Asymptotic stabilization of some equilibria of an underactuated undewater vehicle*

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Abstract

The stabilization problem for selected relative equilibria of an underactuated rigid body, modelling a simple underwater vehicle, moving in an ideal fluid is addressed. State feedback control laws achieving local asymptotic stability of a forward motion and of a diving/rising with forward/reverse motion are proposed. The control design exploits the Hamiltonian nature of the system to be controlled and it is based on the so-called Interconnection and Damping Assignment (IDA) procedure. Simulation results complete the work.

Keywords: Nonlinear systems, passivity, mechanical systems, Kirchoff equations.

1 Introduction

The problem of (asymptotic) stabilization of the dynamic model of rigid bodies, modelled either with the Euler's equations or with the Kirchhoff's equations, has attracted a lot of attention. The Euler's equations describe only the angular velocity of the rigid body, *i.e.* no coupling between linear and angular velocities is present. As a result the Euler's equations are adequate in describing (to the first approximation) the motion of a satellite, *i.e.* of a rigid body moving in free space. On the contrary, the Kirchhoff's equations describe both linear and angular velocities of the rigid body, and can be used to model the behavior of a vehicle moving in an ideal fluid[1].

The stabilization problem for the underactuated Euler's equations has been thoroughly investigated from several points of view. Stabilization using classical nonlinear control methods has been studied in [2, 22, 1, 19], whereas energy based stabilization has been dealt with in the paper [6] and

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[1]As noted in [13] the Kirchhoff's equations provide an adequate description of the dynamics of a neutrally buoyant rigid body, moving in an infinite volume of irrotational, incompressible, inviscid fluid at rest at infinity. We call such a fluid an *ideal* one.

lately in [4]. It is worth noting that robust stabilization problems and stabilization using partial state information have also been addressed, see *e.g.* [15, 5, 3, 14].

On the other hand, stabilization problems for the underactuated Kirchhoff's equations are relatively unexplored. Notable exceptions are the series of papers by A.M. Bloch and co-workers. In [11] the stability properties of the uncontrolled Kirchhoff equations as a function of the system parameters have been studied, while in [13, 23, 24] stabilizing control laws for a class of relative equilibria have been proposed. In all these papers, following the approach developed in [6], it is assumed that the control action is delivered by internal actuators (rotors). This assumption is reasonable for a class of relatively small underwater vehicles, such as those used in oceanographic applications, as detailed in [23], whereas, for more conventional underwater vehicles, a different actuator structure needs to be considered, namely thrusters and planes (fins). This structure has been used, for example, in [20, 12], where the (local) stabilization problem for the complete model (kinematic and dynamic) of an underwater vehicle, in the presence of three, four or six actuators, has been solved by means of (time-varying) continuous control laws. Note that the model considered in [20] contains nonzero hydrodynamic drag, *i.e.* the zero equilibrium of the uncontrolled Kirchhoff's equations is asymptotically stable, whereas hydrodynamic drag is neglected in [12].

In this paper, as in [13, 23], we consider the problem of stabilization of classes of relative equilibria for the underactuated Kirchhoff's equations. More precisely, stabilizing control laws are designed using the IDA methodology for two types of relative equilibria, neglecting hydrodynamic drag and dissipative forces. Similar to [20, 12], we assume that the control action is delivered by thrusters and planes. It is further assumed that only one thruster and two pairs of planes are acting on the system, which is consistent with the simplest actuators pattern found in submarines, see [8], and this is sufficient to ensure local strong accessibility of the system (see Section 2 for further detail).

The paper is organized as follows. In Section 2 the mathematical model of a rigid body moving in an ideal fluid is presented and some properties of the open loop system are discussed. Section 3 contains a brief overview of the general IDA methodology, which has been recently proposed in [18] and references therein. Some links of this methodology with the method of *controlled Lagrangians*, independently developed by N.E. Leonard and co-workers in [7] and used in the control of underwater vehicles in [23], have been established in [18]. In Section 4 the stabilization problem for a forward (or backward) translational motion is analyzed, whereas Section 5 deals with the stabilization problem for a steady diving/rising with forward/reverse motion. Finally, Section 6 presents some simulation results and Section 7 contains concluding remarks and an outlook on future research.

2 The model

Consider an ellipsoidal rigid body completely submerged in an ideal fluid and assume that the center of gravity of the body coincides with the center of buoyancy. Let Π_1, Π_2 and Π_3 (P_1, P_2 and P_3) denote the components of the angular (linear) momentum vector along a body fixed reference frame with the origin at the center of gravity and consisting of the three principal axes. The Kirchhoff equations (see [10, 13] for further detail) for the rigid body subject to external forces and torques are

$$
\begin{aligned}
\dot{\Pi}_1 &= \left(\frac{1}{J_3} - \frac{1}{J_2}\right)\Pi_2\Pi_3 + \left(\frac{1}{M_3} - \frac{1}{M_2}\right)P_2P_3 + B_1^T u \\
\dot{\Pi}_2 &= \left(\frac{1}{J_1} - \frac{1}{J_3}\right)\Pi_3\Pi_1 + \left(\frac{1}{M_1} - \frac{1}{M_3}\right)P_3P_1 + B_2^T u \\
\dot{\Pi}_3 &= \left(\frac{1}{J_2} - \frac{1}{J_1}\right)\Pi_1\Pi_2 + \left(\frac{1}{M_2} - \frac{1}{M_1}\right)P_1P_2 + B_3^T u \\
\dot{P}_1 &= \frac{P_2\Pi_3}{J_3} - \frac{P_3\Pi_2}{J_2} + B_4^T u \\
\dot{P}_2 &= \frac{P_3\Pi_1}{J_1} - \frac{P_1\Pi_3}{J_3} + B_5^T u \\
\dot{P}_3 &= \frac{P_1\Pi_2}{J_2} - \frac{P_2\Pi_1}{J_1} + B_6^T u
\end{aligned}
\tag{1}
$$

where $J_1 > 0$, $J_2 > 0$ and $J_3 > 0$ denote the principal moments of inertia and $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$ denote the diagonal terms of the (diagonal) inertia matrix, $u = \mathrm{col}(T_1, T_2, T_3, F_1, F_2, F_3)$ and

