5: Filters

Difference Equations

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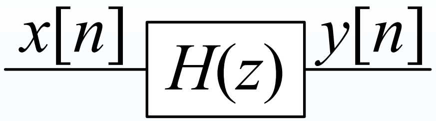

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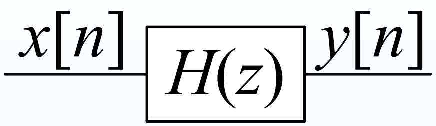

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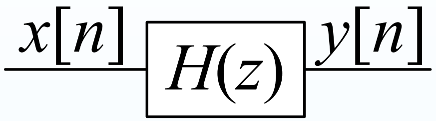

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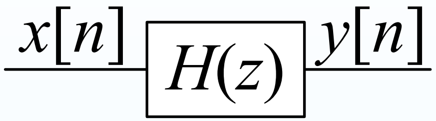

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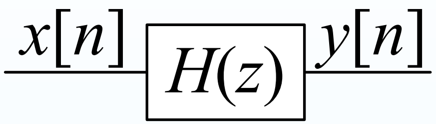

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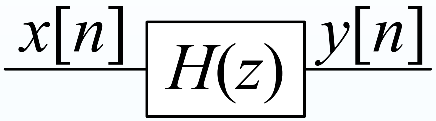

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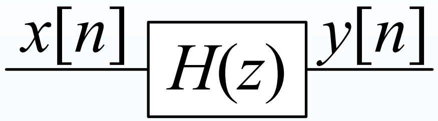

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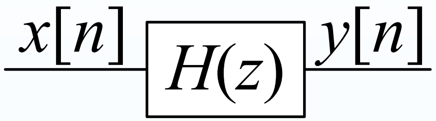

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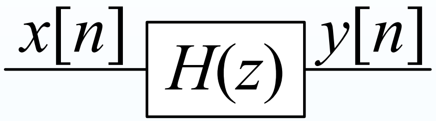

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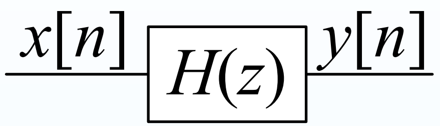

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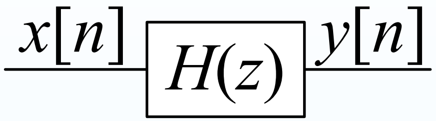

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Note negative sign in first equation.
Authors in some SP fields reverse the sign of the $a[n]$: BAD IDEA.

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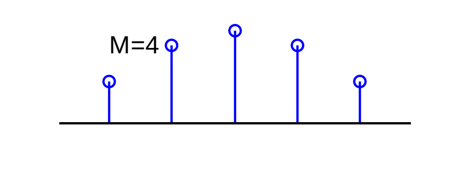

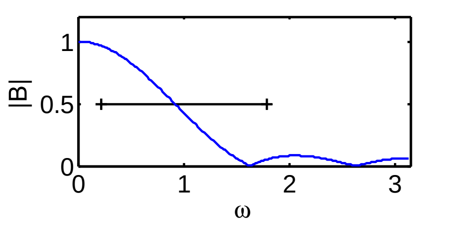

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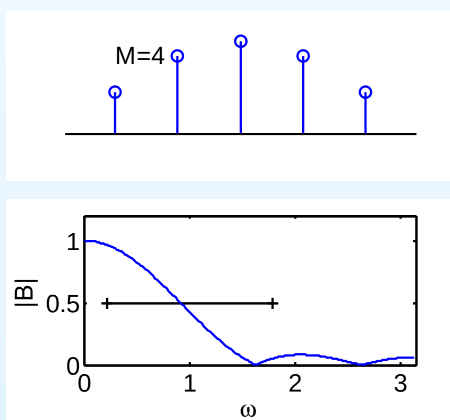

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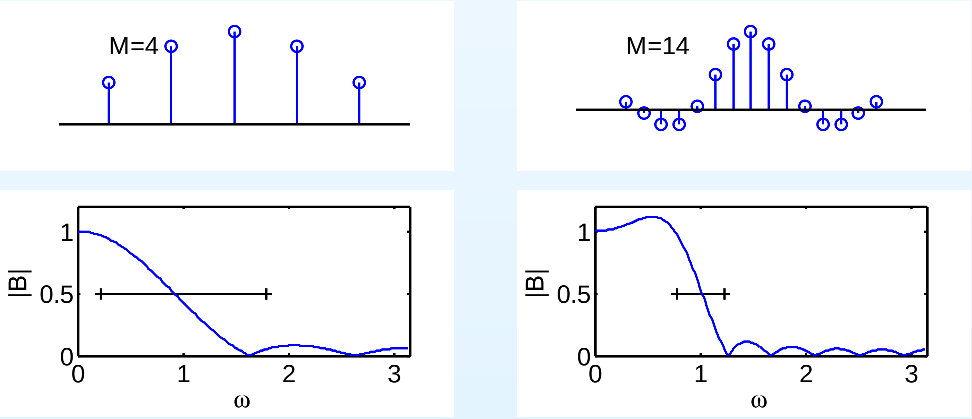

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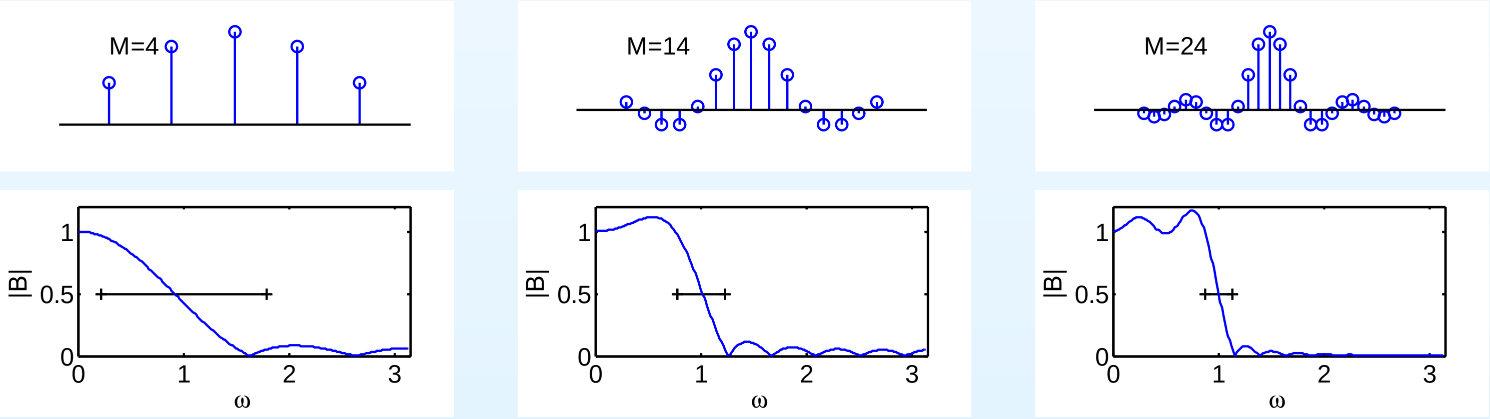

