Immersion and invariance: a new tool for stabilization and adaptive control of nonlinear systems*

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Abstract

A new method to design asymptotically stabilizing and adaptive control laws for nonlinear systems is presented. The method relies upon the notions of system immersion and manifold invariance and, in principle, does not require the knowledge of a (control) Lyapunov function. The construction of the stabilizing control laws resembles the procedure used in nonlinear regulator theory to derive the (invariant) output zeroing manifold and its friend. The method is well suited in situations where we know a stabilizing controller of a nominal reduced order model, which we would like to robustify with respect to higher order dynamics. This is achieved by designing a control law that asymptotically immerses the full system dynamics into the reduced order one. We also show that in adaptive control problems the method yields stabilizing schemes that counter the effect of the uncertain parameters adopting a robustness perspective—this is in contrast with most existing adaptive designs that (relying on certain matching conditions) treat these terms as disturbances to be rejected. It is interesting to note that our construction does not invoke certainty equivalence, nor requires a linear parameterization, furthermore, viewed from a Lyapunov perspective, it provides a procedure to add cross terms between the parameter estimates and the plant states. Finally, it is shown that the proposed approach is directly applicable to systems in feedback and feedforward form, yielding new stabilizing control laws. We illustrate the method with several academic and practical examples, including a mechanical system with flexibility modes, an electromechanical system with parasitic actuator dynamics and an adaptive nonlinearly parameterized visual servoing application.

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1 Introduction

The problems of stabilization and adaptive control of nonlinear systems have been widely studied in the last years and several constructive or conceptually constructive methodologies have been proposed, see, e.g., the monographs [19, 37, 13, 21] for a summary of the state of the art. Most of the nonlinear stabilization methods rely on the use of (control) Lyapunov functions either in the synthesis of the controller or in the analysis of the closed loop system. For systems with Lagrangian or Hamiltonian structures Lyapunov functions are replaced by storage functions with passivity being the sought–after property [27]. Alternatively, the input–to–state stability point of view [38], the concept of nonlinear gain functions and the nonlinear version of the small gain theorem [14, 40] have been used in the study of cascaded or interconnected systems.

At the same time the local/global theory of output regulation has been developed and systematized [6]. This relies upon the solution of the so-called Francis-Byrnes-Isidori (FBI) equations: a set of partial differential equations (PDEs) that must be solved to compute a solution to the regulator problem. It must be noted that the solution of the nonlinear regulator problem requires ultimately the solution of a (output feedback) stabilization problem, whereas, to the best of the authors' knowledge, the tools and concepts exploited in the theory of output regulation have not yet been used in the solution of standard stabilization problems.

In the present work we take a new look at the nonlinear stabilization and adaptive control problems. More precisely, we make use of two classical tools of nonlinear regulator theory and of geometric nonlinear control—(system) Immersion and (manifold) Invariance—to reduce the problem of designing stabilizing and adaptive control laws for general nonlinear systems to other subproblems which, in some instances, might be easier to solve. We call the new methods I&I stabilization and I&I adaptive control.

The concept of invariance has been widely used in control theory. The development of linear and nonlinear geometric control theory (see [44, 26, 11] for a comprehensive introduction) has shown that invariant subspaces, and their nonlinear equivalent, invariant distributions, play a fundamental role in the solution of many design problems. Slow and fast invariant manifolds, which naturally appear in singularly perturbed systems, were used for stabilization [18] and analysis of slow adaptation systems [34]. Relatively recently, it has also been discovered that the notion of invariant manifolds is crucial in the design of stabilizing control laws for classes of nonlinear systems. More precisely, the theory of the center manifold [8] has been instrumental in the design of stabilizing control laws for systems with noncontrollable linear approximation, see, e.g., [1], whereas the concept of zero dynamics and the strongly related notion of zeroing manifold have been exploited in several local and global stabilization methods, including passivity based control, backstepping and forwarding.

The notion of immersion has also a longstanding tradition in control theory. Its basic idea is to *project* the system under consideration into a system with prespecified properties. For example, the classical problem of immersion of a generic nonlinear system into a linear and controllable system by means of static or dynamic state feedback has been extensively studied, see [26, 11] for further detail. State observation has traditionally being formulated in terms of system immersion, see [15] for a recent application. More recently, immersion has been used in the nonlinear regulator theory to derive necessary and sufficient conditions for robust regulation. In [6, 12] it is shown that robust regulation is achievable provided that the exosystem can be immersed into a linear and observable system.

Instrumental for the developments of this paper is to recast stabilization in terms of system immersion.[1] More precisely, we consider the system $\dot{x} = f(x, u)$ and the basic stabilization problem of finding (whenever possible) a state feedback control law $u = u(x)$ such that the closed loop system is locally (globally) asymptotically stable. The procedure that we propose to solve this problem consists of two steps, first, find a target dynamical system $\dot{\xi} = \alpha(\xi)$ which is locally (globally) asymptotically stable and of dimension strictly smaller than the dimension of x, a mapping $x = \pi(\xi)$, and function

[1]In this respect we bring the readers attention to [25], where it is shown that a dynamical system (possibly infinite dimensional) is stable if it can be immersed into another stable dynamical system by means of a so–called stability preserving mapping. The definition of the latter given in [25] is related with some of the concepts used in this paper.

$c(x)$, such that

$$f(\pi(\xi), c(\pi(\xi))) = \frac{\partial \pi}{\partial \xi}(\xi)\alpha(\xi),$$

i.e., any trajectory $x(t)$ of the system $\dot{x} = f(x, c(x))$ is the image through the mapping $\pi(\cdot)$ of a trajectory of the target system. Note that the mapping $\pi : \xi \to x$ is an immersion, i.e., the rank of π is equal to the dimension of ξ. Second, apply a control law that renders the manifold $x = \pi(\xi)$ attractive and keeps the closed–loop trajectories bounded. In this way we have that the closed–loop system will asymptotically behave like the desired target system and stability will be ensured. This reformulation of the stabilization problem is implicit in sliding mode control where the target dynamics are the dynamics of the system on the sliding manifold, which is made attractive with a discontinuous control, while $c(x)$ is the so-called equivalent control [43]. (In this case, due to the discontinuous nature of the control, the manifold is reached in finite time.) A similar procedure is proposed in [16], with the fundamental differences that π is not an immersion, but a change of coordinates, and the applied control is the one that renders the manifold invariant.

From the above discussion it is obvious that the concept of immersion requires the selection of a *target dynamical system*. This is in general a non-trivial task, as the solvability of the underlying control design problem depends upon such a selection. For general nonlinear systems the classical target dynamics are linear, and a complete theory in this direction has been developed both for continuous and discrete time systems [26, 11]. For physical systems the choice of a linear target dynamics is not necessarily the most suitable one because, on one hand, workable designs should respect the constraints imposed by the physical structure. On the other hand, it is well–known that most physical systems are not feedback linearizable. Fortunately, in many cases of practical interest, it is possible to identify a natural (not necessarily linear) target dynamics. For instance, for systems admitting a slow/fast decomposition—which usually appears in applications where actuator dynamics or bending modes must be taken into account—a physically reasonable selection for the target dynamics is the slow (rigid) subsystem, for which we assume known a stabilizing controller. In all these examples the application of the I&I method may be interpreted as a procedure to robustify, with respect to some higher order dynamics, a controller derived from a low order model. Other physical situations include unbalanced electrical systems where their regulated balanced representation is an obvious choice, whereas for AC drives the so–called field oriented behavior is a natural selection for the target dynamics. These problems are currently being studied and will be reported elsewhere.

I&I is also applicable in adaptive control, where a sensible target dynamics candidate is the closed–loop system that would result if we applied the known parameters controller. Clearly, in this case the target dynamics is only partially known but, as we show in the paper, the mapping $x = \pi(\xi)$ mentioned above is *naturally* defined and, under some suitable structural assumptions, it is possible to design non–certainty equivalent controllers such that adaptive stabilization is achieved. It is interesting to note that our construction does not require a linear parameterization. Also, viewed from a Lyapunov function perspective, I&I provides a procedure to add cross terms between the parameter estimates and the plant states. It is widely recognized that the unnatural linear parameterization assumption and our inability to generate non–separable Lyapunov functions have been the major Gordian knots that have stymied the practical application of adaptive control, hence the importance of overcoming these two obstacles. To the best of our knowledge, with the notable exception of [33], the present work constitutes the first general contribution in this direction.

In this work, the I&I method is employed to design stabilizing and adaptive control laws for academic and physical examples in some of the situations described above. In Section 2 the general theory is presented, namely a set of sufficient conditions for the construction of local (global) stabilizing control laws for general nonlinear affine systems. These results are then used in Section 3 to treat examples of actuator dynamics and flexible modes. Section 4 presents the formulation—within the framework of I&I—of the adaptive control problem. As an illustration we solve the long standing problem of adaptive camera calibration for visual servoing, which is a nonlinearly parameterized example. In Section 5 we derive new control laws for systems with special structures, namely the so–called triangular and feedforward forms. Finally, Section 6 gives some summarizing remarks and suggestions for future work.

2 Asymptotic stabilization via immersion and invariance

The present section contains the basic theoretical results of the paper, namely a set of sufficient conditions for the construction of globally asymptotically stabilizing static state feedback control laws for general, control affine, nonlinear systems. Note however, that similar considerations can be done for dynamic output feedback and nonaffine systems, while local versions follow *mutatis mutandis.* We also present two motivating examples.

2.1 Main result

Theorem 1 *Consider the system*[2]

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

with state $x \in I\!R^n$ and control $u \in I\!R^m$, with an equilibrium point $x_* \in I\!R^n$ to be stabilized. Let $p < n$ and assume we can find mappings

$$\alpha(\cdot) : I\!R^p \to I\!R^p \qquad \pi(\cdot) : I\!R^p \to I\!R^n \qquad c(\cdot) : I\!R^p \to I\!R^m$$
$$\phi(\cdot) : I\!R^n \to I\!R^{n-p} \qquad \psi(\cdot,\cdot) : I\!R^{n\times(n-p)} \to I\!R^m$$

such that the following hold.

(H1) (Target system) *The system*

$$\dot{\xi} = \alpha(\xi), \tag{2}$$

with state $\xi \in I\!R^p$, has a globally asymptotically stable equilibrium at $\xi_* \in I\!R^p$ and $x_* = \pi(\xi_*)$.

(H2) (Immersion condition) *For all $\xi \in I\!R^p$*

$$f(\pi(\xi)) + g(\pi(\xi))c(\pi(\xi)) = \frac{\partial \pi}{\partial \xi}\alpha(\xi). \tag{3}$$

(H3) (Implicit manifold) *The following set identity holds*

$$\{x \in I\!R^n \mid \phi(x) = 0\} = \{x \in I\!R^n \mid x = \pi(\xi) \text{ for some } \xi \in I\!R^p\}. \tag{4}$$

(H4) (Manifold attractivity and trajectory boundedness) *All trajectories of the system*

$$\dot{z} = \frac{\partial \phi}{\partial x}\left[f(x) + g(x)\psi(x,z)\right] \tag{5}$$
$$\dot{x} = f(x) + g(x)\psi(x,z) \tag{6}$$

are bounded and satisfy

$$\lim_{t\to\infty} z(t) = 0. \tag{7}$$

Then x_* is a globally asymptotically stable equilibrium of the closed loop system

$$\dot{x} = f(x) + g(x)\psi(x, \phi(x))$$

Proof. We establish the claim in two steps. First, it is shown that the equilibrium x_* is globally attractive, then that the closed loop system possesses a Lyapunov stability property.

By (H4), and the fact that the right hand side of (5) is $\dot{\phi}$, we have that any trajectory of the closed loop system is bounded and it is such that (7) holds, i.e., it converges towards the manifold $\phi(x) = 0$, which is well defined by (H3). Moreover, by (H1) and (H2) the manifold is invariant and internally asymptotically stable, hence all trajectories of the closed loop system converge to the equilibrium x_*.