$$
B = \begin{bmatrix} B_1^T \\ B_2^T \\ \vdots \\ B_6^T \end{bmatrix}
$$

is a matrix dependent upon the actuators configuration. As observed in [13], system (1) is a port-controlled Hamiltonian system, *i.e.* a system described by equations of the form

$$
\dot{x} = (J(x) - R(x))\left(\frac{\partial H}{\partial x}\right)^T + Bu \tag{2}
$$

with state $x = \mathrm{col}(\Pi_1, \Pi_2, \Pi_3, P_1, P_2, P_3)$, control u, internal Hamiltonian

$$
H(x) = \frac{1}{2}\left(\frac{\Pi_1^2}{J_1} + \frac{\Pi_2^2}{J_2} + \frac{\Pi_3^2}{J_3} + \frac{P_1^2}{M_1} + \frac{P_2^2}{M_2} + \frac{P_3^2}{M_3}\right), \tag{3}
$$

interconnection matrix

$$
J(x) = -J^T(x) = \begin{bmatrix}
0 & -\Pi_3 & \Pi_2 & 0 & -P_3 & P_2 \\
\Pi_3 & 0 & -\Pi_1 & P_3 & 0 & -P_1 \\
-\Pi_2 & \Pi_1 & 0 & -P_2 & P_1 & 0 \\
0 & -P_3 & P_2 & 0 & 0 & 0 \\
P_3 & 0 & -P_1 & 0 & 0 & 0 \\
-P_2 & P_1 & 0 & 0 & 0 & 0
\end{bmatrix}, \tag{4}
$$

dissipation $R(x) = R^T(x) = 0$ and input matrix B. In what follows we assume that the only manipulated variables are T_1, T_2 and F_3, which correspond to the simplest possible actuators configuration for an underwater vehicle (see [8]). An important property of system (1) with the considered actuators configuration is expressed in the following statement.

Lemma 1 *Consider system (1) with*

$$
B = [b_1, b_2, b_3] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \tag{5}
$$

i.e. *T_1, T_2 and F_3 are the only manipulated variables.*

Assume $J_1 \neq J_2$. Then the system is locally strongly accessible.

Proof. If $J_1 \neq J_2$, the vector fields

$$[b_1, [f(x), b_2]] = \mathrm{col}(0, 0, \frac{1}{J_1} - \frac{1}{J_2}, 0, 0, 0),$$

$$[b_1, [f(x), b_3]] = \mathrm{col}(0, 0, 0, 0, -\frac{1}{J_1}, 0)$$

and

$$[b_2, [f(x), b_3]] = \mathrm{col}(0, 0, 0, \frac{1}{J_2}, 0, 0)$$

where $f(x) = J(x) \left(\frac{\partial H}{\partial x}\right)^T$, together with the input vector fields b_1, b_2 and b_3, form a set of six constant and linearly independent vectors, all belonging to the strong accessibility distribution. Hence the claim. $\triangleleft$

Remark 1 As well known, strong accessibility neither implies controllability nor feedback stabilizability, however it provides an indication that the actuators pattern considered is not *a priori* inadequate to control the rigid body.

System (1) with B as in (5) and constant u possesses several equilibria. The ones of interest in practical applications are those corresponding to a translation of the rigid body without rotation. To give a precise description of such equilibria and for ease of exposition, an equilibrium point x_e is said to be of *order* i if the vector x_e has precisely i nonzero components.

Lemma 2 *Consider system (1) with B as in (5). Assume $M_1 \neq M_2$. Then all achievable equilibrium points*

$$x_e = \mathrm{col}(\bar{\Pi}_1, \bar{\Pi}_2, \bar{\Pi}_3, \bar{P}_1, \bar{P}_2, \bar{P}_3)$$

corresponding to zero angular momenta, i.e. $\bar{\Pi}_1 = \bar{\Pi}_2 = \bar{\Pi}_3 = 0$, *are such that*

$$\bar{P}_1 \bar{P}_2 = 0.$$

Lemma 2, the proof of which is trivial, hence omitted, lends itself to the following interpretation. If only equilibria corresponding to zero angular momenta, but possibly non-zero constant controls, are considered, the only achievable order two equilibria are

E_2': $x_e = \mathrm{col}(0, 0, 0, \bar{P}_1, 0, \bar{P}_3)$
E_2'': $x_e = \mathrm{col}(0, 0, 0, 0, \bar{P}_2, \bar{P}_3)$,

whereas the only achievable order one equilibria are

E_1': $x_e = \mathrm{col}(0, 0, 0, \bar{P}_1, 0, 0)$,
E_1'': $x_e = \mathrm{col}(0, 0, 0, 0, \bar{P}_2, 0)$,
E_1''':$x_e = \mathrm{col}(0, 0, 0, 0, 0, \bar{P}_3)$.

In the considered case of an ellipsoidal rigid body modelling an underwater vehicle, (see Figure 1 for detail) the control T_1 is associated with the motion of the rudder, the control T_2 is associated with the motion of the bow and stern planes, and the control F_3 results from the action of the propeller. As a result, Lemma 2 states that it is not possible to achieve steady translational motion with a nonzero component along all three directions for zero steady angular momenta. Moreover, the only feasible maneuvers with zero angular momenta are

M_2': steady rising/diving with forward/reverse motion;

M_2'': steady sideways motion with forward/reverse motion;

M_1': steady rising/diving motion;

M_1'': steady sideways motion;

M_1''': steady forward/reverse motion.

In the rest of the paper the focus will be on two specific maneuvers, namely M_2' and M_1'''. This corresponds to the study of the stabilization problem for system (1) around the equilibria E_2' and E_1'''. Note that similar considerations, following the guidelines provided below, can be made for the other maneuvers.