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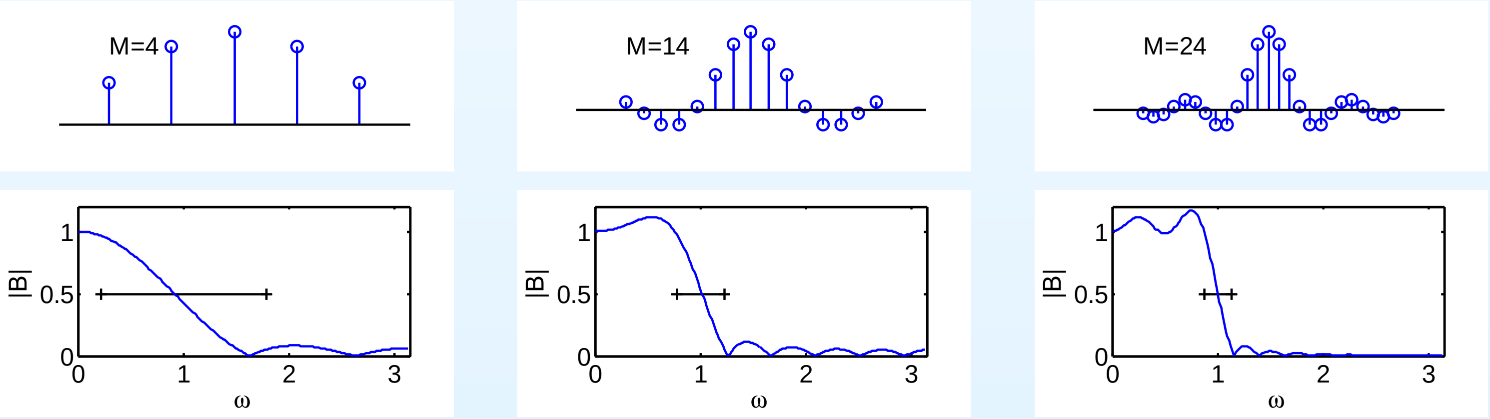

Rule of thumb: Fastest possible transition $\Delta\omega \geq \frac{2\pi}{M}$ (marked line)

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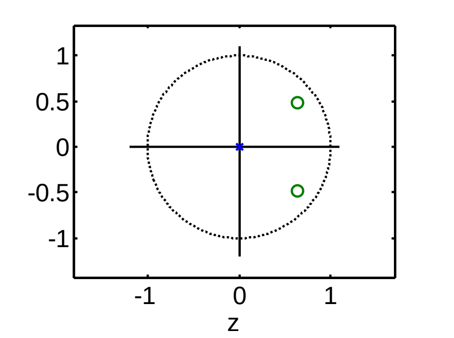
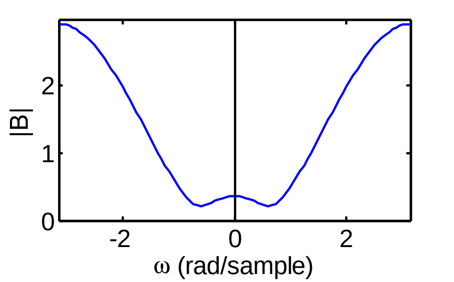
Real:

$[1,\ -1.28,\ 0.64]$





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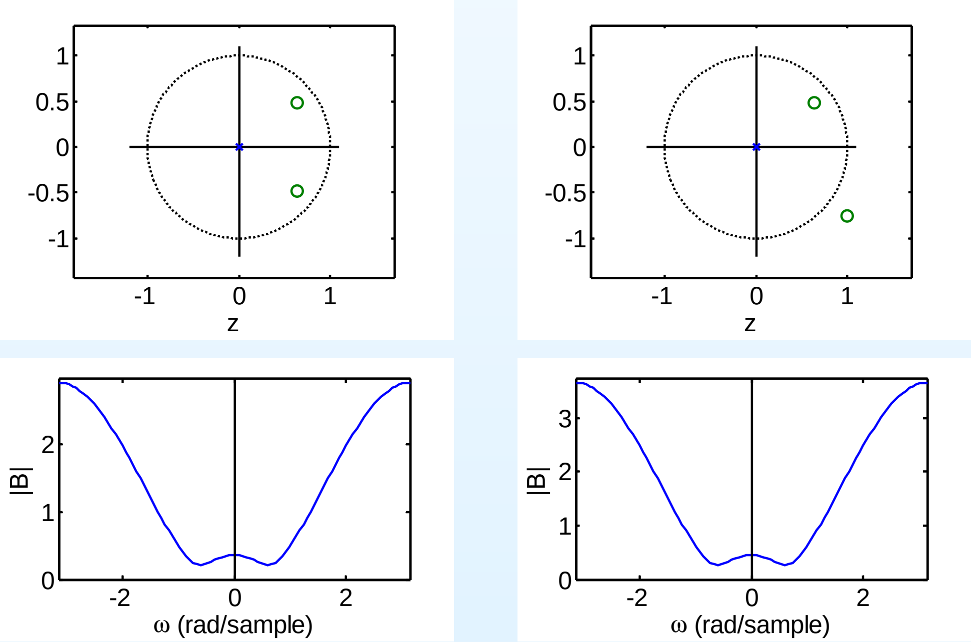

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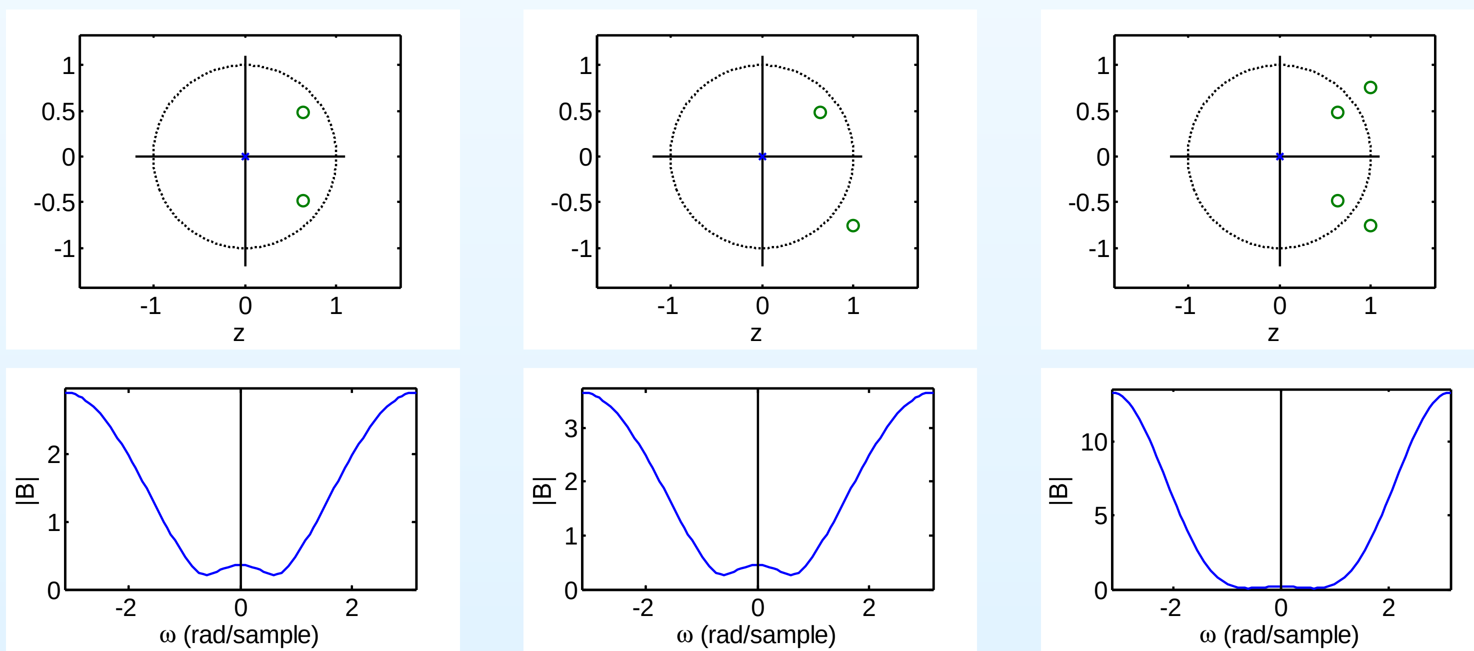
Real + Symmetric $b[n]$ $\Rightarrow$ conjugate+reciprocal groups of four or else pairs on the real axis

Real:	Symmetric:	Real + Symmetric:
$[1,\ -1.28,\ 0.64]$	$[1,\ -1.64+0.27j,\ 1]$	$[1,\ -3.28,\ 4.7625,\ -3.28,\ 1]$



