Chapter 1

Is the conservative wave equation regular?

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We consider an infinite-dimensional system described by the wave equation on an n–dimensional domain, with mixed boundary control and mixed boundary observation, which has been analyzed (as an example for a certain class of conservative linear systems) in [13]. A somewhat simpler version of this system has appeared (also as an example) in the paper [11, Section 7] and a related system has been discussed in [5].

We assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary Γ, as defined in Grisvard [3]. This means that locally, after a suitable rotation of the orthogonal coordinate system, the boundary is the graph of a Lipschitz function defined on an open set in $\mathbb{R}^{n-1}$. Such a boundary admits corners and edges. Γ_0 and Γ_1 are nonempty open subsets of Γ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\overline{\Gamma_0 \cup \Gamma_1} = \Gamma$. We denote by x the space variable ($x \in \overline{\Omega}$). A function $b \in L^\infty(\Gamma_1)$ is given, which intuitively expresses how strongly the input signal acts on different parts of the active boundary Γ_1. We assume that $b(x) \neq 0$ for almost every $x \in \Gamma_1$. The equations of the system are

$$\begin{cases} \ddot{z}(x,t) = \Delta z(x,t) & \text{on } \Omega \times [0,\infty), \\ z(x,t) = 0 & \text{on } \Gamma_0 \times [0,\infty), \\ \frac{\partial}{\partial \nu} z(x,t) + |b(x)|^2 \dot{z}(x,t) = \sqrt{2} \cdot b(x) u(x,t) & \text{on } \Gamma_1 \times [0,\infty), \\ \frac{\partial}{\partial \nu} z(x,t) - |b(x)|^2 \dot{z}(x,t) = \sqrt{2} \cdot b(x) y(x,t) & \text{on } \Gamma_1 \times [0,\infty), \\ z(x,0) = z_0(x), \quad \dot{z}(x,0) = w_0(x) & \text{on } \Omega, \end{cases} \tag{1.1}$$

where u is the input function and y is the output function. The functions z_0 and w_0 are the initial state of the system. The part Γ_0 of the boundary is just reflecting waves, while inputs and outputs act through the part Γ_1.

For every $g \in \mathcal{H}^1(\Omega)$ we denote by γg the Dirichlet trace of g on Γ (for $g \in C^1(\overline{\Omega}) \subset \mathcal{H}^1(\Omega)$ this would simply be the restriction of g to Γ). We regard γg as an element of $L^2(\Gamma)$. We define the Hilbert space

$$\mathcal{H}^1_{\Gamma_0}(\Omega) = \{g \in \mathcal{H}^1(\Omega) \mid \gamma g = 0 \text{ on } \Gamma_0\}, \qquad \|g\|_{\mathcal{H}^1} = \|\nabla g\|_{L^2}.$$

Proposition 1. *The equations* (1.1) *determine a well-posed linear system* Σ *with input space* $U = L^2(\Gamma_1)$*, output space* $Y = L^2(\Gamma_1)$ *and state space*

$$X = \mathcal{H}^1_{\Gamma_0}(\Omega) \times L^2(\Omega).$$

For the precise meaning of a well-posed linear system we refer to [8, 9, 6]. These papers use the same notation and terminology that we use here, but their references will indicate other works in which equivalent definitions can be found. We give a short explanation of what well-posedness means in our case. If we take $x(0) = [z_0 \ w_0]^T \in X$, $u \in L^2([0,\infty); U)$ and we solve the equations (1.1) on the time interval $[0,\infty)$, then we get $x(\tau) = [z(\tau) \ \dot{z}(\tau)]^T \in X$ for every $\tau \geq 0$. $x(\tau)$ is called the state of the system at time τ. Moreover, if we denote the restriction of y to $[0,\tau]$ by $\mathbf{P}_\tau y$, then $\mathbf{P}_\tau y \in L^2([0,\tau]; Y)$. (Note that in our particular case, $U = Y$.) We can introduce four families of bounded operators $\mathbb{T}, \Phi, \Psi$ and $\mathbb{F}$ indexed by $\tau \geq 0$ such that for every such τ,

$$x(\tau) = \mathbb{T}_\tau x(0) + \Phi_\tau \mathbf{P}_\tau u, \qquad \mathbf{P}_\tau y = \Psi_\tau x(0) + \mathbb{F}_\tau \mathbf{P}_\tau u.$$