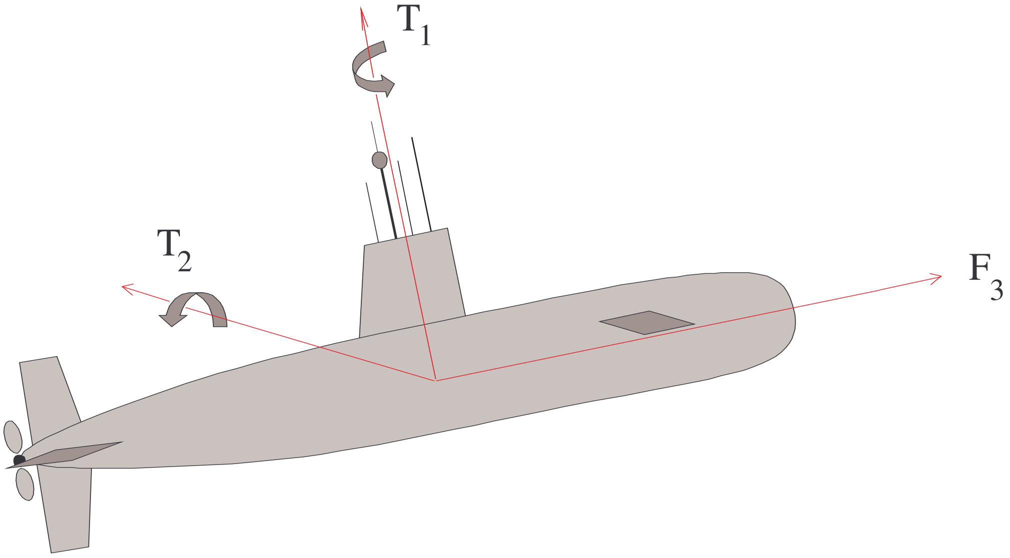

Figure 1: Schematic diagram of the underwater vehicle with the selected actuators configuration.

3 The IDA methodology

In this section we briefly review the so-called IDA control methodology. This is a recently developed control design method, generalizing the classical passivity based control approach, see [18] for a more in-depth description of the general theory and the papers [16, 21, 17, 4] for some applications. The design methodology can be described as follows.

Given a port-controlled Hamiltonian system described by equations of the form (2) and a point x_e, find (if possible) a modification $H_a(x)$ of the internal Hamiltonian, a modification $J_a(x) = -J_a^T(x)$ of the interconnection matrix, a modification $R_a(x) = R_a^T(x)$ of the dissipation and a state feedback control law $u = u(x)$, such that

(C1) for all x one has

$$\dot{x} = (J(x) - R(x)) \left(\frac{\partial H}{\partial x}\right)^T + Bu(x) = (J(x) + J_a(x) - R(x) - R_a(x)) \left(\frac{\partial (H + H_a)}{\partial x}\right)^T ; \quad (6)$$

(C2) x_e is a locally asymptotically stable[2] equilibrium of the closed loop system

$$\dot{x} = (J(x) + J_a(x) - R(x) - R_a(x)) \left(\frac{\partial (H + H_a)}{\partial x}\right)^T . \quad (7)$$

To provide simple statements of the main results of the paper the following definition is introduced.

Definition 1 *Given a port-controlled Hamiltonian system and a point x_e in the state space, we say that the system is locally IDA stabilizable around x_e if there exists $H_a(x)$, $J_a(x) = -J_a^T(x)$, $R_a(x) = R_a^T(x)$, and $u(x)$ such that conditions (C1) and (C2) are satisfied.*

Equation (6) represents a system of partial differential equations in the unknown functions $H_a(x)$ and $u(x)$, once candidates $J_a(x)$ and $R_a(x)$ have been selected, together with some boundary conditions, namely

$$\left.\frac{\partial (H + H_a)}{\partial x}\right|_{x=x_e} = 0$$

$$H(x) + H_a(x) > H(x_e) + H_a(x_e)$$

[2]Depending of the application, diverse stability properties can be considered, see for example Section 4.

for all $x \neq x_e$ close to x_e, and[3]

$$R(x) + R_a(x) \geq 0.$$

In some cases it is possible to derive a simplified version of (6). For example, if $R(x) = 0$ and if the candidates $J_a(x)$ and $R_a(x)$ are such that $J_a(x) = 0$ and $\mathrm{Im}(R_a(x)) \subseteq \mathrm{Im}(B)$, or equivalently $\mathrm{Ker}(R_a(x)) \supseteq \mathrm{Ker}(B)$, and if in addition the matrix B is full column rank[4], then all solutions of equation (6) are given by all pairs $(H_a(x), u(x))$ such that

$$B^{\perp} J(x) \left(\frac{\partial H_a}{\partial x}\right)^T = 0, \tag{8}$$

and

$$u(x) = (B^T B)^{-1} B^T \left[(J(x) - R_a(x)) \left(\frac{\partial H_a}{\partial x}\right)^T - R_a(x) \left(\frac{\partial H}{\partial x}\right)^T \right]. \tag{9}$$

Remark 2 The above discussion is based on the simplifying assumption that hydrodynamic drags are negligible. Note that the presence of a generic nonzero matrix $R(x)$ (modelling nonzero drags) in the model (2) may significantly complicate the solvability of the partial differential equation (6), whereas in the presence of a nonzero $R(x)$ such that $\mathrm{Im}(R(x)) \subseteq \mathrm{Im}(B)$, all results of the paper can be proved with trivial modifications.

In the next two sections the above simplified version of the IDA methodology is employed to derive stabilizing control laws for system (1) around the equilibria E_2' and E_1'''. To this end, we first establish a preliminary fact, which is instrumental in deriving the main results of the paper.

Lemma 3 *Consider system (1). All globally defined differentiable solutions of the partial differential equations (8) can be obtained by composition of the basis functions*

$$\psi_1 = P_3 \qquad \psi_2 = P_1^2 + P_2^2 \qquad \psi_3 = \Pi_1 P_1 + \Pi_2 P_2 + \Pi_3 P_3, \tag{10}$$

i.e. *any function of the form*

$$\prod_{i \in \mathcal{I}} \phi_i(\psi_j),$$

with $\mathcal{I}$ a finite set of indexes, $j = 1, 2, 3$ and differentiable $\phi_i(\cdot)$ is a solution of the partial differential equations (8).