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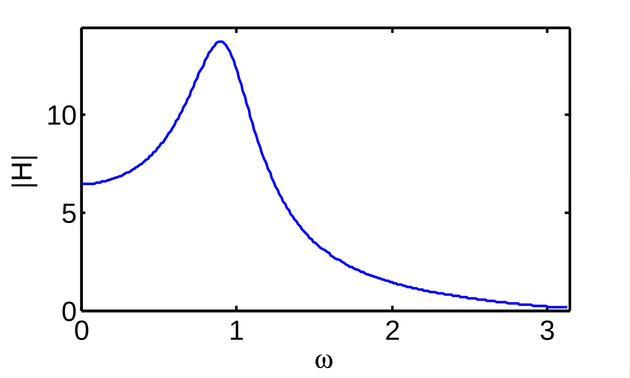
Example:

$H(z) = \frac{2+2.4z^{-1}}{1-0.96z^{-1}+0.64z^{-2}}$



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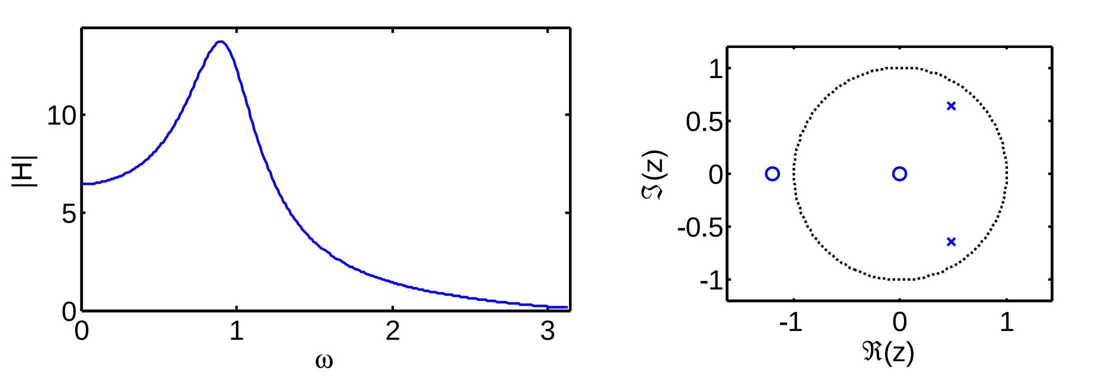

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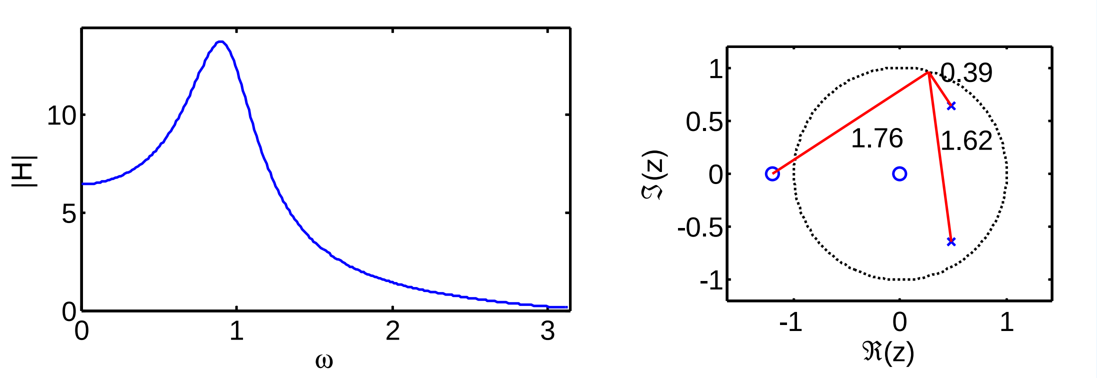
At $\omega = 1.3$: $\left|H(e^{j\omega})\right| = \frac{2\times1.76}{1.62\times0.39}$



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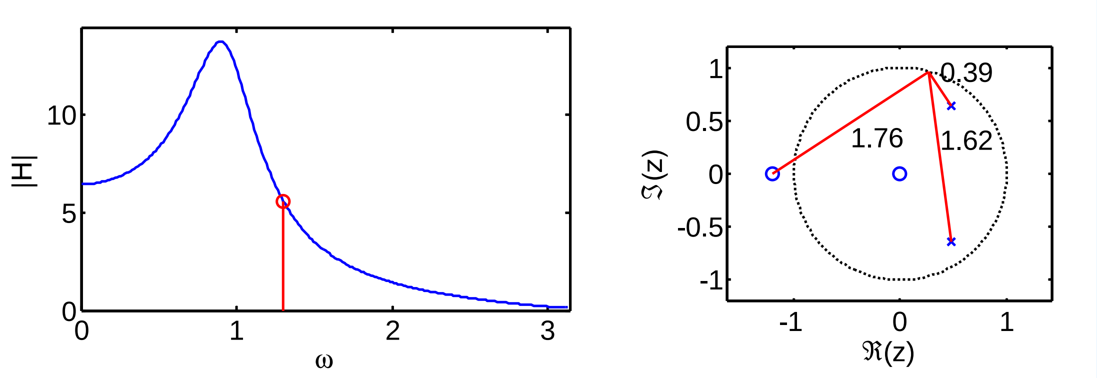
$\left|H(e^{j\omega})\right|=\frac{|b[0]|\left|z^{-M}\right|\prod_{i=1}^{M}|z-q_i|}{\left|z^{-N}\right|\prod_{i=1}^{N}|z-p_i|}$ for $z=e^{j\omega}$

Example:

$H(z)=\frac{2+2.4z^{-1}}{1-0.96z^{-1}+0.64z^{-2}}=\frac{2\left(1+1.2z^{-1}\right)}{(1-(0.48-0.64j)z^{-1})(1-(0.48+0.64j)z^{-1})}$

At $\omega=1.3$: $\left|H(e^{j\omega})\right|=\frac{2\times1.76}{1.62\times0.39}=5.6$



IIR Frequency Response

Factorize $H(z) = \frac{B(z)}{A(z)} = \frac{b[0]\prod_{i=1}^{M}\left(1-q_i z^{-1}\right)}{\prod_{i=1}^{N}\left(1-p_i z^{-1}\right)}$

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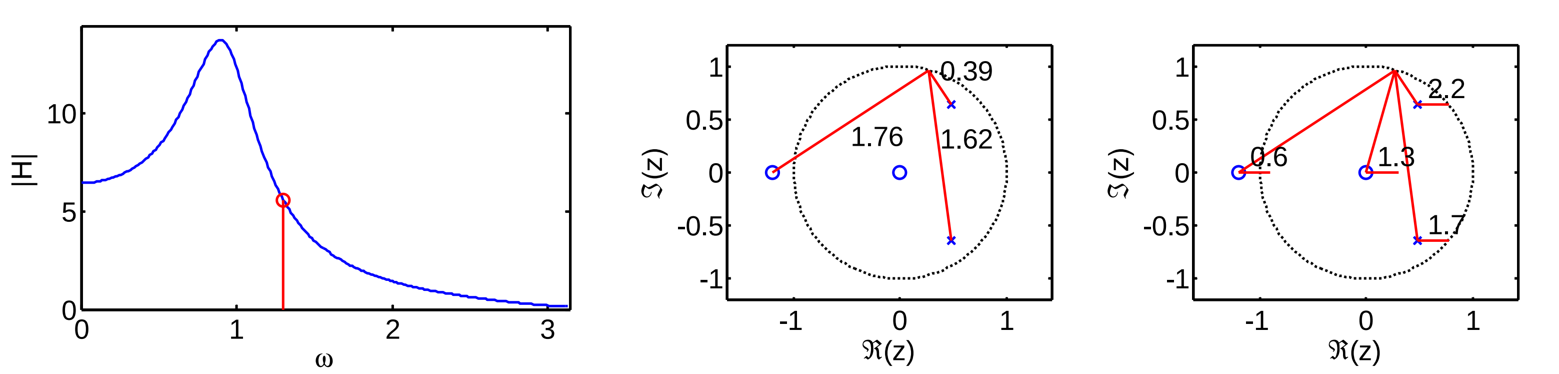
At $\omega = 1.3$: $\left|H(e^{j\omega})\right| = \frac{2\times 1.76}{1.62\times 0.39} = 5.6$

$$\angle H(e^{j\omega}) = (0.6+1.3) - (1.7+2.2) = -1.97$$



Negating z

Given a filter $H(z)$ we can form a new one $H_R(z) = H(-z)$

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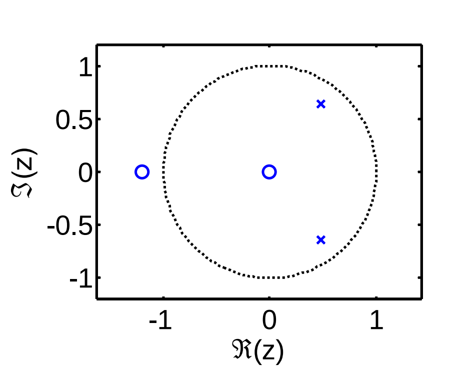