Thus, for every $\tau \geq 0$, the operator matrix

$$\Sigma_\tau = \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix}$$

defines a bounded operator from $X \times L^2([0,\tau]; U)$ to $X \times L^2([0,\tau]; Y)$. This is the essential feature of a well-posed linear system. In fact, in [8, 9, 6], Σ is defined as the family of operators Σ_τ. For a well-posed linear system, the family $\mathbb{T}$ is a strongly continuous semigroup of operators acting on X. Proposition 1 was proved in [13, Section 7], together with the following:

Proposition 2. *The system* Σ *from Proposition* 1 *is conservative.*

The fact that Σ is conservative means that the operators Σ_τ are unitary. In particular, the fact that Σ_τ is isometric means that we have

$$\|x(\tau)\|^2 - \|x(0)\|^2 = \int_0^\tau \|u(t)\|^2 \mathrm{d}t - \int_0^\tau \|y(t)\|^2 \mathrm{d}t,$$

which can be interpreted as an energy balance equation. For background on conservative systems we refer to [1, 2, 4, 7, 12, 13].

The system Σ has, like every conservative system, a transfer function $\mathbf{G}$ which is in the Schur class. This means that $\mathbf{G}$ is analytic on the open right half-plane $\mathbb{C}_0$ and $\|\mathbf{G}(s)\| \leq 1$ for all $s \in \mathbb{C}_0$. For the simple proof of this fact see [13, Theorem 1.3 and Proposition 4.5]. The boundary values $\mathbf{G}(i\omega)$ can be defined for almost every $\omega \in \mathbb{R}$ as nontangential limits, and we have

$$\mathbf{G}(i\omega)^*\mathbf{G}(i\omega) = \mathbf{G}(i\omega)\mathbf{G}(i\omega)^* = I$$

for almost every $\omega \in \mathbb{R}$ (i.e., $\mathbf{G}$ is inner and co-inner). This follows from [10, Proposition 2.1] or alternatively from [7, Corollary 7.3].

Recall that a well-posed linear system with input space U, output space Y and transfer function $\mathbf{G}$ is called *regular* if for every $v \in U$, the limit

$$\lim_{s \to +\infty,\ s \in \mathbb{R}} \mathbf{G}(s)v = Dv$$

exists. In this case, the operator $D \in \mathcal{L}(U, Y)$ is called the feedthrough operator of the system (see [8, 9, 6] for further details). For regular linear systems, the theories of local representation, feedback and dynamic stabilization are much simpler than for well-posed linear systems.

Conjecture. *The system Σ from Proposition 1 is regular and its feedthrough operator is zero.*

Consider the particular situation when Ω is one-dimensional: $\Omega = (0, 1)$, $\Gamma_0 = \{0\}$, $\Gamma_1 = \{1\}$ and $U = Y = \mathbb{C}$. Now the function b becomes a nonzero number, and without loss of generality we may take $b = 1$. It is easy to see that the input signal enters the domain at $x = 1$, propagates along the domain (with unit speed) until it gets reflected at $x = 0$ and then it propagates back to exit (as the output signal) at $x = 1$. If the initial state is zero, then for $t \geq 2$ we have $y(t) = u(t - 2)$, so that the transfer function is $\mathbf{G}(s) = e^{-2s}$. Note that $\mathbf{G}$ is indeed inner and it is regular with feedthrough operator zero.

The author thinks that he can prove the conjecture in the following particular case: the active boundary Γ_1 can be partitioned into a finite union of open subsets which are either planar (i.e., an open subset of an $n-1$ dimensional hyperplane) or spherical (i.e., an open subset of an $n - 1$ dimensional sphere). The idea is to construct solutions of (1.1) which locally (near a boundary point) look like a planar or spherical wave moving into the domain Ω (the initial state is zero) and locally (in time and space), u is a step function. Then locally (in time and space) y is zero, which proves the claim, due to the equivalent characterization of regularity via the step response, see [8, Theorem 5.8].

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