Proof. To begin with observe that, for system (1), equations (8) can be written in matrix form as

$$\frac{\partial H_a}{\partial x}\Gamma(x) = \left[\begin{array}{ccc} \dfrac{\partial H_a}{\partial \Pi_1} & \cdots & \dfrac{\partial H_a}{\partial P_3} \end{array} \right] \left[\begin{array}{ccc} -\Pi_2 & 0 & P_3 \\ \Pi_1 & -P_3 & 0 \\ 0 & P_2 & -P_1 \\ -P_2 & 0 & 0 \\ P_1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] = 0. \tag{11}$$

Moreover, the distribution spanned by the columns of the matrix $\Gamma(x)$ is (generically) three dimensional and involutive. Hence, the claim is a consequence of Frobenius theorem [9]. $\triangleleft$

Remark 3 We point out that the functions ψ_3 and $\psi_1^2 + \psi_2$ are Casimir functions for the port-controlled Hamiltonian system (1), *i.e.* their gradients lie in the null space of the matrix $J(x)$ defined in (4). As a result, they solve the partial differential equation (8). Note, however, that due to the special structure of the matrix B, the set of all solutions of the partial differential equation (8) is larger then the set of functions resulting by composition of the basis functions ψ_3 and $\psi_1^2+\psi_2$.

[3]To be precise, an additional detectability condition is required to achieve asymptotic stability, see [18].
[4]This is without loss of generality.

Remark 4 It is worth noting that the partial differential equations (11) also have nonsmooth solutions. For example, the function

$$\tilde{\psi} = \begin{cases} 0, & \text{if } P_3 \neq 0; \\ \Pi_1^2 + \Pi_2^2, & \text{otherwise} \end{cases}$$

is also a solution of equations (11) in all its points of differentiability.

4 Stabilization of a class of *order one* relative equilibria

This section deals with the stabilization problem via IDA for the maneuver M_1''', namely a steady forward/reverse motion. As a special case, it will also be shown how to stabilize the zero equilibrium of equations (1).

Proposition 1 *Consider system (1) and the equilibrium point* $x_e = \mathrm{col}(0,0,0,0,0,\bar{P}_3)$*,* i.e. *the equilibrium* E_1'''*. Then the system is almost globally IDA stabilizable around* x_e*,* i.e. *the equilibrium* x_e *is globally stable in the sense of Lyapunov and for all initial conditions, except a set of measure zero, trajectories converge asymptotically to* x_e*.*

Proof. Condition (C1). Using Lemma 3 it is straightforward to verify that condition (C1) is satisfied by

$$H_a(x) = -\frac{\bar{P}_3}{M_3} P_3 + \lambda \frac{P_3 - \bar{P}_3}{1 + 2(P_3 - \bar{P}_3)^2} (\Pi_1 P_1 + \Pi_2 P_2 + \Pi_3 P_3), \tag{12}$$

$J_a(x) = 0$, $R_a(x) = \mathrm{diag}(\Pi_1^2, \Pi_2^2, 0, 0, 0, 1) \geq 0$ and $u = \mathrm{col}(u_1, u_2, u_3)$, where

$$\begin{aligned}
u_1 = {} & -\frac{{\Pi_1}^2 \lambda \left(P_3 - \bar{P}_3\right) P_1}{1 + 2\left(P_3 - \bar{P}_3\right)^2} - \frac{{\Pi_1}^3}{J_1} + \\
& + P_2 \left(-\frac{\bar{P}_3}{M_3} + \frac{\lambda \left(\Pi_1 P_1 + \Pi_2 P_2 + \Pi_3 P_3\right)}{1 + 2\left(P_3 - \bar{P}_3\right)^2} - \frac{\lambda \left(P_3 - \bar{P}_3\right)\left(\Pi_1 P_1 + \Pi_2 P_2 + \Pi_3 P_3\right)\left(4 P_3 - 4 \bar{P}_3\right)}{\left(1 + 2\left(P_3 - \bar{P}_3\right)^2\right)^2} \right) \\
u_2 = {} & -\frac{{\Pi_2}^2 \lambda \left(P_3 - \bar{P}_3\right) P_2}{1 + 2\left(P_3 - \bar{P}_3\right)^2} - \frac{{\Pi_2}^3}{J_2} + \\
& - P_1 \left(-\frac{\bar{P}_3}{M_3} + \frac{\lambda \left(\Pi_1 P_1 + \Pi_2 P_2 + \Pi_3 P_3\right)}{1 + 2\left(P_3 - \bar{P}_3\right)^2} - \frac{\lambda \left(P_3 - \bar{P}_3\right)\left(\Pi_1 P_1 + \Pi_2 P_2 + \Pi_3 P_3\right)\left(4 P_3 - 4 \bar{P}_3\right)}{\left(1 + 2\left(P_3 - \bar{P}_3\right)^2\right)^2} \right) \\
u_3 = {} & \frac{\bar{P}_3}{M_3} - \frac{\lambda \left(\Pi_1 P_1 + \Pi_2 P_2 + \Pi_3 P_3\right)}{1 + 2\left(P_3 - \bar{P}_3\right)^2} - \frac{\lambda \left(P_3 - \bar{P}_3\right) \Pi_3}{1 + 2\left(P_3 - \bar{P}_3\right)^2} + \frac{\lambda \left(P_3 - \bar{P}_3\right)\left(\Pi_1 P_1 + \Pi_2 P_2 + \Pi_3 P_3\right)\left(4 P_3 - 4 \bar{P}_3\right)}{\left(1 + 2\left(P_3 - \bar{P}_3\right)^2\right)^2} - \frac{P_3}{M_3}.
\end{aligned} \tag{13}$$

Condition (C2). To begin with note that if λ is sufficiently small and if it is also such that

$$|\lambda \bar{P}_3| < \frac{1}{\sqrt{M_3 J_3}},$$

then the function

$$H_d(x) = H(x) + H_a(x),$$

with $H(x)$ and $H_a(x)$ as in equations (3) and (12), respectively, has a unique stationary point, coinciding with the equilibrium point x_e, which is a global minimum. This implies, by condition (C1), that x_e is stable in the sense of Lyapunov. Furthermore, if $\lambda \neq 0$, a straightforward application of La Salle invariance principle indicates that all trajectories of the closed loop system (1)-(13) converge towards the set

$$M = \{(\Pi_1, \Pi_2, \Pi_3, P_1, P_2, P_3) \mid \Pi_1 = \Pi_2 = P_1 = P_2 = 0\}$$

and, inside this set, the trajectories are such that

$$\dot{\Pi}_3 = \dot{P}_3 = 0,$$

i.e. any trajectory of the closed loop system (1)-(13) converges toward a point.