Note now that any trajectory of the closed loop system is the image through the mapping $\pi(\cdot)$ of a trajectory of the target system, which is globally asymptotically stable by (H1). Moreover, for any $\epsilon_1 > 0$ there exists $\delta_1 > 0$ such that $||\xi(0)|| < \delta_1$ implies $||\xi(t)|| < \epsilon_1$. Hence, by regularity of $\pi(\cdot)$, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$||\pi(\xi(0))|| < \delta \Rightarrow ||\pi(\xi(t))|| < \epsilon,$$

which proves the claim. $\triangleleft$

[2]Throughout the paper, if not otherwise stated, it is assumed that all functions and mappings are C^∞. Note however, that all results can be derived under much weaker regularity assumptions.

Discussion

The following observations concerning the assumptions are in order.

1. In the applications of Theorem 1 described below the target system is a priori defined, hence condition (H1) is automatically satisfied. (See Remark 4 below.)
2. Given the target system, the equation (3) of condition (H2) defines a partial differential equation in the unknown π, where c is a free parameter. Note that, if the linearization of (1) (at $x = x_*$) is controllable (and all functions are locally analytic), it has been shown in [16], using Lyapunov Auxiliary theorem and under some *non-resonance* conditions, that we can always find c such that the solution exists locally. Nevertheless, finding the solution of this equation is—in general—a difficult task.Despite this fact, we will show later that a suitable selection of the target dynamics, i.e., following physical and system theoretic considerations, simplifies this problem. In particular we will prove that, in the adaptive control context, picking the natural target dynamics candidate allows to *obviate* this task.
3. Hypothesis (H3) states that the image of the mapping π can be expressed as the zero of a (smooth) function ϕ. Roughly speaking, this is a condition on the invertibility of the mapping that translates into rank restrictions on $\frac{\partial \pi}{\partial \xi}$. In the linear case when $\pi(\xi) = T\xi$, with T some constant $(n \times p)$–matrix, we have $\pi(x) = T^{\perp} x$, where $T^{\perp} T = 0$, and (H3) will hold if and only if T is full rank. In the general nonlinear case, if $\pi : I\!\!R^p \to I\!\!R^n$ is an injective and proper[3] immersion then the image of π is a submanifold of $I\!\!R^n$. (H3) thus requires that such a submanifold can be described (globally) as the zero level set of the function ϕ.

Note, finally, that if there exists a partition of $x = \mathrm{col}(x_1, x_2)$, with $x_1 \in I\!\!R^p$ and $x_2 \in I\!\!R^{n-p}$, and a corresponding partition of $\pi = \mathrm{col}(\pi_1, \pi_2)$ such that π_1 is a global diffeomorphism, then the function $\phi(x) = x_2 - \pi_2(\pi_1^{-1}(x_1))$ is such that (H3) holds.
4. An important remark is that, since the "basis functions" $\phi(x)$ are not uniquely defined, their choice provides an alternative degree of freedom for the verification of (H4), which becomes in this way a non–standard stabilization problem. This central idea is extensively used in adaptive control as well as in the example of Section 3.2.
5. The hypothesis (H4) is given in terms of trajectories of an extended system with state (x, z). An alternative (simpler) formulation can be given if (as detailed above) it is possible to define $z = \phi(x) = x_2 - \pi_2(\pi_1^{-1}(x_1))$, then the control $\psi(x, z) = \psi(x, x_2 - \pi_2(\pi_1^{-1}(x_1))$ has to be such that the trajectories of the system $\dot{x} = f(x) + g(x)\psi(x, x_2 - \pi_2(\pi_1^{-1}(x_1))$ are bounded and $\lim_{t\to\infty} x_2(t) - \pi_2(\pi_1^{-1}(x_1(t))) = 0$. This formulation will be used when dealing with the adaptive control problem.
6. The convergence condition (7) can be relaxed, i.e., to prove asymptotic convergence of $x(t)$ to $x_\star$ it is sufficient to require

$$\lim_{t\to\infty} g(x(t))(\psi(x(t), z(t)) - \psi(x(t), 0)) = 0.$$

Remark 1 The result summarized in Theorem 1 lends itself to the following interpretation. Given the system (1) and the target dynamical system (2) find, if possible, a manifold $\mathcal{M}$, described implicitly by $\{x \in I\!\!R^n \mid \phi(x) = 0\}$, and in parameterized form by $\{x \in I\!\!R^n \mid x = \pi(\xi), \xi \in I\!\!R^p\}$, which can be rendered invariant and asymptotically stable, and such that the (well defined) restriction of the closed loop system to $\mathcal{M}$ is described by $\dot{\xi} = \alpha(\xi)$. Notice, however, that we do not propose to apply the control $u = c(\pi(\xi))$ that renders the manifold invariant, instead we design a control law $u = \psi(x, z)$ that drives to zero the off–the–manifold coordinate z and keeps the system trajectories bounded. However, since the stability condition (H4) imposes $\psi(\pi, 0) = c(\pi)$, on the manifold we actually apply the control that renders $\mathcal{M}$ invariant—this nice "reduction property" is clearly absent in sliding mode control.

Remark 2 Equation (3) is precisely the PDE arising in nonlinear regulator theory, that we refer here as the FBI equation.[4] However, despite the obvious similarities, the FBI equation, and its

[3]Recall that an immersion is a mapping $\pi : I\!\!R^p \to I\!\!R^n$, with $p < n$. It is injective if rank $\pi = p$, and it is proper if the inverse image of any compact set is also compact.

[4]To be precise, two equations arise in nonlinear regulator theory. The former is equation (3), the latter is an equation expressing the fact that the tracking error is zero on the invariant manifold defined via the solution of equation (3). With abuse of terminology, we refer to equation (3) as the FBI equation.

solution, are used in the present context in a completely new form. First of all, in classical regulator theory, the system $\dot{\xi} = \alpha(\xi)$ is assumed Poisson stable, whereas in the I&I framework, it is required to be asymptotically stable. Second, while in regulator theory the mapping $\pi(\xi)$ is needed to define a controlled invariant manifold for the system composed of the plant and the exosystem, in the present approach the mapping $\pi(\xi)$ is used to define a parameterized controlled invariant manifold, which is a submanifold of the state space of the system to stabilize. Finally, in regulator theory the exosystem $\dot{\xi} = \alpha(\xi)$ is driving the plant to be controlled, whereas in the I&I approach the closed loop system contains a *copy* of the dynamics $\dot{\xi} = \alpha(\xi)$.

Finally, equation (3) arises also in the paper [16]. Therein, similarly to what is done in the present paper, the goal is to obtain a closed loop system which is locally equivalent to a target linear system. However, unlike the present context, the target system has the same dimension as the system to be controlled, i.e., the mapping π is a (local) diffeomorphism rather than an immersion.

Remark 3 Note that, as discussed in point **6** above, to have asymptotic convergence of $x(t)$ to $x_\star$ it is not necessarily required that the manifold is reached. This fact, which distinguishes the present approach to others, such as sliding mode, is instrumental to develop the adaptive control theory in Section 4. Note in fact that, in the adaptive control framework the manifold is not known, hence cannot possibly be reached. However, to have asymptotic regulation of the system state it is not necessary to reach the manifold, see Example 2 for a simple illustration of this fact.

Remark 4 In Theorem 1 a stabilizing control law is derived starting from the selection of a target (asymptotically stable) dynamical system. A different perspective can be taken: given the mapping $x = \pi(\xi)$, hence the mapping $z = \phi(x)$, find (if possible) a control law which renders the manifold $z = 0$ invariant and asymptotically stable, and a globally asymptotically stable vector field $\dot{\xi} = \alpha(\xi)$ such that the FBI equation (3) holds. If this goal is achieved the system (1) with output $z = \phi(x)$ is (globally) minimum phase. It is therefore apparent that the result in Theorem 1 can be regarded as the *dual* of the classical stabilization methods based on the construction of passive or minimum phase outputs, see [5, 27] and the survey paper [3].

Remark 5 We stress that Lyapunov based design methods are somewhat dual to the approach (informally) described above. As a matter of fact, in Lyapunov design one seeks a function $V(x)$, which is positive definite (and proper, if global stability is required), and which is such that the autonomous system $\dot{V} = \alpha(V)$ is locally (globally) asymptotically stable. Note that the function $V : x \to I$, where I is an interval of the real axis, is a submersion and the target dynamics, namely the dynamics of the Lyapunov function, are one dimensional. See also [25] for some related issues.

We conclude this subsection with a definition, which will be used in the rest of the paper to provide concise statements.

Definition 1 *A system described by equations of the form (1) is said to be* I&I-stabilizable with target dynamics $\dot{\xi} = \alpha(\xi)$ *if the hypotheses (H1) to (H4) of Theorem 1 are satisfied.*

2.2 A preliminary example

We present here a simple illustrative example, with the twofold objective of putting in perspective the I&I formulation with respect to the composite control approach of [18], and giving the flavor of the required computations. This approach is applicable for singularly perturbed systems of the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, u) \\ \epsilon \dot{x}_2 &= g(x_1, x_2, u) \end{aligned}$$

for which a slow manifold, defined by the function $x_2 = h(x_1, u, \epsilon)$, exists[5] and results from the solution of the PDE

$$\epsilon \left(\frac{\partial h}{\partial x_1} + \frac{\partial h}{\partial u} \frac{\partial u}{\partial x_1} \right) f(x_1, h, u) = g(x_1, h, u).$$

[5]Remark that in I&I stabilization we address the more demanding task of *creating* a desired manifold, not present in the open–loop plant.

In [18] it is proposed to expand h and u in a power series of ϵ, taking as the zero term in u the control law that stabilizes the slow subsystem. Collecting the terms with the same powers of ϵ we then obtain sets of equations relating the terms h_i and u_i, that can be iteratively solved to approximate (up to any order $\mathcal{O}(\epsilon^k)$) the solution of the PDE. The control is then constructed as

$$u = u_0 + \epsilon u_1 + \cdots + \epsilon^k u_k - K[x_2 - (h_0 + h_1 + \cdots + h_k)]$$

where the last term, with $K > 0$, is a fast control that steers the system into the slow manifold.

Example 1 Let us consider the academic two–dimensional example presented in [18], namely

$$\begin{aligned} \dot{x}_1 &= x_1 x_2^3 && (8)\\ \epsilon\dot{x}_2 &= x_2 + u && (9) \end{aligned}$$

First, we follow [18] and choose the immersion dynamics (2) as $\dot{\xi} = -\xi^5$ and fix $\pi_1 = \xi$. The FBI equations (3) become then

$$\begin{aligned} \xi\pi_2^3 &= -\xi^5 \\ \pi_2 + c(\xi) &= -\epsilon\frac{\partial\pi_2}{\partial\xi}\xi^5. \end{aligned}$$

From the first equation we directly obtain[6] $\pi_2 = -\xi^{\frac{4}{3}}$, while the mapping c is defined by the second equation. Now the manifold $x = \pi(\xi)$ can be implicitly described by $\phi(x) = x_2 + x_1^{\frac{4}{3}} = 0$, and the off–the–manifold dynamics (5) is given as

$$\epsilon\dot{z} = \psi(x,z) + x_2 + \frac{4\epsilon}{3}x_1^{\frac{4}{3}}x_2^3.$$

The I&I design is completed choosing $\psi(x,z) = -x_2 - \frac{4\epsilon}{3}x_1^{\frac{4}{3}}x_2^3 - z$, which yields the closed–loop dynamics

$$\begin{aligned} \dot{x}_1 &= x_1 x_2^3 \\ \epsilon\dot{x}_2 &= -\tfrac{4\epsilon}{3}x_1^{\frac{4}{3}}x_2^3 - z \\ \epsilon\dot{z} &= -z. \end{aligned} \qquad (10)$$

From the last equation above it is clear that (7) holds, hence, to complete the proof it only remains to show that all trajectories of the system (10) are bounded. For, consider the (partial) coordinates transformation $\eta = x_2 + x_1^{4/3}$ yielding

$$\begin{aligned} \dot{x}_1 &= x_1(\eta - x_1^{4/3})^3 \\ \epsilon\dot{\eta} &= -z \\ \epsilon\dot{z} &= -z, \end{aligned} \qquad (11)$$

and note that $z(t)$ and $\eta(t)$ are bounded for all t. Finally, boundedness of $x_1(t)$ can be proved observing that the dynamics of x_1 can be expressed in the form

$$\dot{x}_1 = -x_1^5 + p(x_1,\eta)$$

for some function $p(x_1,\eta)$ satisfying $|p(x_1,\eta)| \le k(\eta)|x_1|^4$, for some $k(\eta) > 0$ and for $|x_1| > 1$. The control law is obtained as

$$u = \psi(x,\phi(x)) = -2x_2 - x_1^{\frac{4}{3}}(1 + \frac{4\epsilon}{3}x_2^3),$$

that clearly satisfies $\psi(x,0) = c(x_1)$, with c defined by the second FBI equation. For the sake of comparison we note that the composite controller obtained in [18] is

$$u = -2x_2 - x_1^{\frac{4}{3}}(1 + \frac{4\epsilon}{3}x_1^4)$$

[6] This function is only defined for $\xi > 0$, which similarly to [18] will determine the domain of operation of the closed–loop system.

which results from the exact solution of the manifold equations.

