gamma_1(Q) \in \mathcal{L}(Z, H^{-1/2}(Q)).$$

Proof. Using Green's formula $\big(\gamma_1(\partial\Omega)f, \gamma_0(\partial\Omega)g\big) = \big(\Delta f, g\big) + \big(\nabla f, \nabla g\big)$, where $f \in H^2(\Omega)$ and $g \in H^1(\Omega)$, we see that $\gamma_1(\partial\Omega)$ can be extended to Z and then $\gamma_1(\partial\Omega) \in \mathcal{L}(Z, H^{-1/2}(\partial\Omega))$. (We have used the well known facts that $\gamma_0(\partial\Omega)$ is onto $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ is the dual of $H^{1/2}(\partial\Omega)$.) For every $f \in Z$, the distribution $\gamma_1(\partial\Omega)f \in H^{-1/2}(\partial\Omega)$ can be restricted to Q, by applying it only to test functions $\varphi \in C_0^\infty(Q)$. We denote this restriction (a distribution on Q) by $\widetilde{\gamma}_1(Q)f$. If M denotes the norm of $\gamma_1(\partial\Omega)f$ in $H^{-1/2}(\partial\Omega)$, then we have $\big|\big(\widetilde{\gamma}_1(Q)f, \varphi\big)\big| \leq M\|\varphi\|_{H^{1/2}}$, for all $\varphi \in C_0^\infty(Q)$. Since $C_0^\infty(Q)$ is dense in $H^{1/2}(Q)$, it follows that $\widetilde{\gamma}_1(Q)f$ can be extended to a distribution in $H^{-1/2}(Q)$ and $\|\widetilde{\gamma}_1(Q)f\|_{H^{-1/2}(\partial\Omega)} \leq M$. Hence, $\widetilde{\gamma}_1(Q) \in \mathcal{L}(Z, H^{-1/2}(Q))$. It can be verified that if $f \in H^2(\Omega)$, then $\widetilde{\gamma}_1(Q)f$ coincides with its usual Neumann trace $\gamma_1(Q)f$, discussed before the proposition. In the sequel, we will use the same notation for the original Neumann trace operator $\gamma_1(Q)$ and for its extension to Z. □

Proposition 7.2. *There exists an operator* $N(Q) \in \mathcal{L}(H^{-1/2}(Q), Z)$ *such that for every* $v \in H^{-1/2}(Q)$,

$$f = N(Q)v \quad \textit{if and only if} \quad \begin{cases} f \in H^1(\Omega), \quad \Delta f - f = 0, \\ \gamma_1(\partial\Omega \setminus Q)f = 0, \quad \gamma_1(Q)f = v\,. \end{cases}$$

Proof. By the Riesz representation theorem, for every $v \in H^{-1/2}(Q)$ there exists an $f \in H^1(\Omega)$ such that $\big(f, \varphi\big)_{H^1} = \langle v, \gamma_0(Q)\varphi\rangle$, for all $\varphi \in H^1(\Omega)$. Here by $\langle\cdot,\cdot\rangle$ we denote the duality pairing between $H^{-1/2}(Q)$ and $H^{1/2}(Q)$. Clearly, the above f depends linearly on v and we denote the operator from v to f by $N(Q)$. Thus, $N(Q) \in \mathcal{L}(H^{-1/2}(Q), H^1(\Omega))$ and

$$\big(N(Q)v, \varphi\big)_{H^1} = \big(v, \gamma_0(Q)\varphi\big) \qquad \text{for any} \quad \varphi \in H^1(\Omega)\,. \tag{7.3}$$

We prove the "only if" part of the proposition. Assume that $f = N(Q)v$. By taking $\varphi \in C_0^\infty(\Omega)$, we have from (7.3) that $\big(f, \varphi\big)_{H^1} = 0$. From here, using (7.1) we see that

$$\big(f, \varphi\big) + \big(\nabla f, \nabla\varphi\big) = 0\,.$$

Hence, we have $\Delta f - f = 0$ (in the sense of distributions), as claimed. In particular, it follows that $\Delta f \in L^2(\Omega)$, so that $f \in Z$. Since $\Delta f = f$ depends continuously on v, we conclude that in fact $N(Q) \in \mathcal{L}(H^{-1/2}(Q), Z)$.

Using the extension of $\gamma_1(\partial\Omega)$ from Proposition 7.1, we have that Green's formula (see the proof of Proposition 7.1) remains valid for all $f \in Z$ and for all $g \in H^1(\Omega)$. Taking $f = N(Q)v$, where $v \in H^{-1/2}(Q)$, and using that $\Delta f = f$, we get

$$\big(\gamma_1(\partial\Omega)f, \gamma_0(\partial\Omega)g\big) = \big(f, g\big) + \big(\nabla f, \nabla g\big) = \big(f, g\big)_{H^1} \qquad \text{for any } g \in H^1(\Omega)\,.$$