Note now that the closed loop system (1)-(13) possesses a manifold of equilibria, namely the set

$$E = \{(\Pi_1, \Pi_2, \Pi_3, P_1, P_2, P_3) \mid \Pi_1 = \Pi_2 = P_1 = P_2 = 0, \Pi_3 = \phi(P_3, \bar{P}_3, \lambda)\},$$

where

$$\phi(P_3, \bar{P}_3, \lambda) = \frac{\left(P_3 - \bar{P}_3\right)\left(1 + 2\,{P_3}^2 - 4\,P_3\,\bar{P}_3 + 2\,\bar{P}_3^2\right)^2}{\left(2\,\bar{P}_3^3 - 4\,P_3\,\bar{P}_3^2 + \bar{P}_3 + 2\,{P_3}^2\bar{P}_3 - 2\,P_3\right)M_3\,\lambda},$$

and observe that for any nonzero λ,

$$\phi(\bar{P}_3, \bar{P}_3, \lambda) = 0.$$

By the above discussion the set E is asymptotically attractive, hence to conclude the proof we need to show that any point in the set $\bar{E} = E \setminus \{x_e\}$ is an unstable equilibrium. For, pick a point $\bar{x}$ in $\bar{E}$, consider the function $\Delta(x) = H_d(x) - H_d(\bar{x})$, and note that $\Delta(\bar{x}) = 0$ and $\dot{\Delta}(x) \leq 0$. Moreover, as x_e is the only stationary point of the function $H_d(x)$, hence of $\Delta(x)$, we conclude that in any neighborhood of $\bar{x}$ the function $\Delta(x)$ takes positive and negative values. Finally, for any neighborhood of $\bar{x}$, the set of points

$$C = \{x \mid \Delta(x) < 0 \cap \dot{\Delta}(x) < 0\}$$

is non-empty and has $\bar{x}$ as an accumulation point. This implies, by Cetaev Instability Theorem, that the point $\bar{x}$ is an unstable equilibrium. Therefore, with the exception of the trajectories starting from a point belonging to the union of the stable manifolds[5] of the equilibria in $\bar{E}$, all other trajectories converge asymptotically towards the desired equilibrium x_e. $\triangleleft$

Remark 5 Note that the result in Proposition 1 does not require the assumption $\bar{P}_3 \neq 0$. Hence, it is also proved that the control law (13) with $\bar{P}_3 = 0$ solves the almost global IDA stabilization problem for the zero equilibrium of system (1).

Remark 6 If one is only interested in almost IDA stabilization of the equilibrium E_1''' with $\bar{P}_3 \neq 0$ then, the proof of Proposition 1 can be repeated setting $\lambda = 0$, hence obtaining much simpler control laws. However, if one is interested in the stabilization of the zero equilibrium, *i.e.* of the equilibrium E_1''' with $\bar{P}_3 = 0$ then, a nonzero λ is needed.

Remark 7 In the proof of Proposition 1 a classical Lyapunov instability theorem has been used to prove that any point in the set $\bar{E} = E \setminus \{x_e\}$ is an unstable equilibrium. However, if numerical values for the masses M_1, M_2 and M_3 and the moments of inertia J_1, J_2 and J_3, are selected, and a specific $\bar{P}_3$ is given, then such instability analysis can be carried out directly on the linear approximation of the closed loop system around any point in the set $\bar{E}$. For example, if $\bar{P}_3 = 0$, and the mechanical parameters are selected as in [23], then a simple computation shows that the closed loop linearized system at any point in the set $\bar{E}$ has one eigenvalue with positive real part. Figure 2 shows the real part of the eigenvalues of such a linearized system. The set E is parameterized by Π_3. Note that for any $\Pi_3 \neq 0$, *i.e.* for any point in the set $\bar{E}$, there is an eigenvalue with positive real part.

5 Stabilization of a class of *order two* relative equilibria

In this section we consider the problem of local asymptotic stabilization via IDA of the maneuver M_2', namely a steady rising/diving with forward/reverse motion.

Proposition 2 *Consider system (1) and the equilibrium point* $x_e = \mathrm{col}(0, 0, 0, \bar{P}_1, 0, \bar{P}_3)$, i.e. *the equilibrium* E_2'. *Suppose* $M_1 > M_2$. *Then the system is locally IDA stabilizable around* x_e.

[5]This set of *bad* initial conditions has measure zero.

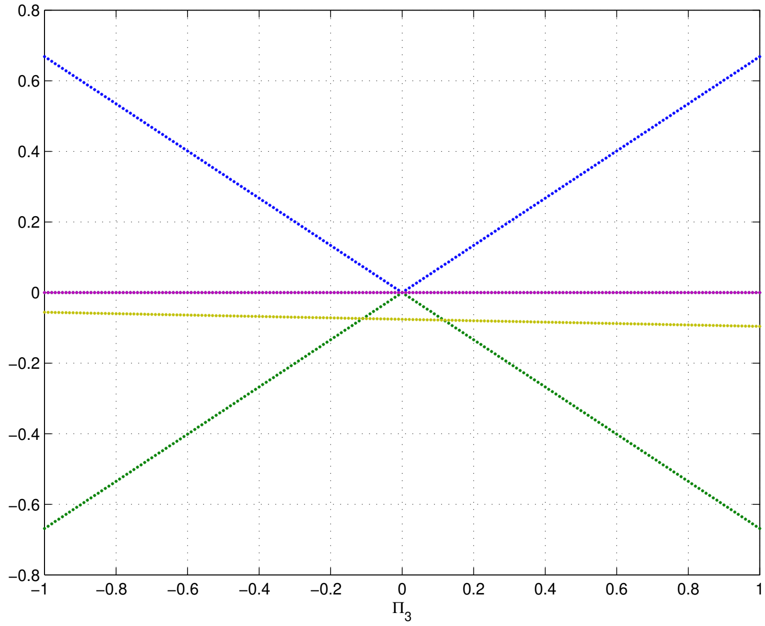

Figure 2: Real part of the eigenvalues of the linearized closed loop system (1)-(13) in the set E parameterized by Π_3.