If instead of $\dot{\xi} = -\xi^5$ we choose another target dynamics we can easily establish the following global stabilization result.

Proposition 1 *System (8) is globally I&I-stabilizable with target dynamics $\dot{\xi} = -|\xi|^3\xi$, where $\xi \in I\!R$.*

Proof. A simple computation shows that a solution of the FBI equations is $\pi_1(\xi) = \xi$ and $\pi_2(\xi) = -|\xi|$. As a result, the manifold equation takes the form $\phi(x) = x_2 + |x_1| = 0$ and this can be made globally attractive, while keeping all trajectories bounded, by the control law $u = -2x_2 - \epsilon|x_1|x_2^3 - |x_1|$. $\triangleleft$

Remark 6 It is worth noting that the function $\phi(x) = x_2 + |x_1|$ is not everywhere C^1. However, despite the lack of regularity, the I&I procedure can be easily applied.

3 Robustification *vis à vis* higher order dynamics

In this section we present two simple examples of application of I&I stabilization where we know a stabilizing controller of a nominal reduced order model, and we would like to robustify it with respect to some higher order dynamics. First, we consider a levitated system with actuator dynamics, then we treat a robot manipulator with joint flexibilities. The first example is presented to highlight the connections of I&I with other existing techniques, in particular we show how our framework allows to recover the composite control and backstepping solutions. On the other hand, for the robot problem we prove that with I&I we can generate a novel family of global *tracking* controllers under the standard assumptions.

3.1 Magnetic levitation system

Consider a magnetic levitation system consisting of an iron ball in a vertical magnetic field created by a single electromagnet. Adopting the standard modelling assumptions for the electromagnetic coupling, see, e.g., [41], we obtain the model (with domain of validity $-\infty < \xi_2 < 1$)

$$\Sigma_T : \begin{cases} \dot{\xi}_1 &= -\dfrac{R_2}{k}(1-\xi_2)\xi_1 + w \\ \dot{\xi}_2 &= \dfrac{1}{m}\xi_3 \\ \dot{\xi}_3 &= \dfrac{1}{2k}\xi_1^2 - mg \end{cases} \tag{12}$$

where the state vector ξ consists of the flux in the inductance, the ball position and its momentum; w is the voltage applied to the electromagnet, m is the mass of the ball, R_2 is the coil resistance, and k is some positive constant that depends on the number of coil turns.

In low–power applications it is typical to neglect the dynamics of the actuator, hence it is assumed that the manipulated variable is w. In this case, it is possible to asymptotically stabilize the aforementioned model at any constant desired ball position $-\infty < \xi_{2*} < 1$ via a nonlinear static state feedback derived using, for instance, feedback linearization [11], backstepping [19] or interconnection and damping assignment control [29], see also [41] for a comparative study. In medium–to–high power applications the voltage w is generated using a rectifier that includes a capacitance. The dynamics of this actuator can be described by the RC circuit shown in Fig 1, where the actual control voltage is u, while R_1 and C model the parasitic resistance and capacitance, respectively. The full order model of the levitated ball system, including the actuator dynamics, is then given by

$$\Sigma : \begin{cases} \dot{x}_1 &= -\dfrac{R_2}{k}(1-x_3)x_1 + x_2 \\ \dot{x}_2 &= -\dfrac{1}{Ck}(1-x_3)x_1 - \dfrac{1}{R_1C}x_2 + \dfrac{1}{R_1C}u \\ \dot{x}_3 &= \dfrac{1}{m}x_4 \\ \dot{x}_4 &= \dfrac{1}{2k}x_1^2 - mg \end{cases} \tag{13}$$

where we have added to the flux, position and momentum, x_1, x_3, x_4, respectively, the coordinate x_2 that represents the voltage across the capacitor.[7]

Designing a new controller for the full model—if at all possible—could be a time consuming task, hence we might want to simply modify the one we already have to make it robust with respect to the actuator dynamics. This is a typical scenario where I&I stabilization can provide a workable design. In particular, we have the following proposition.

Proposition 2 *The full order model of the levitated ball system (13) is I&I stabilizable (at a constant equilibrium corresponding to $x_3 = x_{3*}$) with target dynamics (12), where $w = w(\xi)$ is any stabilizing state feedback for Σ_T with the property that Σ_T in closed loop with $w(\xi, t) + c(t)$, where $c(t)$ is a bounded signal, has bounded trajectories.*

Proof. As (H1) is automatically satisfied, we will only verify the remaining conditions (H2)–(H4) of Theorem 1. First, some simple calculations show that a solution of the FBI equations (3) is given by the map $\pi(\xi) = \mathrm{col}(\xi_1, w(\xi), \xi_2, \xi_3)$. This solution can be easily obtained *fixing* $\pi_3 = \xi_2$ and $\pi_1 = \xi_1$, a choice which captures our control objective. Now, the parameterized manifold $x = \pi(\xi)$ can be implicitly defined as $\phi(x) = x_2 - w(x_1, x_3, x_4) = 0$, hence condition (H3) is also satisfied. (Notice that the reduced model control w is evaluated on the manifold.) Finally, we have to choose a function $\psi(x, z)$ that preserves boundedness of trajectories and asymptotically stabilizes to zero the off–the–manifold dynamics (5), which in this example are described by

$$\dot{z} = \frac{1}{Ck}(1 - x_3)x_1 - \frac{1}{R_1 C}[x_2 - \psi(x, z)] - \dot{w} \tag{14}$$

where $\dot{w}$ is evaluated on the manifold, hence is computable from the full system dynamics Σ. An obvious simple selection, which yields $\dot{z} = -\frac{1}{R_1 C} z$, is

$$\psi(x, z) = x_2 - z + R_1 C \dot{w} + \frac{R_1}{k}(1 - x_3)x_1.$$

To complete the proof of the proposition it only remains to prove that, for the given $w(x_1, x_3, x_4)$, the trajectories of

$$\begin{aligned}
\dot{z} &= -\frac{1}{R_1 C} z \\
\dot{x}_1 &= -\frac{R_2}{k}(1 - x_3)x_1 + x_2 \\
\dot{x}_2 &= -\tfrac{1}{R_1 C} z + \dot{w}(x) \\
\dot{x}_3 &= \frac{1}{m} x_4 \\
\dot{x}_4 &= \frac{1}{2k} x_1^2 - mg
\end{aligned}$$

are bounded. For, note that in the coordinates (z, η, x_1, x_3, x_4), with $\eta = x_2 - w(x_1, x_3, x_4)$, the system is described by the equations

$$\begin{aligned}
\dot{z} &= -\frac{1}{R_1 C} z \\
\dot{\eta} &= -\tfrac{1}{R_1 C} z \\
\dot{x}_1 &= -\frac{R_2}{k}(1 - x_3)x_1 + w(x_1, x_3, x_4) + \eta \\
\dot{x}_3 &= \frac{1}{m} x_4 \\
\dot{x}_4 &= \frac{1}{2k} x_1^2 - mg,
\end{aligned}$$

from which we see that z converges to zero exponentially fast, hence η is bounded. The proof is completed invoking the robustness (with respect to bounded input disturbances) property assumed

[7] Clearly, Σ_T is the slow model of Σ taking as small parameter the parasitic time constant $R_1 C$.

in the Proposition. The control law is finally obtained as

$$u = w(x_1, x_3, x_4) + R_1 C[\dot{w}(x) + \frac{1}{Ck}(1 - x_3)x_1]$$

$\triangleleft$

Remark 7 The full order system Σ is a port–controlled Hamiltonian system, that is the class for which the interconnection and damping assignment technique has been developed [29], however, the inclusion of the actuator dynamics stymies the application of the method. On the other hand, we have seen above that the I&I approach trivially yields a solution.

Remark 8 We can use this example to compare again the I&I formulation with the composite control approach of [18]. Taking x_2 as the slow variable and $R_1 C$ as the small parameter, the problem reduces to finding a function $h(x_1, x_3, x_4, u)$ and a control u such that $x_2 = h$ describes an invariant manifold. This requires the solution of a PDE of the form

$$-\frac{1}{Ck}(1 - x_3)x_1 + \frac{1}{R_1 C}(u - h) = \frac{\partial h}{\partial x_1}[-\frac{R_2}{k}(1 - x_3)x_1 + h] + \frac{1}{m}\frac{\partial h}{\partial x_3}x_4 + \frac{\partial h}{\partial x_4}(\frac{1}{2k}x_1^2 - mg) + \frac{\partial h}{\partial u}\dot{u}$$

In this simple case an exact solution is possible setting $h = w$ and the first correcting term

$$u_1 = \dot{w} + \frac{1}{Ck}(1 - x_3)x_1.$$

This choice ensures $h_k = 0$ for all $k \geq 1$. The resulting invariant manifold ($x_2 = w$) and the controller are the same we obtained, in a straightforward manner, via I&I stabilization! Furthermore, while in the composite control approach the function u_1 above—that determines the controller—is essentially imposed, in I&I we have great freedom in the choice of $\psi(x, z)$ to stabilize (14). Finally, notice that this controller also results from direct application of backstepping [19] to the system Σ.

3.2 Global tracking for flexible joints robots

As a second example of robustification we consider the problem of global tracking of the n–degrees of freedom flexible joint robot model

$$\Sigma : \left\{ \begin{array}{rcl} D(q_1)\ddot{q}_1 + C(q_1, \dot{q}_1)\dot{q}_1 + g(q_1) + K(q_1 - q_2) & = & 0 \\ J\ddot{q}_2 + K(q_2 - q_1) & = & u \end{array} \right. \tag{15}$$

where $q_1, q_2 \in I\!R^n$ are the link and motor shaft angles, respectively, $D(q_1) = D^\top(q_1) > 0$ is the inertia matrix of the rigid arm, J is the constant diagonal inertia matrix of actuators, $C(q_1, q_2)$ is the matrix related to the Coriolis and centrifugal terms, $g(q_1)$ is the gravity vector of the rigid arm, $K = diag\{k_i\} > 0$ is the joint stiffness matrix, and u is the n dimensional vector of torques. See, e.g., [27] for further details on the model and a review of the recent literature.

We present below a procedure to robustify an arbitrary static–state feedback global *tracking* controller designed for the rigid robot.

Proposition 3 *The flexible joint robot model (15) is globally I&I stabilizable along an arbitrary (four times differentiable) trajectory for the links $q_{1*}(t)$ with target dynamics*

$$\Sigma_T : \left\{ \begin{array}{rcl} \dot{\xi}_1 & = & \xi_2 \\ \dot{\xi}_2 & = & -D(\xi_1)^{-1}[C(\xi_1, \xi_2)\xi_2 + g(\xi_1) - w(\xi, t)], \end{array} \right. \tag{16}$$

where $w(\xi, t)$ is any time–varying state feedback that ensures that the solutions $\xi_1(t)$ of Σ_T globally track (any bounded four times differentiable trajectory $\xi_{1*}(t)$), and with the additional property that in closed loop with $w(\xi, t) + c(t)$, where $c(t)$ is a bounded signal, trajectories remain bounded.[8]

[8]For instance, we can take the well–known Slotine and Li controller

$$w(\xi, t) = D(\xi_1)[\ddot{\xi}_{1*} - \Lambda\dot{\tilde{\xi}}_1] + C(\xi_1, \xi_2)[\dot{\xi}_{1*} - \Lambda\tilde{\xi}_1] - K_p(\dot{\tilde{\xi}} + \Lambda\tilde{\xi}_1) + g(\xi_1),$$

where $K_p = K_p^\top > 0$, $\Lambda = \Lambda^\top > 0$ and $\tilde{\xi}_1 = \xi_1 - \xi_{1*}$.