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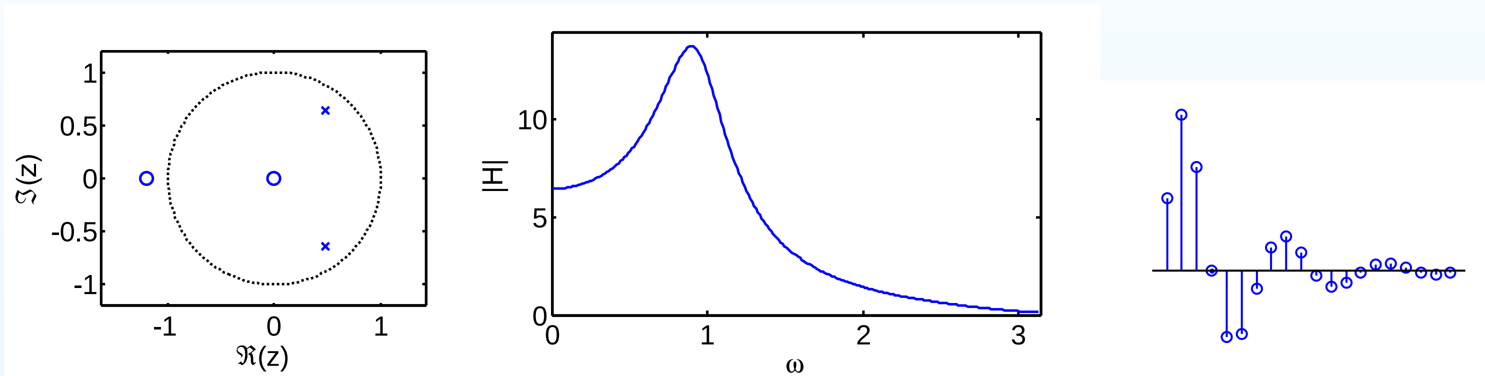

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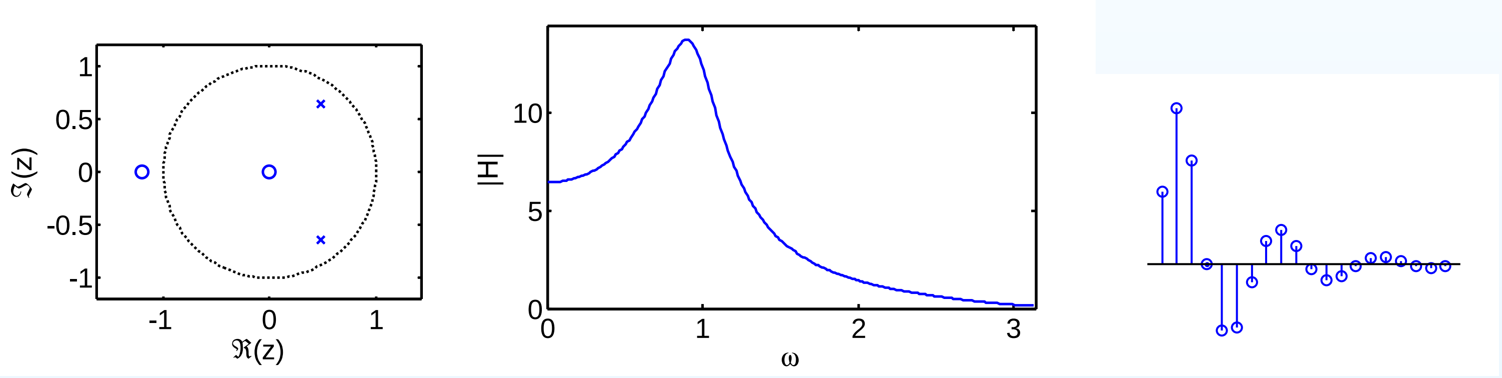

Negate z: $H_R(z) = \frac{2-2.4z^{-1}}{1+0.96z^{-1}+0.64z^{-2}}$

Negate odd coefficients

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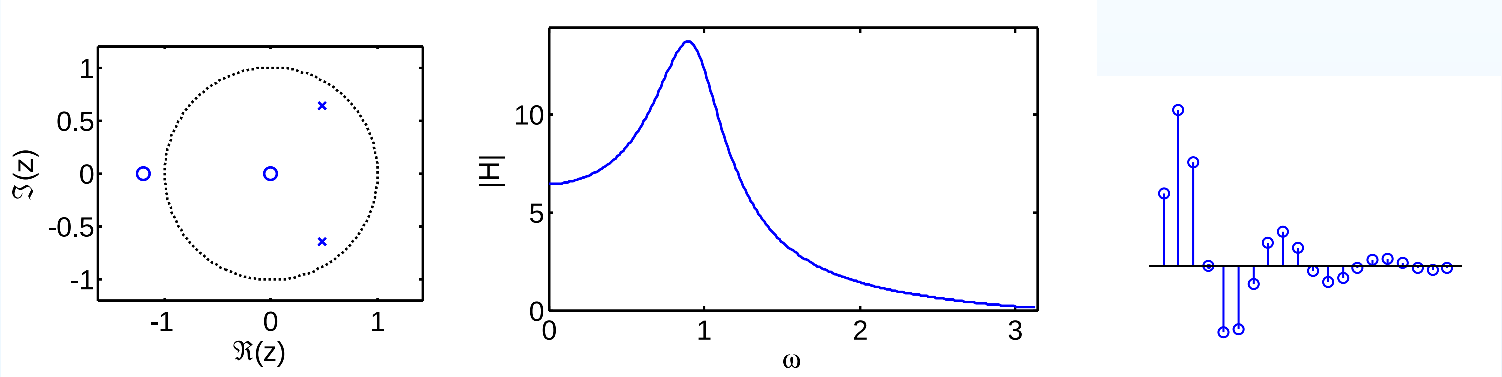

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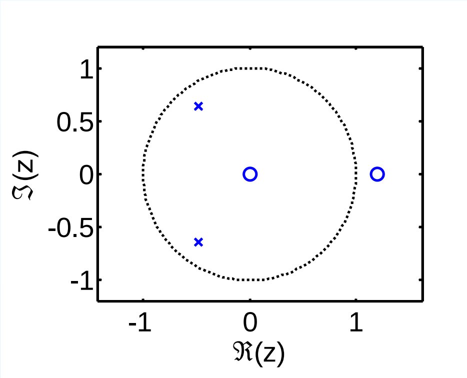