Using (7.3), we now obtain

$$\big(\gamma_1(\partial\Omega)f, \gamma_0(\partial\Omega)g\big) \,=\, \langle v, \gamma_0(Q)g\rangle \qquad \text{for any } \; g\in H^1(\Omega)\,. \tag{7.4}$$

Let $P(Q)$ denote the truncation operator which maps $H^{1/2}(\partial\Omega)$ onto $H^{1/2}(Q)$, by taking the restriction to Q. Its adjoint $P(Q)^* \in \mathcal{L}(H^{-1/2}(Q), H^{-1/2}(\partial\Omega))$ is the operator of extension by zero. Now (7.4) can be rewritten as

$$\begin{aligned}\big(\gamma_1(\partial\Omega)f, \gamma_0(\partial\Omega)g\big) \,&=\, \langle v, P(Q)\gamma_0(\partial\Omega)g\rangle \\ &=\, \langle P(Q)^*v, \gamma_0(\partial\Omega)g\rangle \qquad \text{for any } \; g\in H^1(\Omega)\,.\end{aligned}$$

Here on the second step $\langle\cdot,\cdot\rangle$ denotes, as above, the duality pairing between $H^{-1/2}(Q)$ and $H^{1/2}(Q)$, while on the third step $\langle\cdot,\cdot\rangle$ is the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$.

Since $\gamma_0(\partial\Omega)$ is onto $H^{1/2}(\partial\Omega)$, we conclude that $\gamma_1(\partial\Omega)f = P(Q)^*v$. This is equivalent to the fact that

$$\gamma_1(Q)f \,=\, v\,, \qquad \gamma_1(\partial\Omega\setminus Q)f \,=\, 0\,,$$

as claimed in the proposition. The "if" part of the proposition is a corollary of the following well known fact (see, e.g., [11, Chapter 2]). The weak solution in the space $H^1(\Omega)$ of the Poisson equation $\Delta f - f = 0$ with the above Neumann boundary conditions is unique. Moreover, it is regular in the sense that $\Delta f \in L^2(\Omega)$ (and even $H^1(\Omega)$) and $f\in H^2(\Omega')$ for any strictly inner domain $\Omega'\subset\Omega$. □

Recall that $\widetilde{H}^{-1}(\Omega)$ is the dual of $H^1(\Omega)$ (with respect to the pivot space $L^2(\Omega)$) and $A\in\mathcal{L}(H^1(\Omega), \widetilde{H}^{-1}(\Omega))$. If we regard $N(Q)$ as an operator in $\mathcal{L}(H^{-1/2}(Q), H^1(\Omega))$, then

$$N^*(Q) \,\in\, \mathcal{L}(\widetilde{H}^{-1}(\Omega), H^{1/2}(Q))\,.$$

Proposition 7.3. *We have*

$$N^*(Q)(I-A)g \,=\, \gamma_0(Q)g \qquad \textit{for any } \; g\in H^1(\Omega)\,.$$

Proof. Using (7.2), we rewrite (7.3) in the form

$$\langle N(Q)v, (I-A)g\rangle \,=\, \langle v, \gamma_0(Q)g\rangle \qquad \text{for any } \; v\in H^{-1/2}(Q), \;\; g\in H^1(\Omega)$$

(on the left-hand side we have used the duality pairing between $H^1(\Omega)$ and $\widetilde{H}^{-1}(\Omega)$, while on the right we have used the duality pairing between $H^{-1/2}(Q)$ and $H^{1/2}(Q)$). From here, the desired identity follows easily. □

After all this preparation, we move back to our heat equation system, more precisely to the system of type $(1,1)$ (see Section 2). Choose $U = H^{-1/2}(\mathcal{S})$ and $Y = H^{1/2}(\widetilde{\mathcal{S}})$, so that the system is well-posed according to Theorem 2.6. The equations of this system can be written, for sufficiently smooth input functions, in the form

$$\dot z(t) \,=\, Az(t) + Bu(t)\,, \qquad y(t) \,=\, \overline{C}z(t)\,,$$

where $\overline{C} = \gamma_0(\widetilde{\mathcal{S}}) \in \mathcal{L}(H^1(\Omega), Y)$, is an extension of the observation operator C of the system (C is defined on X_1 only). The transfer function of the system is

$$\mathbf{G}(s) = \overline{C}(sI - A)^{-1}B \qquad \text{for any } s \in \mathbb{C}_0 \,.$$

Proof of Theorem 2.8. According to (6.6), the control operator B is the adjoint of $\gamma_0(\mathcal{S})$, so that by Proposition 7.3 it can be written in the form

$$B = (I - A)N(\mathcal{S}) \,.$$

Hence, we have for $s \in \mathbb{C}_0$ that $\mathbf{G}(s) = \overline{C}(sI - A)^{-1}(I - A)N(\mathcal{S})$. Using the formula for $\overline{C}$ and Proposition 7.3, this becomes

$$\mathbf{G}(s) = N^*(\widetilde{\mathcal{S}})(I - A)(sI - A)^{-1}(I - A)N(\mathcal{S}) \,.$$