Proof. To establish the proof we will again verify that the conditions (H1)–(H4) of Theorem 1 are satisfied. First, it is easy to see that a solution of the FBI equations (3) is given by the map

$$\pi(\xi,t) = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_1 + K^{-1}w(\xi,t) \\ \pi_4(\xi,t) \end{bmatrix}$$

where $\pi_4 = \frac{\partial \pi_3}{\partial \xi}\dot{\xi}$, with $\dot{\xi}$ as defined in (16). This solution follows immediately taking the natural state space realization of (15), i.e., $x = \mathrm{col}(q_1, \dot{q}_1, q_2, \dot{q}_2)$, and *fixing* $\pi_1 = \xi_1$, as required by our control objective. Now, an implicit definition of the manifold[9] $\phi(x,t) = 0$ is obtained with

$$\phi(x,t) = \begin{bmatrix} x_3 - x_1 - K^{-1}w(x,t) \\ x_4 - x_2 - K^{-1}\left\{ \frac{\partial w}{\partial x_1}x_2 - \frac{\partial w}{\partial x_2}D^{-1}(x_1)[C(x_1,x_2)x_2 + g(x_1) + K(x_1 - x_3)] - \frac{\partial w}{\partial t} \right\} \end{bmatrix}.$$

It is important to underscore that, while ϕ_1 is obtained from the obvious choice

$$\phi_1(x,t) = x_3 - \pi_3(\xi,t)|_{\xi_1 = x_1, \xi_2 = x_2}$$

the term ϕ_2 is not defined in this manner. However, the set identity (4) is satisfied for the definition above as well. To verify this observe that

$$x_4 - \pi_4(\xi,t)|_{\xi_1 = x_1, \xi_2 = x_2} = \phi_2(x,t) + K^{-1}\frac{\partial w}{\partial x_2}D^{-1}(x_1)K[x_3 - x_1 - K^{-1}w(x,t)]$$

but the last right–hand term in square brackets is precisely ϕ_1. The interest in defining ϕ_2 in this way is that $\dot{\phi}_1 = \phi_2$, simplifying the task of stabilizing the off–the–manifold dynamics, which is given by

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= m(x,t) + J^{-1}\psi(x,z,t) \end{aligned}$$

where $m(x,t)$ can be computed via differentiation of $\phi_2(x,t)$. It is then a trivial task to select a control law $\psi(x,z,t)$ that asymptotically stabilizes z to zero. An obvious simple selection being

$$\psi(x,z,t) = -J[m(x,t) + K_1 z_1 + K_2 z_2],$$

with K_1, K_2 arbitrary positive definite matrices. To complete the proof, it is necessary to show that all trajectories of the closed loop system with state (x, z_1, z_2) are bounded. For, it is sufficient to rewrite the system in the coordinates $(x_1, x_2, \phi(x,t), z_1, z_2)$ and use arguments similar to those in the proof of Proposition 2. $\triangleleft$

Remark 9 In [39] the composite control approach is used to derive approximate feedback linearizing asymptotically stabilizing controllers for the full–inertia model. In the case of block–diagonal inertia considered here the slow manifold equations can be exactly solved and the stabilization is global. It is interesting to note that, as in the previous example, the resulting control law is different from the I&I stabilizer proposed here.

Remark 10 The target dynamics Σ_T *is not* the rigid model obtained from a singular perturbation reduction of the full model Σ with small parameters $\frac{1}{k_i}$. In the latter model the inertia matrix of Σ_T is $D + J$, and not simply D as here, see [39]. Our motivation to choose this target dynamics is clear noting that, if we take the rigid model resulting from a singular perturbation reduction the solution of the FBI equations leads to $\pi_3 = \xi_1 + K^{-1}(w - \dot{\xi}_2)$, significantly complicating the subsequent analysis.

Remark 11 The system (15) is feedback linearizable. Note, in fact, that the system with output $y = q_1$ has a well-defined vector relative degree $(r_1, \cdots, r_n) = (4, \cdots, 4)$ and $\sum r_i = 4n$, i.e., $y = q_1$ are flat outputs. The present design does not exploit this property, and it is such that the system (15) with output $y = x_3 - x_1 - K^{-1}w(x,t) (= \phi_1(x))$ has a well-defined vector relative degree $(r_1, \cdots, r_n) = (2, \cdots, 2)$, with $\sum r_i = 2n$, and the zero dynamics are exactly given by the target dynamics (16).

[9]Note that in this case the target dynamics and the equations defining the invariant manifold depend explicitly on t.

4 Adaptive control via immersion and invariance

In this section we show how the general I&I theory of Section 2 can be used to develop a novel framework for adaptive stabilization of nonlinear systems. A key step for our developments is to add to the classical certainty–equivalent control a new term that, together with the parameter update law, will be designed to satisfy the conditions of I&I. In particular, this new term will *shape the manifold* into which the adaptive system will be immersed. We present first a general theorem for nonlinearly parameterized controllers with linearly parameterized plants. Then, we consider the case when the controller depends linearly in the parameters, and the plant satisfies a matching assumption. As an example of nonlinear parameterization we present, and solve, a prototype robotic vision problem, which has attracted the attention of several researchers in the last years, and proved unsolvable with existing adaptive techniques.

4.1 Problem formulation

We consider the problem of stabilization of systems of the form (1) under the following assumption.

(H5) *(Stabilizability)* There exists a parameterized function $\Psi(x,\theta)$, where $\theta \in \mathbb{R}^q$, such that for some *unknown* $\theta_* \in \mathbb{R}^q$ the system

$$\dot{x} = f_*(x) := f(x) + g(x)\Psi(x,\theta_*) \tag{17}$$

has a globally asymptotically stable equilibrium at $x = x_*$.

The I&I adaptive control problem is then formulated as follows:

Definition 2 *The system (1) with assumption (H5) is said to be adaptively I&I stabilizable if the system*

$$\Sigma : \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)\Psi(x,\hat{\theta} + \beta_1(x)) \\ \dot{\hat{\theta}} & = & \beta_2(x,\hat{\theta}) \end{array} \right. \tag{18}$$

with extended state $x, \hat{\theta}$, and "controls" β_1 and β_2, is I&I stabilizable with target dynamics

$$\Sigma_T : \dot{\xi} = f_*(\xi) \tag{19}$$

From the first equation in (18) we see that in the I&I approach we depart from the certainty–equivalent philosophy and do not apply directly the estimate coming from the update law in the controller. It is important to recall that, in general, $f(x)$, and possibly $g(x)$, will depend on θ_* and are therefore only partially known.

4.2 Classical approaches revisited

To put in perspective the I&I approach—underscoring its new features— it is convenient to recall first the "classical" procedures to address the problem of adaptive control of nonlinear systems. The main difference with respect to the case of linear plants is that the system in closed–loop with the known parameter controller will (in general) still depend on the unknown parameters. Consequently, the Lyapunov function that establishes stability of this system will also be a function of the parameters, and is hence unknown.

The four approaches

There are four different ways to try to bypass this difficulty. (A fifth, practically appealing, alternative is the supervisory control of Morse [9], which is however formulated under different assumptions.)

The first, simplest, way to deal with the presence of unknown parameters in the Lyapunov function is to assume the following.

(A1) *(Lyapunov function matching)* There exists a Lyapunov function $V(x,\theta)$ for the equilibrium x_* of the ideal system $\dot{x} = f_*(x)$ such that $\frac{\partial V}{\partial x}(x,\theta)g(x)$ is *independent* of the parameters, hence computable.[10]

[10] The most notable example where this assumption is satisfied is in the Slotine and Li controller for robot manipulators, see, e.g., [27].

If the structural assumption (A1) holds and if, in addition, we assume that

(H6) *(Linearly parameterized control)* the function $\Psi(x, \theta)$ may be written as

$$\Psi(x, \theta) = \Psi_0(x) + \Psi_1(x)\theta \tag{20}$$

for some known functions $\Psi_0(x)$ and $\Psi_1(x)$;

then, invoking standard arguments, it is possible to propose a certainty–equivalent adaptive control of the form $u = \Psi_0(x) + \Psi_1(x)\hat{\theta}$, and postulate a classical separate Lyapunov function candidate $W(x, \tilde{\theta}) = V(x, \theta) + \frac{1}{2}\tilde{\theta}^\top \Gamma^{-1}\tilde{\theta}$, where $\Gamma = \Gamma^\top > 0$ and $\tilde{\theta} := \hat{\theta} - \theta_*$ is the parameter error. Choosing the estimator that cancels the $\tilde{\theta}$–dependent term in $\dot{W}$ yields the error dynamics as

$$\begin{aligned} \dot{x} &= f_*(x) + g(x)\Psi_1(x)\tilde{\theta} \\ \dot{\tilde{\theta}} &= -\Gamma \Psi_1^\top(x) \left(\frac{\partial V}{\partial x} g(x) \right)^\top \end{aligned} \tag{21}$$

and $\dot{W} = \frac{\partial V}{\partial x} f_*(x) \leq 0$.

If the Lyapunov function V is strict, i.e., $\frac{\partial V}{\partial x} f_*(x)$ is a negative definite function of x, then LaSalle's invariance principle allows to conclude global regulation to x_* of the systems state. Otherwise, it is necessary to add the following, rather restrictive, detectability assumption.

(A2) *(Detectability)* Along the trajectories of the *overall* system (21), the following implication holds

$$\frac{\partial V}{\partial x} f_*[x(t)] \equiv 0 \;\Rightarrow\; \lim_{t\to\infty} x(t) = x_*. \tag{22}$$

It is important to underscore that (22) may hold in the known parameter case, when $\tilde{\theta} = 0$, but be violated when considering the full $(x, \hat{\theta})$ state. (This is the case of gravity compensation of the simple pendulum example, see Example 3 below.) It is worth noting that in [32] a procedure that combines direct and indirect adaptive control to overcome, in some cases, this obstacle has been proposed.

When (A1) is not satisfied the Lyapunov function is expressed with the parameter estimates, instead of the actual unknown parameters, but this brings along a new term to the derivative, i.e., $\frac{\partial V}{\partial \hat{\theta}}\dot{\hat{\theta}}$, that has to be countered.

The second approach assumes that the effect of the parameters can be *rejected* when considered as disturbances with known derivative. The key assumption here is as follows (see Section 4.1.1. of [19]).

(A3) There exists a function $\Psi_\tau(x, \hat{\theta})$, to be added to the stabilizing term $\Psi(x, \hat{\theta})$, such that

$$\frac{\partial V}{\partial x} g(x) \Psi_\tau(x, \hat{\theta}) + \frac{\partial V}{\partial \hat{\theta}} \Gamma \left(\frac{\partial V}{\partial x} f_1(x) \right)^\top = 0.$$

As pointed out in [19] such a function is unlikely to exist since $\frac{\partial V}{\partial x} g(x)$ will, in general, be zero at some points. However, (A3) holds in the so–called "extended matching" case, which leads to the by–now classical adaptive backstepping theory for systems in triangular forms, see, e.g., [19], [21].

In the third approach robustness, instead of disturbance rejection is utilized, thus the matching conditions are replaced by growth conditions—either on the Lyapunov function [33] or the system vector field [36]. Unfortunately, to date there is no characterization of the systems for which the required Lyapunov function growth conditions hold. It should be pointed, however, that systems in so–called strict feedback form with polynomial nonlinearities satisfy the Lyapunov growth condition of [33]. See also [7] for an interesting application to a nonlinearly parameterized nonlinear system.