Pole and zero positions are negated

Negating z

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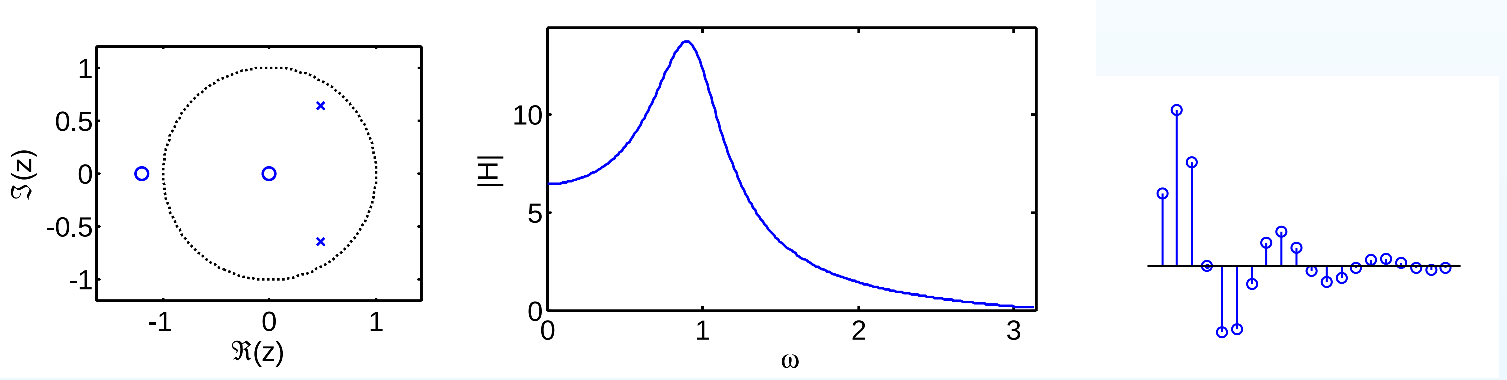

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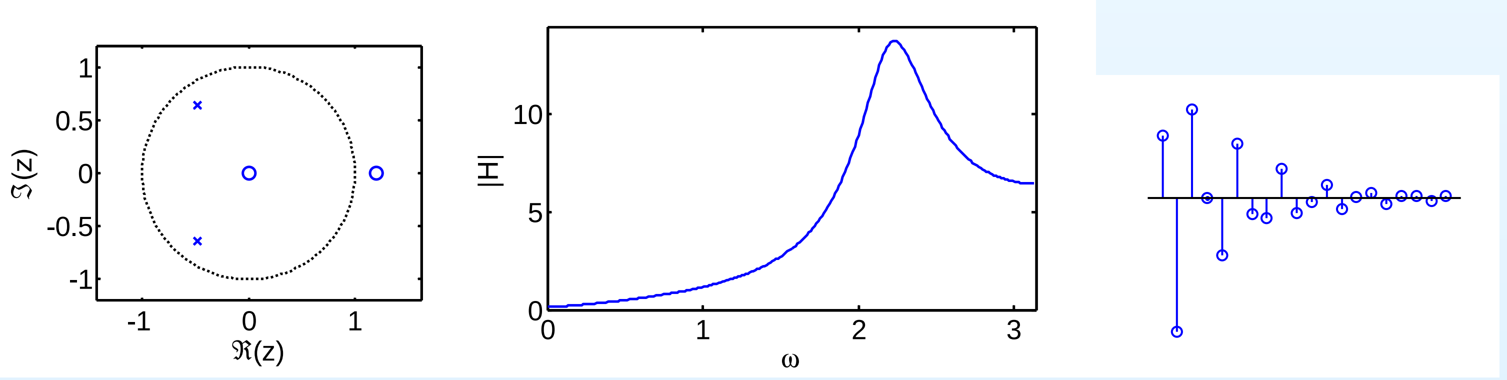

Pole and zero positions are negated, response is flipped and conjugated.

Cubing z

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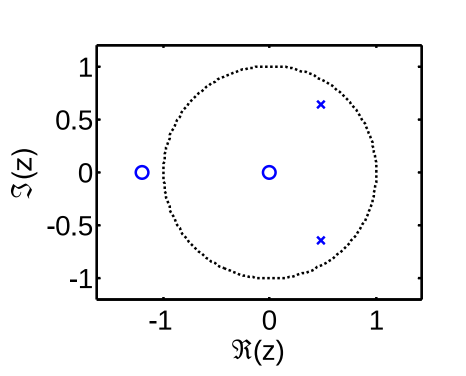

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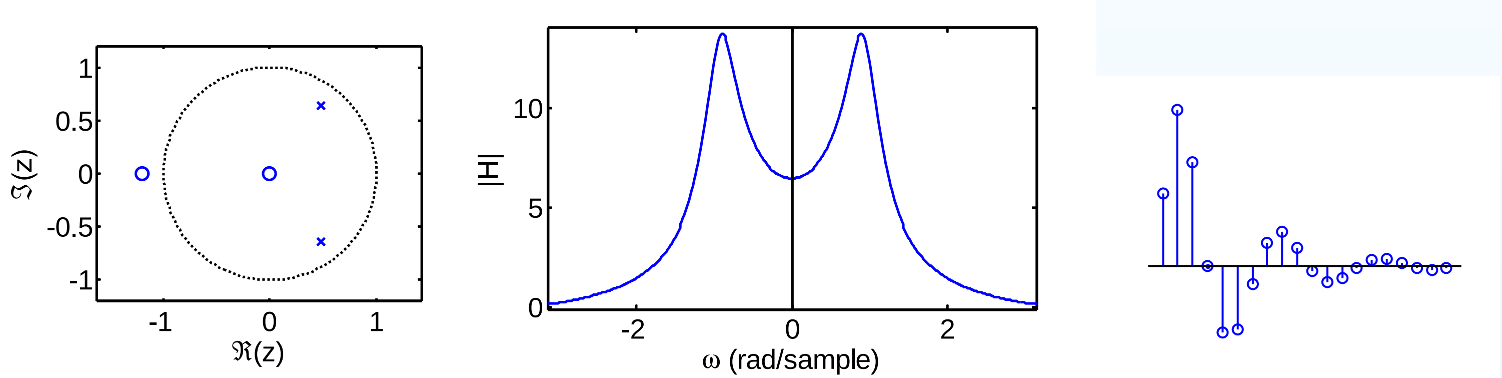

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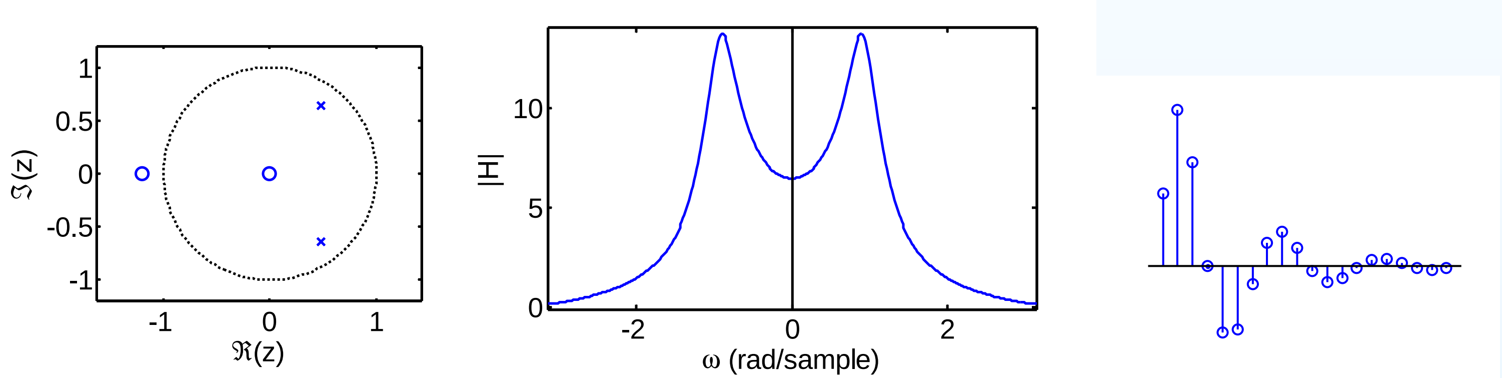