Proof. Condition (C1). Using Lemma 3 it is straightforward to verify that condition (C1) is satisfied by

$$H_a(x) = -\frac{1}{4}\frac{1+2\mu M_1}{\bar{P}_1^2 M_1}(P_1^2+P_2^2)^2 + \mu(P_1^2+P_2^2) - \frac{\bar{P}_3}{M_3}P_3,$$

$J_a(x) = 0$, $R_a(x) = \mathrm{diag}(R_1, R_2, 0, 0, 0, R_6)$ and

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -P_3\left(-\frac{(1+2\,\mu\,M_1)\left({P_1}^2+{P_2}^2\right)P_2}{\bar{P}_1^2 M_1} + 2\,\mu\,P_2\right) - \frac{P_2\,\bar{P}_3}{M_3} - \frac{R_1\,\Pi_1}{J_1} \\ P_3\left(-\frac{(1+2\,\mu\,M_1)\left({P_1}^2+{P_2}^2\right)P_1}{\bar{P}_1^2 M_1} + 2\,\mu\,P_1\right) + \frac{P_1\,\bar{P}_3}{M_3} - \frac{R_2\,\Pi_2}{J_2} \\ \frac{R_6}{M_3}\left(\bar{P}_3 - P_3\right) \end{bmatrix}. \quad (14)$$

Condition (C2). To begin with, note that the modified Hamiltonian

$$\begin{aligned} H_d(x) = H(x) + H_a(x) = &\ \frac{1}{2}\left(\frac{\Pi_1^2}{J_1} + \frac{\Pi_2^2}{J_2} + \frac{\Pi_3^2}{J_3} + \frac{P_1^2}{M_1} + \frac{P_2^2}{M_2} + \frac{P_3^2}{M_3}\right) \\ & -\frac{1}{4}\frac{1+2\mu M_1}{\bar{P}_1^2 M_1}(P_1^2+P_2^2)^2 + \mu(P_1^2+P_2^2) - \frac{\bar{P}_3}{M_3}P_3 \end{aligned}$$

has a stationary point for $x = x_e$. Moreover, under the assumption $M_1 > M_2$, the Hessian matrix is positive definite for any $\mu < -\frac{1}{2M_1}$ and, by condition (C1)

$$\dot{H}_d \leq 0,$$

if R_1, R_2 and R_6 are positive. As a result $H_d(x)$ qualifies as a (local) Lyapunov function for the considered equilibrium. Finally, note that $\dot{H}_d = 0$, implies $\Pi_1 = \Pi_2 = 0$ and $P_3 = \bar{P}_3$, and, by La Salle invariance principle, along the trajectories of the closed loop system one has

$\dot{\Pi}_1 = \dot{\Pi}_2 = \dot{P}_3 = 0$. This implies $P_1 = \bar{P}_1$ and $P_2 = 0$, which in turn yields $\dot{P}_1 = \dot{P}_2 = 0$, and $\Pi_3 = 0$, which proves the claim. $\triangleleft$

Remark 8 It must be noted that the condition $M_1 > M_2$ does not imply Lyapunov stability of the equilibrium E_2', namely $x_e = \mathrm{col}(0, 0, 0, \bar{P}_1, 0, \bar{P}_3)$, with constant (nonzero) input.

Remark 9 A simple analysis reveals that the closed loop system (1)-(14) has multiple equilibria. Besides the equilibrium of interest $x_e = \mathrm{col}(0, 0, 0, \bar{P}_1, 0, \bar{P}_3)$, the point $\tilde{x}_e = \mathrm{col}(0, 0, 0, -\bar{P}_1, 0, \bar{P}_3)$ is also a locally asymptotically stable equilibrium. Further isolated equilibria, depending on the value of the constant μ may be present. However, if[6]

$$\mu \in \mathcal{M} = \left[-\frac{1}{2M_2}, -\frac{1}{2M_1}\right]$$

no additional isolated equilibria exist. Finally, the hyperplane

$$\{(\Pi_1, \Pi_2, \Pi_3, P_1, P_2, P_3) \mid \Pi_1 = \Pi_2 = P_1 = P_2 = 0, P_3 = \bar{P}_3\}$$

is also an equilibrium manifold. Simple computations show that, if $\mu < -\frac{1}{2M_1}$, the points of this manifold are unstable equilibria.

6 Simulation results

Simulations have been carried out to demonstrate the properties of the proposed control laws. The parameters of the underwater vehicle have been taken from [23]. Two prototype maneuvers are considered, namely the maneuver M_1''' and the maneuver M_2'.

Figures 3 and 4 display the state and control histories of system (1) in closed loop with the control law derived in Proposition (1) for $\bar{P}_3 = 1$ and $\bar{P}_3 = 0$, respectively. Note the non-exponential convergence rate.

Figure 5 displays the state and control histories of system (1) in closed loop with the control law (14). The steady rising and forward linear momenta are $\bar{P}_1 = 1/2$ and $\bar{P}_3 = 1$, respectively.

7 Conclusions

The problem of asymptotic stabilization of the *underactuated* Kirchhoff's equations, *i.e.* of the equations describing the linear and angular velocities of a rigid body in an ideal fluid in presence of partial actuation, has been addressed. It has been shown that, using the IDA methodology developed in [18], a class of physically motivated equilibria can be (almost) asymptotically stabilized. The result in this paper complements the study in [13, 23, 24, 11], where the same stabilization problem has been addressed by use of internal rotors. It must be noted that both the approach pursued in this paper and that in [24], and related papers, are of interest in applications, depending on the structure and the nature of the actuation devices. The theoretical developments have been complemented by simulations.

Work is under way to improve the dynamic behavior of the closed loop system, *e.g.* the convergence rate, and to design stabilizing control law with other actuators configuration.