We should remark that, both the second and third routes assume, instead of (H6), the following.

(H7) *(Linearly parameterized plant)* The vector field $f(x)$ can be written in the form

$$f(x) = f_0(x) + f_1(x)\theta_* \tag{23}$$

for some known functions $f_0(x) : \mathbb{R}^n \to \mathbb{R}^n$ and $f_1(x) : \mathbb{R}^n \to \mathbb{R}^{(n\times q)}$.[11]

[11] For the sake of clarity we present only the case when the uncertain parameters enter in $f(x)$. The design when they enter (linearly) also in $g(x)$ follows *verbatim*. This remark applies also to the I&I scheme of Proposition 4.

Also, it is interesting to note that, similarly to our adaptive I&I scheme, the controllers reported in [19, 21, 33] are non-certainty equivalent and (may be) nonlinear in the parameters.

Finally, the adaptive stabilization problem can also be approached, as done in indirect adaptive control, adopting a (plant) identification perspective. Namely, under assumption (H7), it is easy to show [32] that, with the filtered signals

$$
\begin{aligned}
\dot{w} &= -\lambda(x+w) - [f_0(x) + g(x)u] \\
\dot{\Phi} &= -\lambda\Phi(x) + f_1(x)
\end{aligned}
$$

where $\lambda > 0$ is a design parameter, the identification error $e_I = \Phi(x)\hat{\theta} - (x+w)$ exponentially converges to $\Phi(x)\tilde{\theta}$, at a speed determined by λ. Suggesting the parameter update law $\dot{\hat{\theta}} = -\Phi^\top(x)e_I$, which ensures the parameter error is nonincreasing. The control may be nonlinearly parameterized, but its computation may cross to singularities that may be difficult to avoid. Further, as the estimation of the parameters is decoupled from the control, the only way to guarantee that the "quality of the control" improves with time is by ensuring the parameter error actually decreases. This, in its turn, imposes an excitation restriction on the regressor matrix $\Phi(x(t))$ that is hard to verify *a priori*. These two points are, of course, the Achilles heel of indirect adaptive control.

Discussion

For the purposes of comparison with the I&I approach the following remarks are in order.

1. The analysis of the standard schemes invariably relies on the construction of error equations (similar to (21)), where the x–dynamics is decomposed into a stable portion and a perturbation due to the parameter mismatch. In the I&I formulation we will also write the x–dynamics in an error equation form but with the ideal system perturbed by the off–the–manifold coordinate z.
2. The first and second procedures are based on the cancellation of the perturbation term in the Lyapunov function derivative. It is well–known that cancellation is a very fragile operation, which generates a manifold of equilibria and is at the core of many of the robustness problems of classical adaptive control, see Section 2.2.2 of [30], and Chapter 5 of [20] for additional details. In the third approach this term is not canceled but dominated. In the I&I formulation we will not try to cancel the perturbation term coming from z but—rendering the manifold attractive—only make it sufficiently small. In this respect I&I resembles the third approach as well as indirect adaptive control.
3. Compared to the indirect method, I&I provides additional flexibility to reparameterize the plant so as to avoid controller singularities. This feature has been instrumental in [28] to reduce the prior knowledge on the high frequency gain for linear multivariable plants.
4. The assumption that the *controller* is linearly parameterized is critical in some of the classical procedures. (In some cases it may be replaced by a convexity condition, but this is very hard to verify for more than one uncertain parameter.) As we will see below, in adaptive I&I we do not, *a priori*, require this assumption.

Illustrative example

Before closing this subsection let us illustrate, with an example, some of the difficulties encountered by standard adaptive techniques, that—as shown later—can be overcome with adaptive I&I. For, consider the (normalized) generalized averaging model of a thyristor–controlled series capacitor system used in flexible AC transmission systems to regulate the power flow in a distribution line [22]:

$$
\begin{aligned}
\dot{x}_1 &= x_2 u \\
\dot{x}_2 &= -x_1 u - \theta_1 x_2 - \theta_2
\end{aligned}
\tag{24}
$$

where x_1, x_2 are the dynamic phasors of the capacitor voltage, $u > 0$ is the control signal which is directly related to the thyristor firing angle, and $\theta = [\theta_1, \theta_2]^\top$ are unknown positive parameters representing the nominal action of the control and one component of the phasor of the line current, respectively. The control objective is to drive the state to an equilibrium $x_* = [0, -1]^\top$ with a positive control action.

Note that the known parameter controller $u = \theta_2$ achieves the desired objective with the (non–strict) Lyapunov function $V(x) = \frac{1}{2}(x_1+1)^2 + \frac{1}{2}x_2^2$. To make this controller adaptive we first try the direct approach and, as (A1) is verified, propose $u = \hat{\theta}_2$, which following the calculations above with the Lyapunov function $V(x)$ suggests the estimator $\dot{\hat{\theta}} = -x_2$. It is easy to show that, unfortunately, the detectability assumption (A2) is not satisfied, and actually the closed–loop system has a manifold of equilibria described by $x_1(\tilde{\theta}_2 + \theta_2) - \theta_2 = 0$, hence asymptotic stability is impossible. The second approach does not apply either because assumption (A3) is not satisfied.

The indirect approach is also hampered by the detectability obstacle. Indeed, reparameterizing the second systems equation with the regressor $\Phi(x) = -[x_2, 1]^\top$ and implementing the estimator

$$\begin{aligned}
\dot{\hat{x}}_2 &= -x_1 u + \hat{\theta}^\top \Phi(x) - (\hat{x}_2 - x_2) \\
\dot{\hat{\theta}} &= -\Phi(x)(\hat{x}_2 - x_2)
\end{aligned} \tag{25}$$

ensures that the prediction error $\hat{x}_2 - x_2$ converges to zero, but again, the overall system is not detectable with respect to this output. Furthermore, as $\Phi(x)$ contains a constant term it cannot be persistently exciting, and parameter convergence will not be achieved. (The same scenario appears if we use filtered signals in the estimator as suggested above.) Finally, the combined direct–indirect scheme proposed in [32] to overcome the detectability obstacle results in

$$\dot{\hat{\theta}} = -\Phi(x)(\hat{x}_2 - x_2) - \begin{bmatrix} 0 \\ x_2 \end{bmatrix},$$

and the derivative of the Lyapunov function $V(x) + \frac{1}{2}(\hat{x}_2 - x_2)^2 + \frac{1}{2}|\tilde{\theta}|^2$ yields $-(\hat{x}_2 - x_2)^2 - \theta_1 x_2^2$, but this signal still does not satisfy (A2).

In summary, to the best of our knowledge, none of the existing methods will provide a solution to the adaptive stabilization problem for system (24).

4.3 Adaptive I&I with a linearly parameterized plant

We will present now a procedure to design an adaptive I&I scheme when the plant is linearly parameterized. Although the construction is totally different the resulting error equations are similar to the one obtained with indirect adaptive control, but obviating the need of the linear filters.

Proposition 4 *Assume (H5), (H7) and the following hold.*

(H8) (Manifold attractivity and trajectory boundedness) *There exists a function $\beta_1 : I\!R^n \to I\!R^q$ such that all trajectories of the error system*

$$\dot{x} = f_*(x) + g(x)[\Psi(x, z+\theta_*) - \Psi(x, \theta_*)] \tag{26}$$

$$\dot{z} = -\left[\frac{\partial \beta_1}{\partial x} f_1(x)\right] z \tag{27}$$

are bounded and satisfy $\lim_{t\to\infty} g(x(t))[\Psi(x(t), z(t)+\theta_*) - \Psi(x(t), \theta_*)] = 0$.

Then the system (1) is adaptively I&I stabilizable.

Moreover, if the controller satisfies the Lipschitz condition[12]

$$|\Psi(x, z+\theta_*) - \Psi(x, \theta_*)| \leq M(x)|z|$$

for all $z \in I\!R^q$ and for some function $M(x) : I\!R^n \to I\!R_{>0}$, then (H8) may be replaced by the following two assumptions.

(H8') There exists a function $\beta_1 : I\!R^n \to I\!R^q$ such that

$$\frac{\partial \beta_1}{\partial x} f_1(x) + \left[\frac{\partial \beta_1}{\partial x} f_1(x)\right]^\top \geq M^2(x) I > 0 \tag{28}$$

[12] This assumption clearly holds for linearly parameterized controls.

(H8") There exists *a radially unbounded function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *such that*

$$\begin{aligned} \frac{\partial V}{\partial x} f_*(x) &\leq 0 \\ \limsup_{\|x\|\to\infty} \frac{\left\| \frac{\partial V}{\partial x} f_*(x) \right\|}{\left\| \frac{\partial V}{\partial x} g(x) \right\|^2} &\leq K < \infty. \end{aligned}$$

Proof. Similarly to the nonlinear stabilization problem we have to verify the conditions (H1)–(H3) of Theorem 1, however, instead of (H4) we will directly prove that, under the conditions of the proposition, we have global convergence of x to x_*, with all signals bounded. First, (H1) is automatically satisfied from (H5). Second, for the immersion condition (3) we are looking for mappings $\pi(\xi)$ and $c(\pi(\xi))$, with

$$\begin{bmatrix} x \\ \hat\theta \end{bmatrix} = \pi(\xi) = \begin{bmatrix} \pi_1(\xi) \\ \pi_2(\xi) \end{bmatrix}, \qquad c(\pi(\xi)) = \begin{bmatrix} c_1(\pi(\xi)) \\ c_2(\pi(\xi)) \end{bmatrix},$$

where, for notational convenience we have introduced the partitions, that solve the FBI equations

$$\begin{aligned} \frac{\partial \pi_1}{\partial \xi} f_*(\xi) &= f(\pi_1(\xi)) + g(\pi_1(\xi))\Psi[\pi_1(\xi), \pi_2(\xi) + c_1(\pi_1(\xi))] \\ \frac{\partial \pi_2}{\partial \xi} f_*(\xi) &= c_2(\pi(\xi)). \end{aligned}$$

For any function $c_1(\pi(\xi))$ a solution of these equations is clearly given by

$$\begin{aligned} \pi_1(\xi) &= \xi \\ \pi_2(\xi) &= \theta_* - c_1(\pi(\xi)) \end{aligned}$$

while $c_2(\pi(\xi))$ is defined by the last identity. This verifies condition (H2) and, setting $\beta_1(\xi) = c_1(\pi(\xi))$, we get the implicit manifold condition (4) of (H3) as

$$\phi(x, \hat\theta) := \hat\theta - \theta_* + \beta_1(x) = 0. \tag{29}$$

Now, replacing the control law in (18) we get $\dot x = f(x) + g(x)\Psi(x, \hat\theta + \beta_1(x))$. Writing this equation in terms of the off–the–manifold coordinates $z = \hat\theta - \theta_* + \beta_1(x)$ and adding and subtracting $g(x)\Psi(x, \theta_*)$ yields the first error equation (26). The off–the–manifold dynamics is obtained differentiating the variable z taking into account (18), (23), and the fact that θ_* is constant, as

$$\dot z = \beta_2(x) + \frac{\partial \beta_1}{\partial x}[f_0(x) + f_1(x)\theta_* + g(x)u].$$

Thus, selecting the parameter update law as

$$\beta_2(x) = -\frac{\partial \beta_1}{\partial x}\left(f_0(x) + f_1(x)[\hat\theta + \beta_1(x)] + g(x)u\right) \tag{30}$$

yields (27). The proof of the first part of the proposition is completed with the use of assumption (H8).