Cube z: $H_C(z) = \frac{2+2.4z^{-3}}{1-0.96z^{-3}+0.64z^{-6}}$

Insert 2 zeros between coefs

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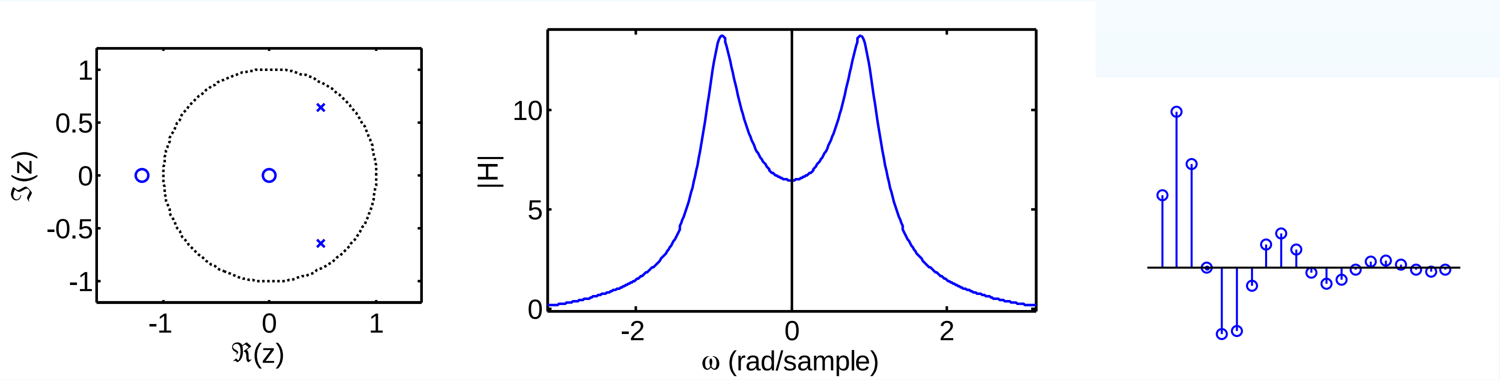

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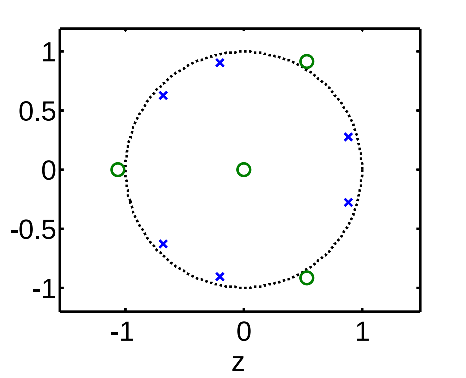

Pole and zero positions are replicated

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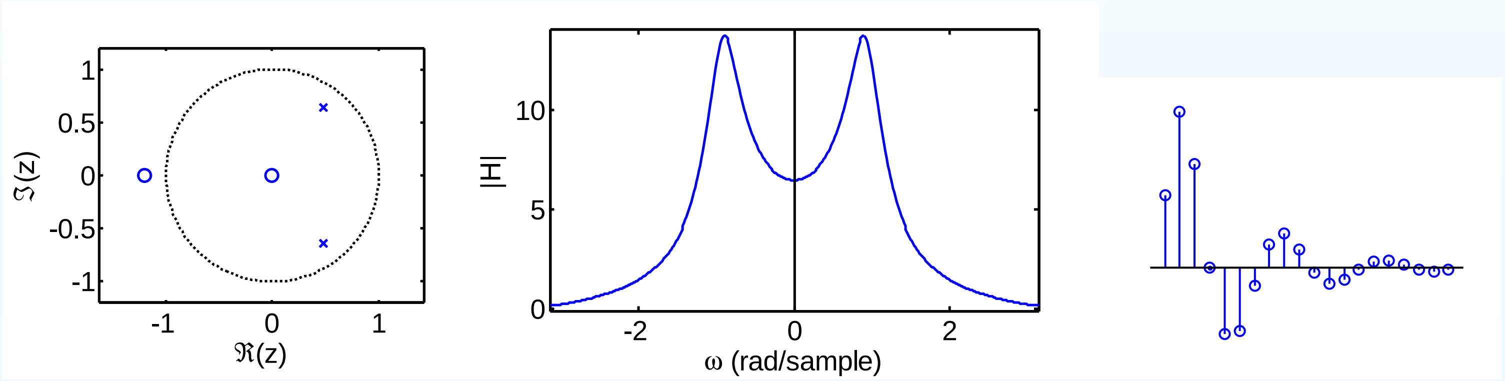

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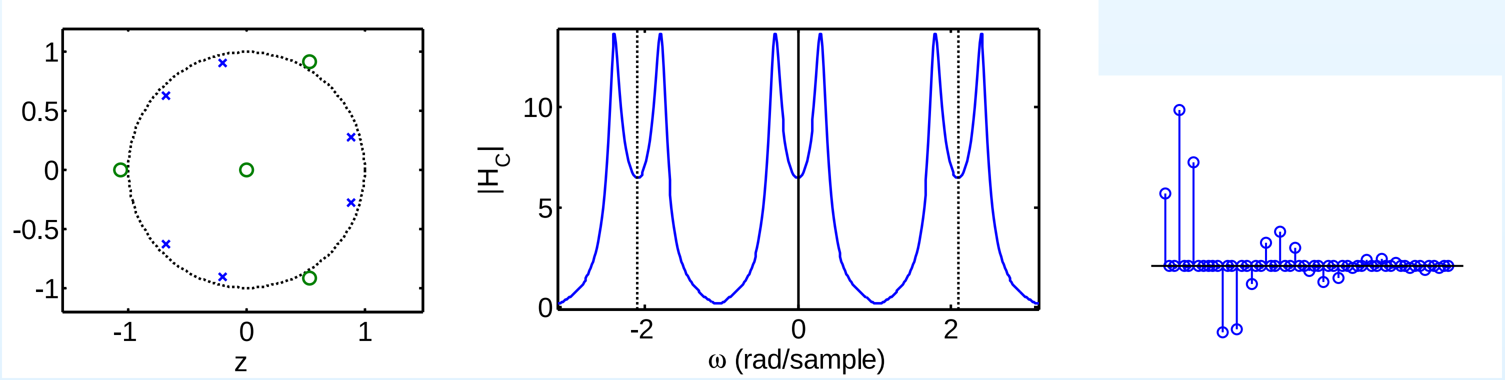

Pole and zero positions are replicated, magnitude response replicated.

Scaling z

+

Given a filter $H(z)$ we can form a new one $H_S(z) = H(\frac{z}{\alpha})$

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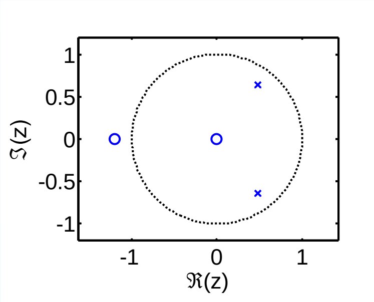

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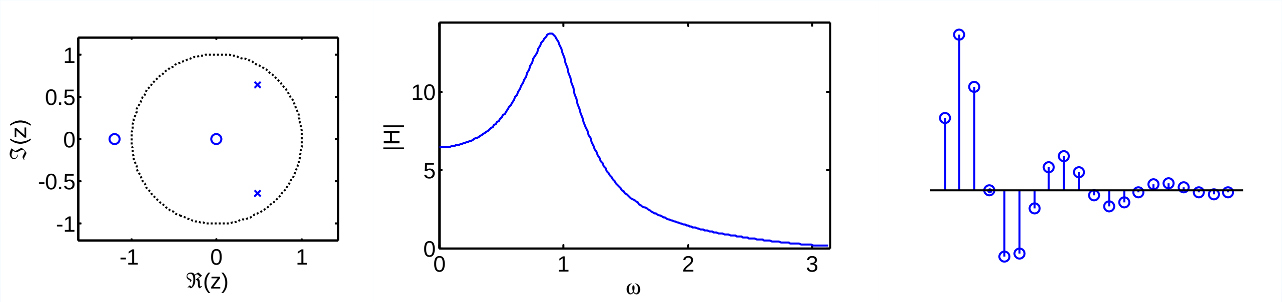