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[6] By the assumption $M_1 > M_2$ the set $\mathcal{M}$ is not empty.

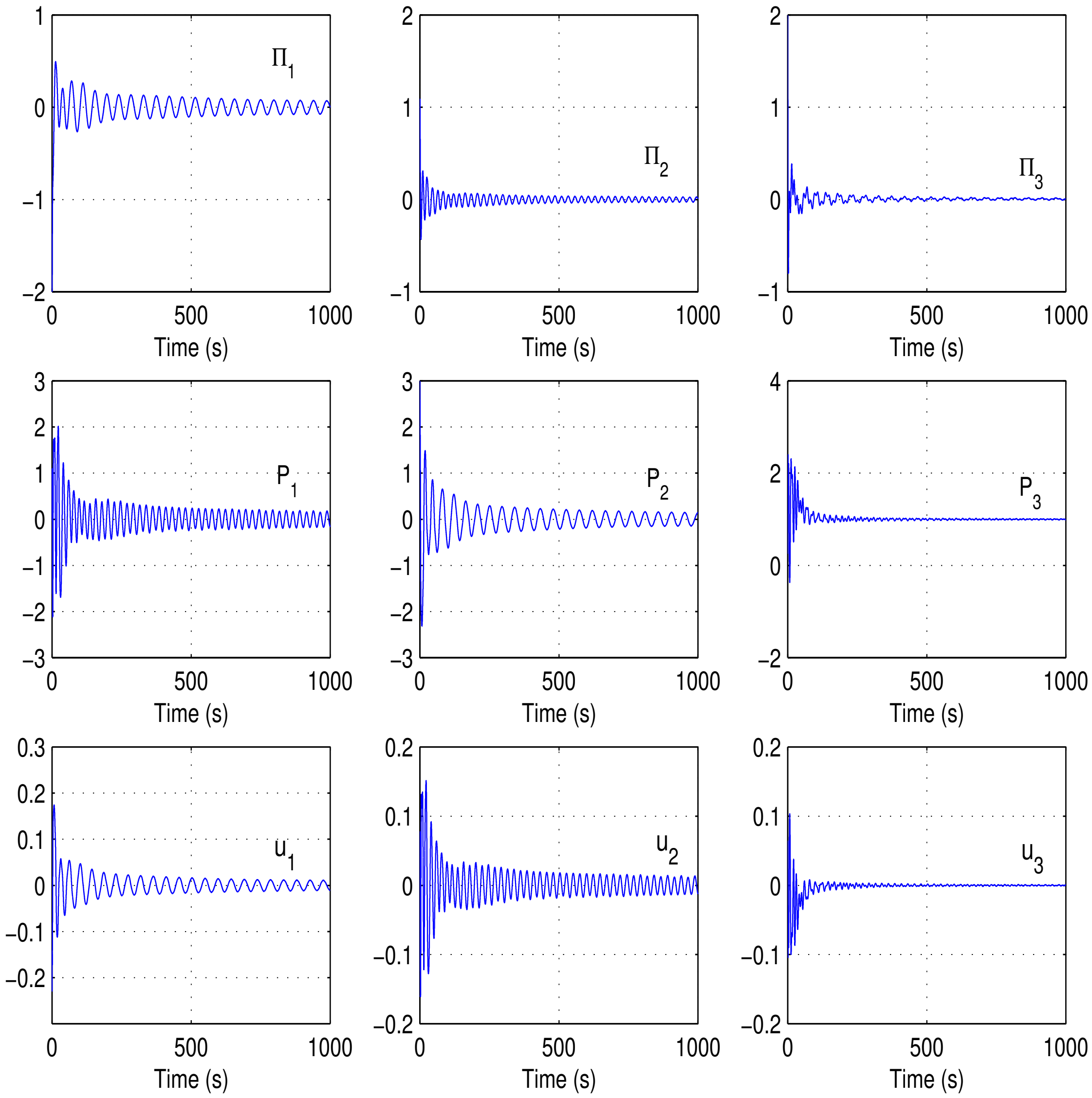

Figure 3: State and control histories of the underwater vehicle for maneuver M_1''' with $\bar{P}_3 = 1$.

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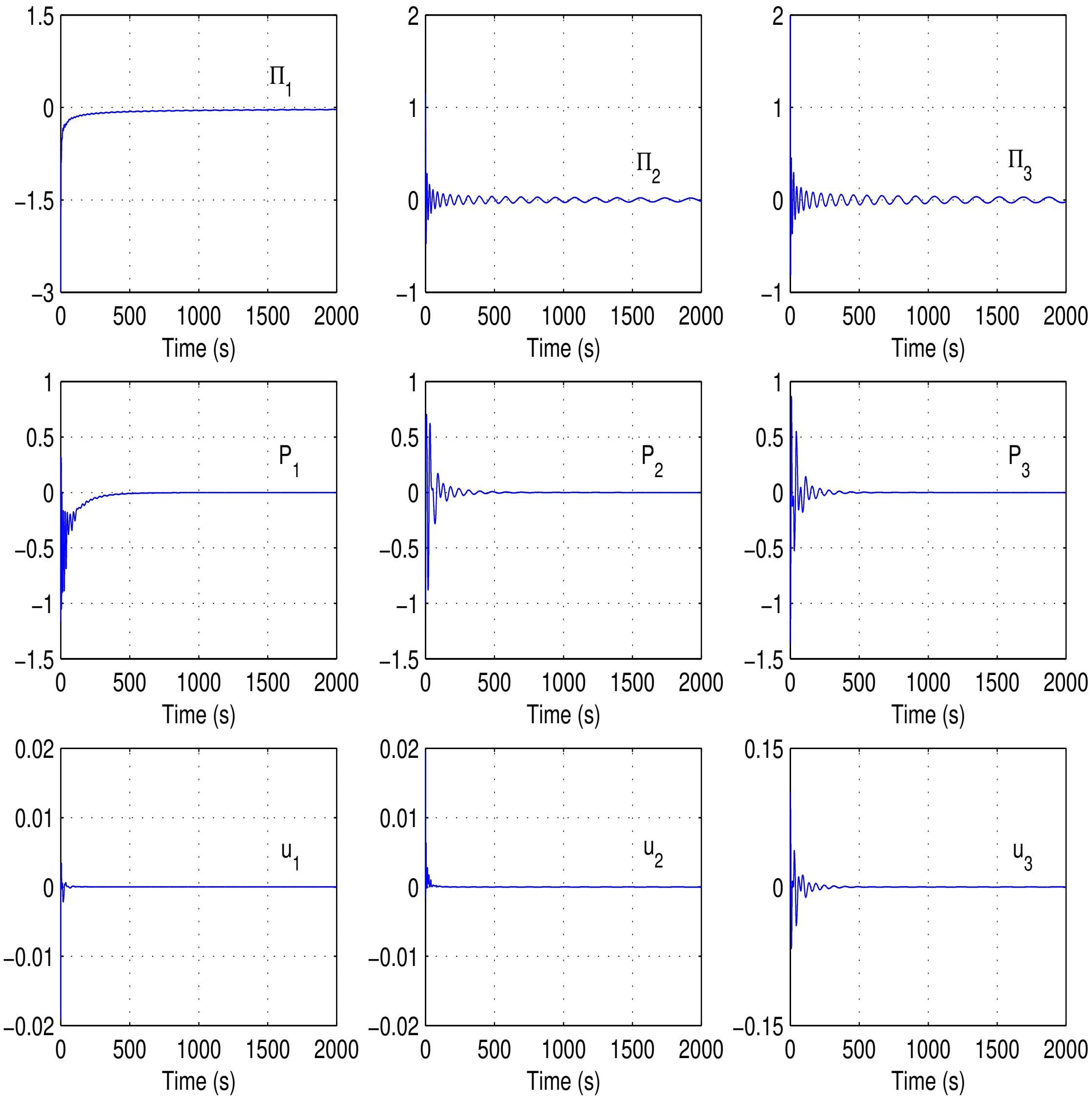