To establish the second part consider a Lyapunov function candidate for the error system (26), (27) of the form $W(x,z) = V(x) + \frac{\rho}{2}|z|^2$, where $V(x)$ is a Lyapunov function for the ideal system $\dot x = f_*(x)$ that furthermore satisfies (H8"), and $\rho > 0$. Invoking assumption (H8'), and using Young's inequality, the derivative of $W(x,z)$ can be bounded, for any $\alpha > 0$, as

$$\dot W \leq \frac{\partial V}{\partial x} f_*(x) + \frac{1}{2\alpha}\left|\frac{\partial V}{\partial x} g(x)\right|^2 + \frac{\alpha}{2} M^2(x)|z|^2 - \rho z^\top \frac{\partial \beta_1}{\partial x} f_1(x) z.$$

The conclusion then follows from (H8') and (H8") selecting $\rho > \alpha$ and α sufficiently large. ◁

Remark 12 From inspection of (26) it is clear that if z decreases then—under some weak conditions on $\Psi(x,\theta)$, including that singularities in the controller computation are avoided—the disturbance term $|\Psi(x, z+\theta_*) - \Psi(x,\theta_*)|$ will also decrease, if it is eventually "dominated" by the stability margin of the ideal system $\dot{x} = f_*(x)$, then stabilization will be achieved. It becomes evident that ensuring stability of the z–subsystem (26), uniformly in x, is the main structural constraint imposed by the proposition. When $q = 1$ it is always possible to find a function $\beta_1(x)$ that ensures z is (locally around zero) nondecreasing.[13] To make z globally nondecreasing it suffices that

$$\frac{\partial f_1}{\partial x}(x) = \left[\frac{\partial f_1}{\partial x}(x)\right]^\top,$$

which is a necessary and sufficient condition for $f_1(x)$ to be a gradient function, that is, for the existence of a function $H : \mathbb{R}^n \to \mathbb{R}$ such that $f_1(x) = \frac{\partial H}{\partial x}(x)$. In this case, we simply take $\beta_1(x) = H(x)$ and the z–dynamics becomes $\dot{z} = -|f_1(x)|^2 z$. In this respect, it is useful to compare (26), (27) with the error system that results from application of indirect adaptive control, which is given by

$$\begin{aligned} \dot{x} &= f_*(x) + g(x)[\Psi(x, \tilde{\theta} + \theta_*) - \Psi(x,\theta_*)] \\ \dot{\tilde{\theta}} &= -\Phi^\top(x)\Phi(x)\tilde{\theta} \end{aligned}$$

where the regressor matrix $\Phi(x)$ is generated using a procedure similar to the one used to get (25). See [32] for details.

Remark 13 The fact that the Lyapunov function $V(x)$ for the ideal system verifies the second condition in (H8") does not seem to be restrictive. (We recall that the Lyapunov function does not need to be known.) Moreover, if

$$\frac{\partial V}{\partial x} f_*(x) = 0 \Rightarrow \frac{\partial V}{\partial x} g(x) = 0,$$

then it is possible, as discussed in Section 9.5 of [11] in a different framework, to define a new function $U(V(x))$ such that the second condition in (H8") holds for $U(V(x))$.

Remark 14 From a Lyapunov analysis perspective, the I&I procedure automatically includes cross terms between plant states and estimated parameters, as suggested in the proof of the second part of the proposition above with $W(x,z)$. Also, we reiterate the fact that the stabilization mechanism does not rely on cancellation of the perturbation term $g(x)[\Psi(x, z+\theta_*) - \Psi(x,\theta_*)]$ in (26). Also, an important advantage of the I&I method is that the detectability assumption (A2) required to handle non–strict Lyapunov functions is, in principle, not needed.

Remark 15 It is clear from the construction of the adaptive I&I control laws that, besides the classical "integral action" of the parameter estimator, through the action of $\beta_1(x)$ we have introduced in the control law a "proportional" term. (See also [33] for an non–certainty equivalent adaptive algorithm that also includes this term.) These kind of parameter update laws were called in early references *PI adaptation*. Although it was widely recognized that PI update laws were superior to purely integral adaptation, except for the notable exception of output error identification, their performance improvement was never clearly established, see, e.g., [30] for a tutorial account of these developments. The I&I framework contributes then to put PI adaptation in its proper perspective.

4.4 Adaptive I&I with a linearly parameterized control

We will consider now the case of linearly parameterized control, that is, we assume the state feedback has the form (20). We will present a proposition, similar in spirit to the first approach presented in Subsection 4.2, where we do not make any explicit assumptions on the dependence of the system with respect to the unknown parameters, but assume instead a condition akin to the "realizability" of $\frac{\partial V}{\partial x} g(x)$.

[13]This is achieved ensuring $\frac{\partial \beta_1}{\partial x} f_1(x) + \left[\frac{\partial \beta_1}{\partial x} f_1(x)\right]^\top \geq 0$.

Proposition 5 *Assume (H5), (H6) hold and there exists a function $\beta_1 : I\!R^n \to I\!R^m$ such that:*

(H9) (Realizability) $\frac{\partial \beta_1}{\partial x} f_*(x)$*, with $f_*(x)$ defined in (17), is independent of the unknown parameters.*

(H10) (Manifold attractivity and trajectory boundedness) *All trajectories of the error system*

$$\dot{x} = f_*(x) + g(x)\Psi_1(x)z \tag{31}$$

$$\dot{z} = \left[\frac{\partial \beta_1}{\partial x} g(x)\Psi_1(x)\right] z \tag{32}$$

are bounded and satisfy $\lim_{t\to\infty} g(x(t))\Psi_1(x(t))z(t) = 0$.

Then the system (1) is adaptively I&I stabilizable.

Proof. In view of the derivations in the proof of Proposition 4 we only need to establish that with (H5), (H6) and (H9) we can obtain the error equations (31), (32). The first equation follows immediately from (17), (18), (20) and (29) setting $z = \hat{\theta} - \theta_* + \beta_1(x)$. The second error equation (32) is obtained from

$$\dot{z} = \beta_2(x) + \frac{\partial \beta_1}{\partial x}[f_*(x) + g(x)\Psi_1(x)z],$$

and selecting the parameter update law as

$$\beta_2(x) = -\frac{\partial \beta_1}{\partial x} f_*(x).$$

◁

Remark 16 Similarly to the discussion in Remark 12, it is clear that the success of the proposed design hinges upon our ability to assign the sign of the (symmetric part of the) matrix $\frac{\partial \beta_1}{\partial x} g(x)\Psi_1(x)$. Comparing with the corresponding matrix in (32) we see that $g(x)\Psi_1(x)$ and $f_1(x)$ play the same role, and the discussion about solvability of the problem carried out in Remark 12 applies as well here. In particular, for the case of single input systems with one uncertain parameter, assumption (H10) is easily satisfied. However, it is true that—besides (H10)—$\beta_1(x)$ should also ensure (H9).

Remark 17 The realizability assumption (H9) is strictly weaker than the strict matching assumption discussed in, e.g., [19], which requires that the uncertain parameters enter in the image of $g(x)$. It is clear that in this case, we can always find $\Psi_1(x)$ such that $f_*(x)$ is independent of the parameters. It is also less restrictive than Assumption (A1) of the first classical approach since, in contrast to the I&I approach where $\beta_1(x)$ is chosen by the designer, in the former case $V(x)$ is essentially fixed by the closed–loop dynamics.

Remark 18 If both the plant and the controller are linearly parameterized, it is possible to combine the procedures described in Sections 4.3 and 4.4 to obtain stable off–the–manifold dynamics.

Remark 19 To study the stability of the error system (31), (32) we can replace (H10) by the following assumptions

(H10') *(Stability)* For all $x \in I\!R^n$, $z \in I\!R^q$ we have

$$Q(x,z) := z^\top \left[\frac{\partial \beta_1}{\partial x} g(x)\Psi_1(x) + \left(\frac{\partial \beta_1}{\partial x} g(x)\Psi_1(x)\right)^\top\right] z + \frac{\partial V}{\partial x}[f_*(x) + g(x)\Psi_1(x)z] \le 0,$$

where $V(x)$ is a Lyapunov function for the target system $\dot{x} = f_*(x)$.

(H10") *(Convergence)* The following implication is true: $Q(x,z) = 0 \;\Rightarrow\; x = x_*$.

The proof is completed considering the Lyapunov function $V(x) + |z|^2$, and using LaSalle's invariance principle. The key requirement here is the attractivity of the manifold, which is ensured with the (restrictive) condition that the matrix in square brackets in $Q(x,z)$ is negative semidefinite (uniformly in x). However, exploiting the particular structure of the system we can choose $\beta_1(x)$ to stabilize the error system under far weaker conditions.

4.5 Examples

To illustrate the (rather non–standard) I&I adaptive design procedure let us consider first two simple (exhaustively studied) examples, for which adaptive I&I generate new control and adaptation laws, and gives place to new non–separate Lyapunov functions. Then, we show that I&I applies to systems in triangular form, and study some asymptotic properties of the resulting controllers. Finally, we solve the problem stated at the end of Subsection 4.2.

Example 2 Consider the stabilization to zero of an unstable first order linear system $\dot{x} = \theta_* x + u$, with $\theta_* > 0$. In this (matched) case we can fix a target dynamics independent of θ_*, for instance as $\dot{\xi} = -\xi$. This choice yields the adaptive I&I control law $u = -x - x(\hat{\theta} + \beta_1(x))$. Selecting the update law as (30), that is $\beta_2(x) = \frac{\partial \beta_1}{\partial x} x$, yields the error equations (31), (32), which in this case become

$$\begin{aligned} \dot{x} &= -x - xz \\ \dot{z} &= -\frac{\partial \beta_1}{\partial x} xz \end{aligned}$$

The problem then boils down to finding a function $\beta_1(x)$ such that the *perturbation* term xz is asymptotically dominated by the "good" term $-x$. The function $\beta_1(x) = \frac{\gamma}{2}x^2$, with $\gamma > 0$, clearly does the job, and yields the "classical" estimator $\dot{\hat{\theta}} = \gamma x^2$. Notice however that the I&I control

$$u = -x - x(\hat{\theta} + \frac{\gamma}{2}x^2),$$

automatically incorporates a cubic term in x that will speed–up the convergence. A Lyapunov function for $f_*(x)$ such that conditions (H8') and (H8") of Remark 19 hold is $V(x) = \frac{1}{2}\log(1 + x^2)$, which yields

$$Q(x,z) = \frac{-x^2}{1+x^2}[(1+x^2)z^2 + z + 1] \leq 0.$$

Other, more practical choices, are immediately suggested, for instance $\beta_1(x) = \frac{\gamma}{2}\log(1 + x^2)$, gives a normalized estimator

$$\dot{\hat{\theta}} = \gamma \frac{x^2}{1+x^2},$$

a feature that is desirable to normalize the signals.

Example 3 Let us study the problem of adaptive I&I stabilization to a non–zero position $(x_{1*}, 0)$ of the basic pendulum

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\theta_* \sin(x_1) - (x_1 - x_{1*}) - x_2 + u \end{aligned}$$

where we assume that a PD controller has already been applied. (As discussed in [27] the classical schemes with the pendulum energy as Lyapunov function are inefficient because of the detectability obstacle. This difficulty can be overcome introducing unnatural cross terms, that get very complicated for the general robot example.) The immersion dynamics can be chosen again independent of θ_*, say as $f_*(x) = [x_2, \ -(x_1 - x_{1*}) - x_2]^\top$, hence the adaptive I&I controller becomes $u = \sin(x_1)(\hat{\theta} + \beta_1(x))$. The error equations take the form

$$\begin{aligned} \dot{x} &= f_*(x) + \begin{bmatrix} 0 \\ \sin(x_1) \end{bmatrix} z \\ \dot{z} &= -\frac{\partial \beta_1}{\partial x_2} \sin(x_1) z \end{aligned}$$

which immediately suggests $\beta_1(x) = -x_2 \sin(x_1)$. The final control and adaptation laws are

$$\begin{aligned} u &= \hat{\theta} \sin(x_1) - x_2 \sin^2(x_1) \\ \dot{\hat{\theta}} &= -((x_1 - x_{1*}) + x_2)\sin(x_1) - x_2^2 \cos(x_1). \end{aligned}$$