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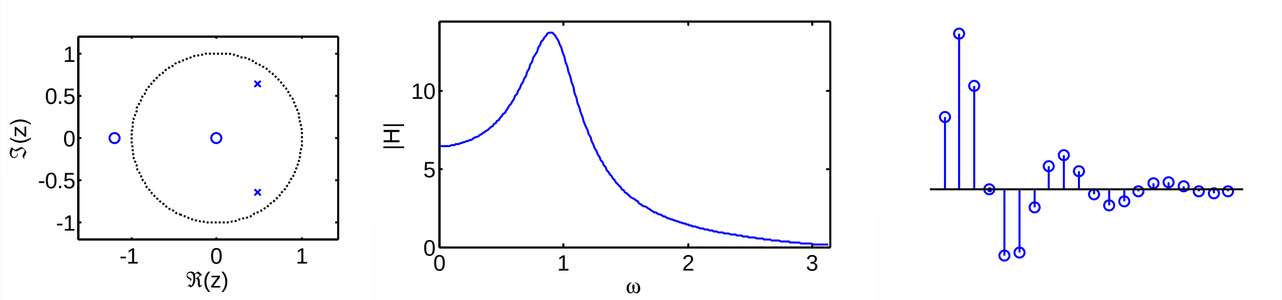

Scale z: $H_S(z) = H\left(\frac{z}{1.1}\right) = \frac{2+2.64z^{-1}}{1-1.056z^{-1}+0.7744z^{-2}}$

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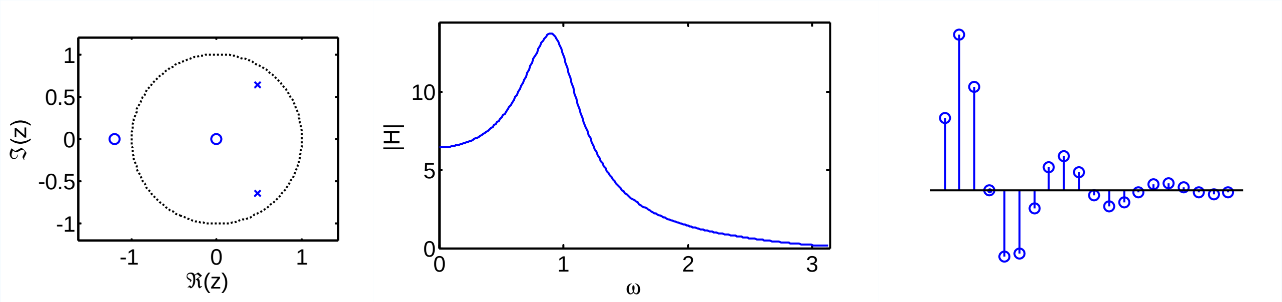

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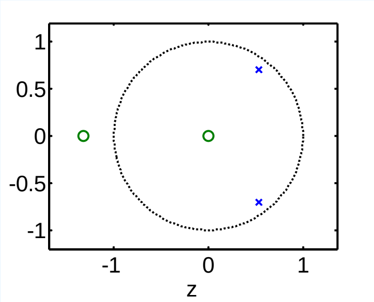

Pole and zero positions are multiplied by α

Scaling z

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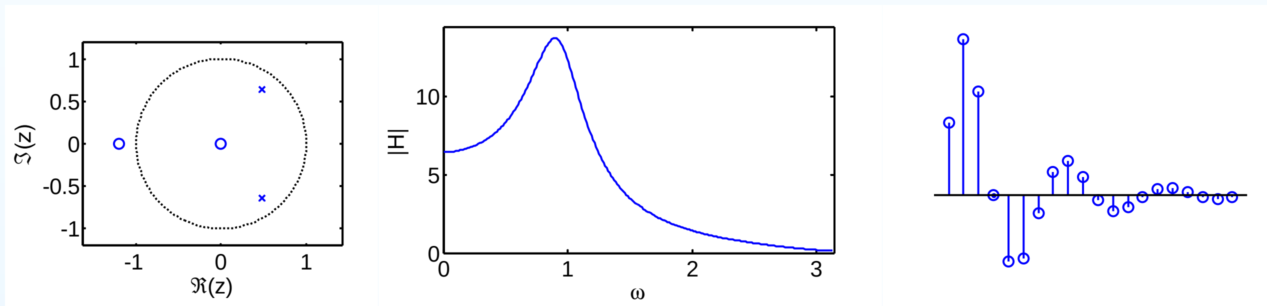

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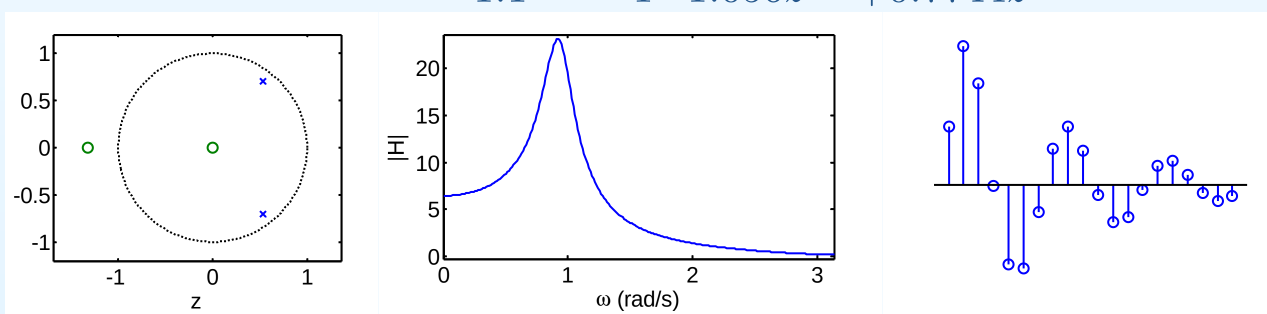

Pole and zero positions are multiplied by α, $\alpha > 1 \Rightarrow$peaks sharpened.

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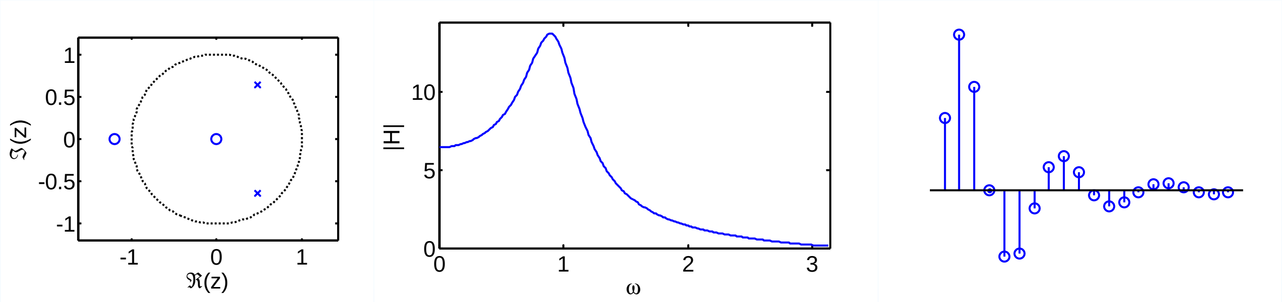