If we introduce the notation

$$\mathcal{B} = (I - A)^{1/2}N(\mathcal{S}) \in \mathcal{L}(U, X) \,,$$

$$\mathcal{C} = N^*(\widetilde{\mathcal{S}})(I - A)^{1/2} \in \mathcal{L}(X, Y) \,,$$

then we obtain

$$\mathbf{G}(s) = \mathcal{C}(I - A)(sI - A)^{-1}\mathcal{B} \,.$$

We decompose $(I - A)(sI - A)^{-1} = I - (s - 1)(sI - A)^{-1}$. The second term is well known to converge strongly to I as $s \to +\infty$ (for real s), so that $\lim_{s\to+\infty}(I - A)(sI - A)^{-1}z = 0$ for every $z \in X$. Thus, we have $\lim_{s\to+\infty} \mathbf{G}(s)v = 0$ for every $v \in U$. $\square$

References

[1] G. Avalos, I. Lasiecka and R. Rebarber, "Lack of time-delay robustness for dynamic stabilization of a structural acoustic model," *SIAM J. Control and Optim.* **37** (1999), 1394-1418.

[2] A. Bensoussan, G. Da Prato, M.C. Delfour and S.K. Mitter, *Representation and Control of Infinite Dimensional Systems,* vol. I and II, Progress in Systems and Control, Birhkäuser, Boston, 1992 and 1993.

[3] C.I. Byrnes, D.S. Gilliam, I.G. Laukó and V.I. Shubov, "Output regulation for linear distributed parameter systems," to appear in *IEEE Trans. Aut. Control.*

[4] C.I. Byrnes, D.S. Gilliam, I.G. Laukó and V.I. Shubov, "Output regulation for parabolic distributed parameter systems: Set point control," *Proceedings 36th IEEE CDC*, 1997, 2225-2230.

[5] C.I. Byrnes, D.S. Gilliam, and V.I. Shubov, "Example of Output Regulation for a System with Unbounded Inputs and Outputs," *Proceedings 38th IEEE Conf. Dec. Control*, 1999, 4280-4284.

[6] C.I. Byrnes, D.S. Gilliam, V.I. Shubov and G. Weiss, "The output regulator problem for regular linear systems," preprint.

[7] R.F. Curtain and G. Weiss, "Well posedness of triples of operators (in the sense of linear systems theory)," *Control and Estimation of Distributed Parameter Systems*, eds: F. Kappel, K. Kunisch, W. Schappacher, pp. 41-59, Basel: Birkhäuser, 1989.

[8] R.F. Curtain, G. Weiss and M. Weiss, "Coprime factorization for regular linear systems," *Automatica* **32** (1996), 1519-1531.

[9] R.F. Curtain and H.J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems,* Springer-Verlag, New York, 1995.

[10] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of AMS, Vol. 23, 1968.

[11] O.A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics*, Applied Mathematical Sciences 49, Springer-Verlag, 1985.

[12] J.L. Lions and E. Magenes, *Nonhomogeneous Boundary Value Problems & Applications, V. 1*, Springer-Verlag, New York, 1972.

[13] H. Logemann, R. Rebarber and G. Weiss, "Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop," *SIAM J. Control and Optim.*, **34** (1996), 572-600.

[14] H. Logemann, E.P. Ryan and S. Townley, "Integral control of infinite-dimensional linear systems subject to input saturation," *SIAM J. Control and Optim.* **36** (1998), 1940–1961.

[15] H. Logemann and S. Townley, "Low gain control of uncertain regular linear systems," *SIAM J. Control and Optimization* **35** (1997), 78-116.

[16] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Springer-Verlag, New York, 1983.

[17] D. Salamon, "Infinite-dimensional linear systems with unbounded control and observation: a functional analytic approach," *Trans. Amer. Math. Soc.* **300** (1987), 383-431.

[18] A. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*, Springer-Verlag, London, 2000.

[19] V.I. Shubov, *An Introduction to Sobolev Spaces and Distributions,* Lecture Notes, Texas Tech University, 1996.

[20] O.J. Staffans, "Coprime factorizations and well-posed linear systems," *SIAM J. Control Optim.* **36** (1998), 1268-1292.

[21] O.J. Staffans, “Passive and lossless infinite-dimensional systems,” submitted in 2001.

[22] V.S. Vladimirov, *Equations of Mathematical Physics*, Mir, Moscow, 1984.

[23] G. Weiss, “Admissible observation operators for linear semigroups,” *Israel J. Math.* **65** (1989), 17-43.

[24] G. Weiss, “Transfer functions of regular linear systems, part I: Characterizations of regularity,” *Trans. of the AMS* **342** (1994), 827-854.

[25] G. Weiss, “Regular linear systems with feedback,” *Math. of Control, Signals and Systems* **7** (1994), 23-57.

[26] G. Weiss and R.F. Curtain, “Dynamic stabilization of regular linear systems,” *IEEE Trans. Aut. Control* **42** (1997), 4-21.

[27] J. Wloka, *Partial Differential Equations*, Cambridge University Press, 1987.