Example 4 As pointed out in Subsection 4.2 the second—so–called backstepping—approach applies (almost exclusively) to systems in triangular form. This class of systems are also adaptively I&I stabilizable, and we illustrate this with the simplest second order example

$$\begin{aligned} \dot{x}_1 &= x_2 + \psi^\top(x_1)\theta \\ \dot{x}_2 &= u \end{aligned}$$

where $\theta \in I\!R^q$ is a vector of unknown parameters and the vector function $\psi(x_1)$ is assumed known. Introduce the classical backstepping error variable $\eta = x_2 - \alpha(x_1, \hat{\theta})$, where $\alpha(x_1, \hat{\theta})$ is a stabilizing function for the first equation that, following the I&I procedure, we choose as

$$\alpha(x_1, \hat{\theta}) = -\lambda_1(x_1) - \psi^\top(x_1)(\hat{\theta} + \beta(x_1)),$$

with $\lambda_1(x_1)$ to be defined below. Proceeding as suggested in the proof of Proposition 4 we obtain the error equations, expressed in terms of the off–the–manifold coordinate $z = \hat{\theta} - \theta + \beta(x_1)$, as

$$\begin{aligned} \dot{x}_1 &= \eta - \lambda_1(x_1) - \psi^\top(x_1)z & (33) \\ \dot{\eta} &= -\lambda_2(x_1, \eta) + \frac{\partial \alpha}{\partial x_1}\psi^\top(x_1)z & (34) \\ \dot{z} &= -\frac{\partial \beta}{\partial x_1}\psi^\top(x_1)z & (35) \end{aligned}$$

where we have selected

$$\begin{aligned} u &= -\lambda_2(x_1, \eta) + \frac{\partial \alpha}{\partial x_1}x_2 + \frac{\partial \alpha}{\partial \hat{\theta}}\dot{\hat{\theta}} + \frac{\partial \alpha}{\partial x_1}\psi^\top(x_1)[\hat{\theta} + \beta(x_1)] \\ \dot{\hat{\theta}} &= \frac{\partial \beta}{\partial x_1}(\lambda_1 - \eta). \end{aligned}$$

and $\lambda_2(x_1, \eta)$ is a function to be defined below. Let $\beta(x_1) = \int_0^{x_1}\psi(s)ds$, hence (35) becomes $\dot{z} = -\psi(x_1)\psi^\top(x_1)z$, and $\lim_{t\to\infty}\psi^\top(x_1(t))z(t) = 0$. The design is completed selecting $\lambda_1(x_1)$ and $\lambda_2(x_1, \eta)$ such that (33), (34) have bounded solutions for any bounded z, which entails convergence to zero of $x_1(t)$ and $\eta(t)$, and consequently of $x_1(t)$ and $x_2(t)$.

For the sake of comparison we present below the closed–loop equations obtained from the direct application of the backstepping procedure

$$\begin{aligned} \dot{x}_1 &= \eta - \lambda_1(x_1) - \psi^\top(x_1)\tilde{\theta} & (36) \\ \dot{\eta} &= -\lambda_2(x_1, \eta) - x_1 + \frac{\partial \alpha}{\partial x_1}\psi^\top(x_1)\tilde{\theta} & (37) \\ \dot{\tilde{\theta}} &= \psi(x_1)x_1 - \frac{\partial \alpha}{\partial x_1}\psi(x_1)\eta & (38) \end{aligned}$$

where typically $\lambda_i = c_i x_i,\ c_i > 0,\ i = 1, 2$. The equations are arranged, in backstepping, in such a way as to cancel the $\tilde{\theta}$–dependent terms from the derivative of the Lyapunov function $\frac{1}{2}(x_1^2 + \eta^2 + \tilde{\theta}^2)$. In I&I this (non–robust) cancellation operation is avoided, as the system (33)–(35) exhibits a cascaded structure.

One final remark concerning this class of systems (that has attracted an enormous amount of attention lately) is that the well–known counterexample of [42] concerning the asymptotic behavior of backstepping controllers is not applicable to adaptive I&I. More precisely, we have the following property, whose proof is omitted for brevity.

Proposition 6 *Consider the scalar system $\dot{x} = u + (x - bx^2)\theta$, with b a known real number, in closed loop with the adaptive I&I controller*

$$\begin{aligned} u &= -x - (x - bx^2)[\hat{\theta} + \beta(x)] \\ \beta &= \frac{1}{2}x^2 - \frac{b}{3}x^3 \\ \dot{\hat{\theta}} &= x(x - bx^2). \end{aligned}$$

The set of initial conditions leading to non–stabilizing asymptotic controllers has zero Lebesgue measure. Moreover, the system in closed loop with any asymptotic controller has bounded trajectories.

Example 5 Let us now look back at the thyristor–controlled series capacitor system (24).

Proposition 7 *The system (24) is "almost" globally I&I stabilizable at the equilibrium point $x_* = [-1, 0]^\top$, where the qualifier "almost" means that for all initial conditions, except a set of zero measure, the trajectories of the closed loop system converge to x_*.*

Proof. Direct application of the construction in the proof of Proposition 4 with

$$\begin{aligned} u &= \hat{\theta}_2 + \beta_1(x) \\ \beta_1 &= -\lambda x_2 \\ \beta_2 &= -\lambda(x_1 u + \hat{\theta}_2 + \lambda x_2) \end{aligned}$$

where $\lambda > 0$, yields the error equations

$$\begin{aligned} \dot{x}_1 &= x_2(z + \theta_2) \\ \dot{x}_2 &= -x_1(z + \theta_2) - \theta_2 - \theta_1 x_2 \\ \dot{z} &= -\lambda(z + \theta_1 x_2) \end{aligned}$$

Notice that, due to the presence of the term $\theta_1 x_2$, in this case we do not obtain a cascade system like in Proposition 4. However, we can still carry out the stability analysis with the Lyapunov function $W(x, z) = \frac{1}{2}(x_1 + 1)^2 + \frac{1}{2}x_2^2 + \frac{1}{2\lambda\theta_1}z^2$, whose derivative is

$$\dot{W} = -\theta_1(x_2 - \frac{1}{\theta_1}z)^2$$

which establishes Lyapunov stability of x_* and global boundedness of solutions. The proof is completed verifying that, besides (x_*, θ_2), the closed–loop equations admit the equilibrium manifold $\{(x, z) \mid x_2 = -\frac{\theta_2}{\theta_1}, z = -\theta_2\}$, but that this equilibria are unstable. (Notice that z does not converge to zero in this case.) $\triangleleft$

4.6 Adaptive visual servoing: A nonlinearly parameterized problem

It this subsection we illustrate with a visual servoing problem how adaptive I&I stabilization can be applied in the nonlinearly parameterized case. We consider the visual servoing of planar robot manipulators under a fixed–camera configuration with *unknown orientation.*[14] The control goal is to place the robot end-effector over a desired static target by using a vision system equipped with a fixed camera to 'see' the robot end-effector and target.

Invoking standard time–scale separation arguments we assume an inner fast loop for the robot velocity control, and concentrate on the kinematic problem where we must generate the references for the robot velocities. The robot is then described by a simple integrator $\dot{q} = v$, where $q \in I\!R^2$ are the joint displacements and v the applied joint torques. We model the action of the camera as a static mapping from the joint positions q to the position (in pixels) of the robot tip in the image output denoted $x \in I\!R^2$. This mapping is described by [17]

$$x = ae^{J\theta_*}[k(q) - \vartheta_1] + \vartheta_2 \tag{39}$$

where θ_* is the orientation of the camera with respect to the robot frame, $a > a_{min} > 0$ and ϑ_1, ϑ_2 denote intrinsic camera parameters (scale factors, focal length and center offset, respectively), $k : I\!R^2 \to I\!R^2$ stands for the robot direct kinematics, and

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The direct kinematics yields $\dot{k} = \mathcal{J}(q)\dot{q}$, where $\mathcal{J} \in I\!R^{2\times 2}$ is the analytic robot Jacobian, which we assume nonsingular. Differentiating (39), and replacing the latter expression, we obtain the dynamic model of the overall system of interest

$$\dot{x} = ae^{J\theta_*}u, \tag{40}$$

[14] We refer the interested reader to [10, 17, 2] for further details on this problem.

where we have introduced the input change of coordinates $u := \mathcal{J}(q)v$. The problem is then to find u such that $x(t)$ converges to x_* in spite of the lack of knowledge of a and θ_*. The task is, of course, complicated by the highly nonlinear dependence on the unknown parameters, in particular θ_*. To the best of our knowledge all existing results require some form of over–parameterization and/or persistency of excitation assumptions, which as is well–known, significantly degrades performance. Using adaptive I&I stabilization it is possible to prove the following result.

Proposition 8 *Assume a strict lower bound on a is known. Then, the system (40) is adaptively I&I stabilizable with target dynamics $\dot{\xi} = -a\xi + ax_*$.*

Proof. First, we observe that we can design a stabilizing law for (40) without the knowledge of the uncertain parameter a. Indeed, the feedback $\Psi(x, \theta_*) = -e^{-J\theta_*}\tilde{x}$, where we have defined $\tilde{x} := x - x_*$ is a global stabilizer that matches the target dynamics. Therefore, we will take our I&I adaptive control of the form

$$u = -e^{-J(\hat{\theta}+\beta_1(x))}\tilde{x}.$$

The adaptive system (18) becomes

$$\Sigma : \left\{ \begin{array}{rcl} \dot{x} & = & -ae^{-J(\hat{\theta}-\theta_*+\beta_1(x))}\tilde{x} \\ \dot{\hat{\theta}} & = & \beta_2(x) \end{array} \right.$$

Some simple calculations show that, in spite of the nonlinear parameter dependence, the corresponding FBI equations (3) can still be solved as in the linear case. That is, for all functions $c_1(\xi)$, $\pi_1(\xi) = \xi$, and $\pi_2(\xi) = \theta_* - c_1(\xi)$ are solutions of the FBI equations with $c_2(\xi) = a\frac{\partial \pi_2}{\partial \xi}(-\xi + x_*)$. The implicit manifold equation takes the form $\phi(x) = \hat{\theta} - \theta_* + \beta_1(x)$, and consequently the error equations become

$$\begin{array}{rcl} \dot{x} & = & -ae^{-Jz}\tilde{x} \\ \dot{z} & = & \beta_2(x) - a\dfrac{\partial \beta_1}{\partial x}e^{-Jz}\tilde{x} \end{array}$$

To stabilize the off–the–manifold dynamics we take $\beta_1(x) = \frac{1}{2}|\tilde{x}|^2$ and $\beta_2(x) = a_{min}|\tilde{x}|^2$, where $a > a_{min}$, yielding

$$\begin{array}{rcl} \dot{z} & = & -\tilde{x}^\top(ae^{-Jz} - a_{min})\tilde{x} \\ & = & -a|\tilde{x}|^2[\cos(z) - \dfrac{a_{min}}{a}], \end{array}$$

where we have used the identity $e^{-Jz} = \cos(z)I - \sin(z)J$, and the skew–symmetry of J, to obtain the last equation. Plotting the graph $\cos(z) - \frac{a_{min}}{a}$, and taking into account that $\frac{a_{min}}{a} < 1$, we see that all trajectories of the z subsystem will converge towards the points $z = \arccos(\frac{a_{min}}{a})$. To complete the proof we observe that the LTI system $\dot{y} = -ae^{-J\bar{z}}y$, with $\bar{z}$ constant, is asymptotically stable if and only if $\cos(\bar{z}) > 0$.[15] ◁

5 Stabilization of systems in feedback and feedforward form

In this section the focus is on systems with special structure, namely systems in feedback form and in feedforward form.[16] Stabilizing control laws for such systems can be constructed using the so-called backstepping and forwarding methodologies, see [19, 37, 13] and references therein for an in–depth description. Aim of this section is to show how these systems are (globally) I&I-stabilizable. Moreover, it is also shown that the present point of view permits to construct control laws different from those obtained by the aforementioned approaches.

[15] It is also possible to complete this proof with standard Lyapunov function arguments taking as Lyapunov function candidate $W(\tilde{x}, z) = |\tilde{x}|^2 + \sin(z)$.

[16] For ease of exposition we deal with the simplest possible feedback and feedforward forms, however more general systems can be studied using similar arguments.

5.1 Systems in feedback form

Proposition 9 *Consider a system described by equations of the form*

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= u \end{aligned} \tag{41}$$

with $x = \mathrm{col}(x_1, x_2) \in (I\!\!R^n \times I\!\!R)$, $u \in I\!\!R$ *and* $f(0,0) = 0$. *Suppose the system*

$$\dot{x}_1 = f(x_1, 0)$$

has a globally asymptotically stable equilibrium at zero.