Figure 4: State and control histories of the underwater vehicle for maneuver M_1''' with $\bar{P}_3 = 0$.

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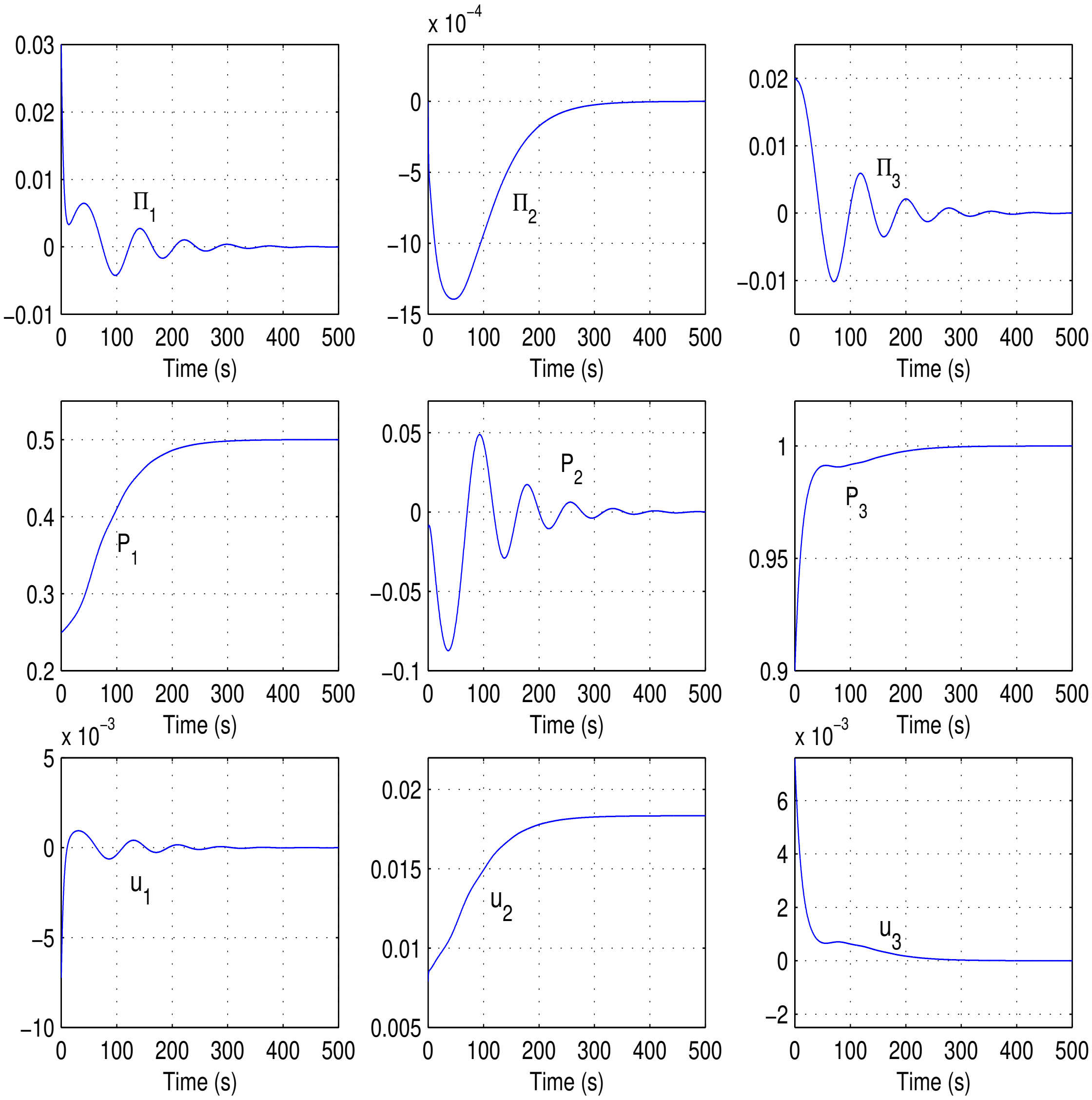

Figure 5: State and control histories of the underwater vehicle for maneuver M_2' with $\bar{P}_1 = 1/2$ and $\bar{P}_3 = 1$.