Then the system (41) is (globally) I&I-stabilizable with target dynamics $\dot{\xi} = f(\xi, 0)$.

Proof. To establish the claim we need to prove that conditions (H1) to (H4) of Theorem 1 hold. For, note that (H1) is trivially satisfied, whereas the mappings

$$x = \begin{bmatrix} \pi_1(\xi) \\ \pi_2(\xi) \end{bmatrix} = \begin{bmatrix} \xi \\ 0 \end{bmatrix} \qquad u = c(\xi) = 0 \qquad \phi(x_1, x_2) = x_2$$

are such that conditions (H2) and (H3) hold.

Note now that the off–the–manifold variable $z = x_2$ can be used as a partial coordinate, hence instead of verifying (H4) we simply need to show that it is possible to select u such that the trajectories of the closed loop system are bounded and $z(t) = x_2(t)$ converges to zero.

To this end, let $u = -K(x_1, x_2)z$, with $K(x_1, x_2) \geq k > 0$ for any (x_1, x_2) and for some $k > 0$, and consider the system

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= -K(x_1, x_2)x_2. \end{aligned} \tag{42}$$

Note that x_2 converges to zero. To prove boundedness of x_1 pick any $M > 0$ and let $V(x_1)$ be a positive definite and proper function such that

$$V_{x_1} f(x_1, 0) < 0 \tag{43}$$

for all $\|x_1\| > M$. Note that such a function $V(x_1)$ exists, by global asymptotic stability of the zero equilibrium of the system $\dot{x}_1 = f(x_1, 0)$, but $V(x_1)$ is not necessarily a Lyapunov function for $\dot{x}_1 = f(x_1, 0)$. Consider now the positive definite and proper function

$$W(x_1, x_2) = V(x_1) + \frac{1}{2}x_2^2$$

and note that, for some function $F(x_1, x_2)$ and for any smooth function $\gamma(x_1) > 0$, one has

$$\begin{aligned} \dot{W} &= V_{x_1} f(x_1, 0) + V_{x_1} F(x_1, x_2) x_2 - K(x_1, x_2) x_2^2 \\ &\leq V_{x_1} f(x_1, 0) + \frac{\|V_{x_1}\|^2}{\gamma(x_1)} + \gamma(x_1)\|F(x_1, x_2)\|^2 x_2^2 - K(x_1, x_2) x_2^2. \end{aligned}$$

As a result, setting $\gamma(x_1)$ such that

$$V_{x_1} f(x_1, 0) + \frac{\|V_{x_1}\|^2}{\gamma(x_1)} < 0$$

for all $\|x_1\| > M$, and selecting

$$K(x_1, x_2) > \gamma(x_1)\|F(x_1, x_2)\|^2$$

yields the claim. $\triangleleft$

Remark 20 It is worth noting that although system (41) is stabilizable using standard backstepping arguments, the control law obtained using backstepping is substantially different from the control law suggested by Proposition 9. The former requires the knowledge of a Lyapunov function for the system $\dot{x}_1 = f(x_1, 0)$ and it is such that, in closed loop, the manifold $x_2 = 0$ is not invariant, whereas the latter requires only the knowledge of the function $V(x_1)$ satisfying equation (43) for sufficiently large $\|x_1\|$ and renders the manifold $x_2 = 0$ invariant and (globally) attractive.

The result in Proposition 9 is illustrated through two simple examples. In the first example the I&I globally stabilizing control law is compared with the control law resulting from standard backstepping arguments.

Example 6 Consider the system

$$\begin{array}{rcl} \dot{x}_1 & = & -x_1 + \lambda x_1^3 x_2 \\ \dot{x}_2 & = & u \end{array}$$

with $x_1 \in \mathbb{R}$. A backstepping based stabilizing control law is

$$u_{BS} = -\lambda x_1^4 - x_2$$

whereas a direct application of the procedure described in the proof of Proposition 9 shows that the I&I stabilizing control law is

$$u_{\mathrm{I\&I}} = -(2 + x_1^8)x_2.$$

The latter does not require the knowledge of the parameter λ, however it is (in general) more *aggressive* because of the higher power in x_1.

Example 7 Consider a system described by equations of the form

$$\begin{array}{rcl} \dot{x}_1 & = & A(\theta)x_1 + F(x_1, x_2)x_2 \\ \dot{x}_2 & = & u \end{array}$$

with $x_1 \in \mathbb{R}^n$ and $\theta \in \mathbb{R}^s$ an unknown constant vector. Assume that the matrix $A(\theta)$ is Hurwitz. Then the system is (globally) I&I-stabilizable with target dynamics $\dot{\xi} = A(\theta)\xi$. A simple computation shows that a stabilizing control law is

$$u_{\mathrm{I\&I}} = -(1 + x_1^2)\|F(x_1, x_2)\|^2 x_2 - x_2,$$

and this does not require the knowledge of a Lyapunov function for the system $\dot{x}_1 = A(\theta)x_1$ neither the knowledge of the parameter θ.

5.2 Systems in feedforward form

We now consider a class of systems in feedforward form. These can be globally asymptotically stabilized using the forwarding method, as described in [37, 24]. We now show that, using the approach pursued in the paper, a new class of control laws can be constructed.

Proposition 10 *Consider a system described by equations of the form*

$$\begin{array}{rcl} \dot{x}_1 & = & h(x_2) \\ \dot{x}_2 & = & f(x_2) + g(x_2)u \end{array} \tag{44}$$

with $x = \mathrm{col}(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^n$, $u \in \mathbb{R}$, $h(0) = 0$, $f(0) = 0$. *Suppose moreover that the zero equilibrium of the system*

$$\dot{x}_2 = f(x_2)$$

is globally asymptotically stable, there exists a smooth function $M(x_2)$ *such that, for all* x_2,

$$L_f M(x_2) = h(x_2),$$

the set

$$S = \{x_2 \in I\!R^n \mid L_g M(x_2) = 0\}$$

is composed of isolated points, and $0 \notin S$.

Then the system (44) is globally I&I-stabilizable with target dynamics $\dot{\xi} = f(\xi)$.

Proof. To begin with note that (H1) in Theorem 1 is trivially satisfied and that the mappings

$$\begin{bmatrix} \pi_1(\xi) \\ \pi_2(\xi) \end{bmatrix} = \begin{bmatrix} M(\xi) \\ \xi \end{bmatrix} \qquad c(\xi) = 0$$

are such that condition (H2) holds. The implicit description of the manifold in (H3) is $z = \phi(x) = x_1 - M(x_2)$ and the off–the–manifold dynamics are $\dot{z} = -L_g M(x_2) u$.

To complete the proof, it remains to verify condition (H4). For, let

$$u = \epsilon \frac{1}{1 + \|g(x_2)\|} \frac{L_g M(x_2)}{1 + \|L_g M(x_2)\|} \frac{z}{1 + \|z\|},$$

with $\epsilon = \epsilon(x_2) > 0$, and consider the closed loop system

$$\begin{aligned} \dot{x}_1 &= h(x_2) \\ \dot{x}_2 &= f(x_2) + \epsilon \frac{g(x_2)}{1 + \|g(x_2)\|} \frac{L_g M(x_2)}{1 + \|L_g M(x_2)\|} \frac{z}{1 + \|z\|} \\ \dot{z} &= -\epsilon \frac{1}{1 + \|g(x_2)\|} \frac{(L_g M(x_2))^2}{1 + \|L_g M(x_2)\|} \frac{z}{1 + \|z\|}. \end{aligned}$$

Note now that $L_g M(x_2) z$ converges to zero. Moreover, x_2 is bounded, provided that ϵ is sufficiently small, and converges to zero. Note now that, if ϵ is sufficiently small, z converges exponentially to zero. As a result, $\eta = x_1 - M(x_2)$ is bounded, hence x_1 is also bounded for all t, which proves the claim. $\triangleleft$

Example 8 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_{21}^3 \\ \dot{x}_{21} &= x_{22}^3 \\ \dot{x}_{22} &= v \end{aligned} \tag{45}$$

and let $v = -x_{21}^3 - x_{22}^3 + u$. The system satisfies all the assumptions of Proposition 10, with $M(x_2) = M(x_{21}, x_{22}) = -x_{21} - x_{22}$. As a result it is globally I&I stabilizable by the control $u = -z$ with $z = x_1 + x_{21} + x_{22}$.

Remark 21 The system (45) has been studied in several papers. In [4] a globally stabilizing control law has been designed through the construction of a minimum-phase relative degree one output map, whereas in [31] a global stabilizing control law has been obtained using a control Lyapunov function. The control law in [31] is

$$u = -x_{21}^3 - x_{22}^3 - x_1 - x_{22}$$

and it is similar to the one proposed above. Note that the former requires the knowledge of a (control) Lyapunov function, whereas the latter does not. Finally, system (45) can be stabilized with the modified version of forwarding proposed in [23], which is again a Lyapunov based design methodology, and (obviously) with homogeneous feedback laws.

Example 9 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= -x_2^3 + (1 - x_2^2) u. \end{aligned}$$

A simple computation shows that $M(x_2) = x_2$ satisfies the assumptions of Proposition 10 and that $S = \{x_2 = -1, x_2 = 1\}$. Hence, a direct application of the procedure outlined in the proof of Proposition 10 shows that the control law

$$u = -\frac{1 - x_2^2}{1 + x_2^4} z$$

with $z = x_1 + x_2$ achieves global I&I stabilization of the closed loop system.

6 Conclusions and outlook

The stabilization problem for general nonlinear systems has been addressed from a new perspective that provides alternative "tuning knobs" for the controller design.

It has been shown that the classical notions of invariance and immersion, together with tools from the nonlinear regulator theory, can be used to design globally stabilizing control laws for general nonlinear systems. The proposed approach is well suited in applications where we can define a—structurally compatible—"desired" reduced order dynamics.

We have explored the applicability of the method to adaptive stabilization, even in the case of nonlinearly parameterized systems where classical adaptive control is severely limited. The main distinguishing feature of adaptive I&I is that, in contrast with most existing adaptive designs, it does not rely on cancellation of terms (in the Lyapunov function derivative). The latter operation is akin to disturbance rejection that, as is well known, is intrinsically fragile and imposes certain matching conditions that restrict its application domain. In adaptive I&I, instead, the deleterious effect of the uncertain parameters is countered adopting a robustness perspective, i.e., generating cascaded structures. Further, in adaptive I&I control laws, besides the classical "integral action" of the parameter estimator, we have introduced in the control law a "proportional" term. As discussed in [28], as illustrated in various examples here, the inclusion of this term enhances the robustness of the design, via the incorporation of additional zero dynamics.

For systems in triangular and in feedforward form, which are classically handled using backstepping or forwarding, respectively, we showed that the I&I framework yields new control laws.

Finally, it is worth noting that, although some of the problems discussed in the paper can be dealt with using other methods (this is the case for the Examples in Section 3, for some of the adaptive control problems in Section 4, and for systems in feedback and feedforward form) the proposed approach provides on one hand new control laws, with extra tuning knobs, and a unique way to address all such problems; on the other hand solutions to some open problems, which cannot be solved using classical tools.

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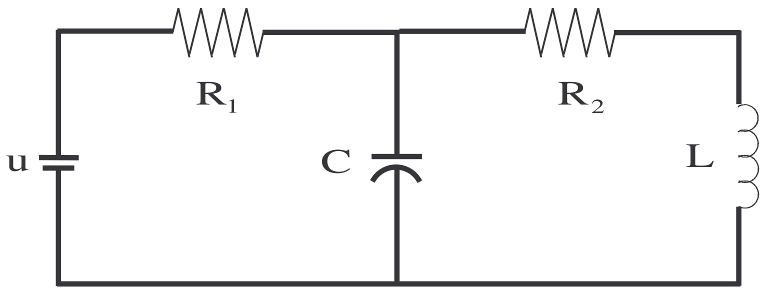

Figure 1: Electrical circuit of the levitated ball and the actuator.