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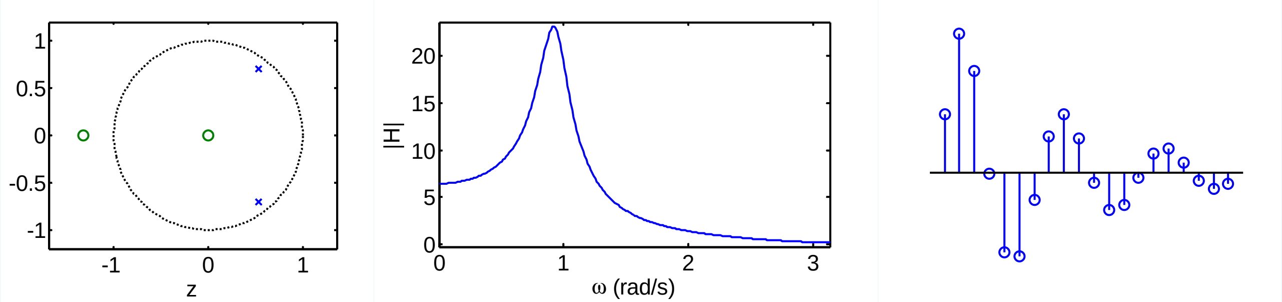

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Pole at $z = p$ gives peak bandwidth $\approx 2\,|\log|p|| \approx 2\,(1 - |p|)$

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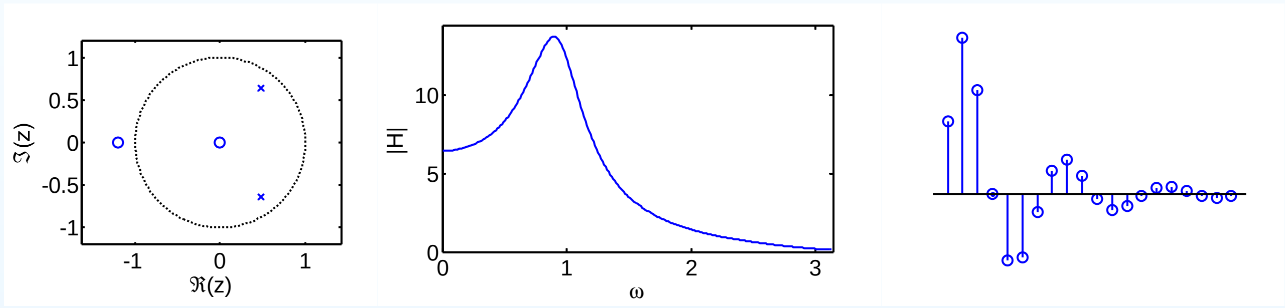

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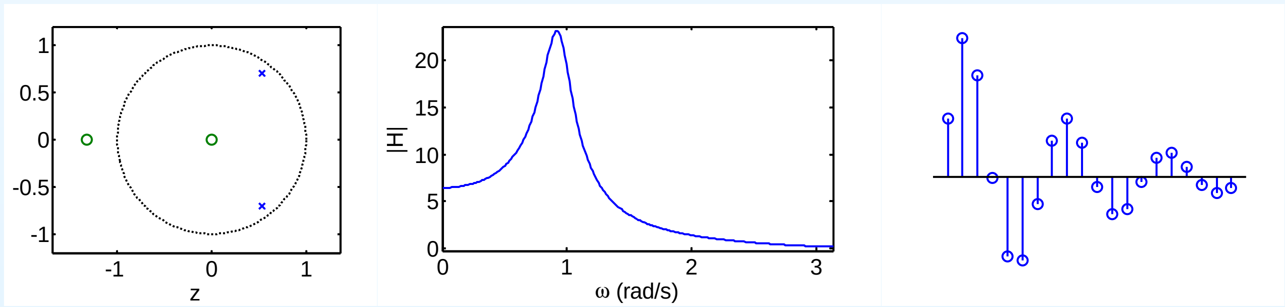

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For pole near unit circle, decrease bandwidth by $\approx 2\log\alpha$

Low-pass filter

+

1st order low pass filter: extremely common

$$y[n] = (1-p)x[n] + py[n-1]$$

Low-pass filter

1st order low pass filter: extremely common

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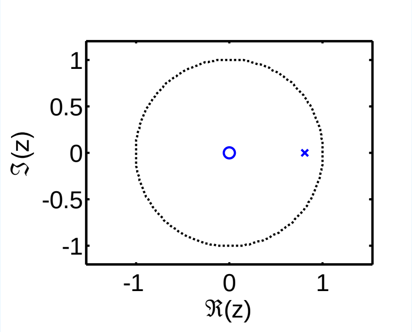

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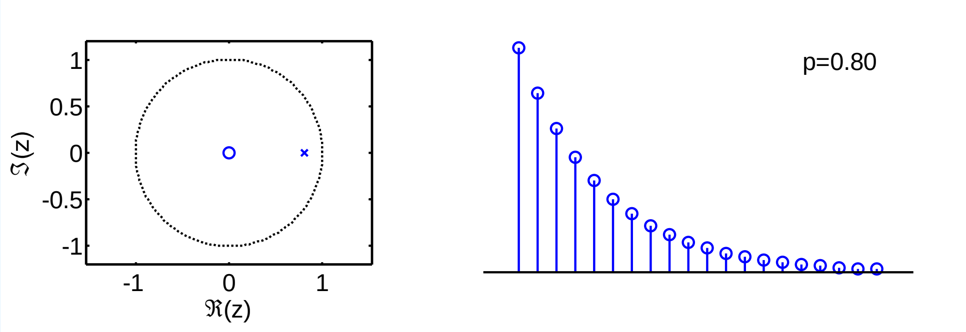

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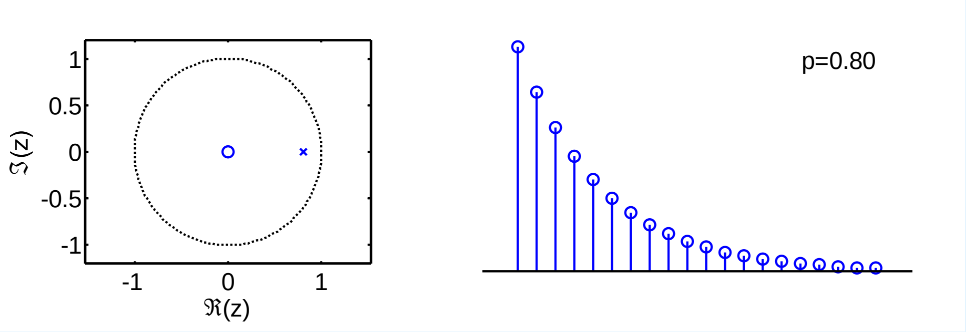
where $\tau = \frac{1}{-\ln p}$ is the time constant in samples.



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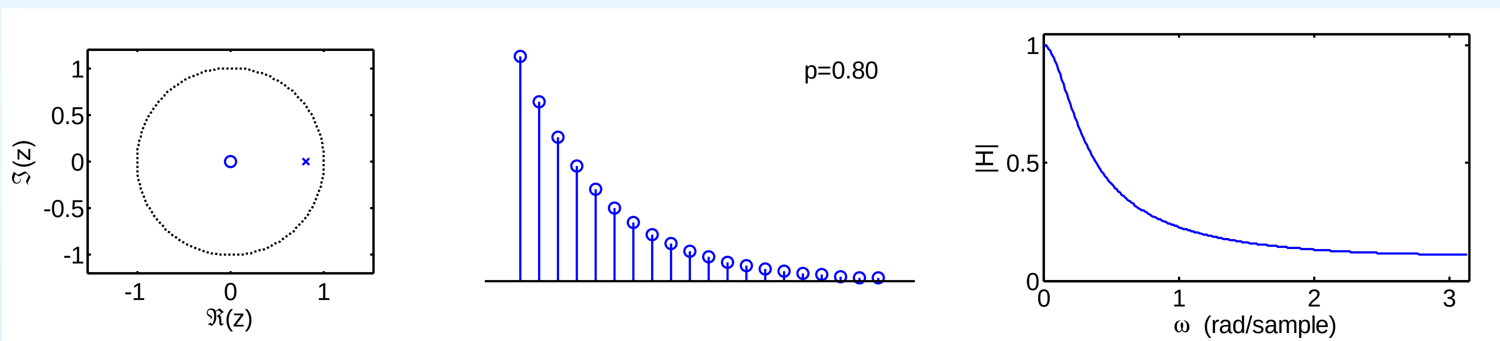

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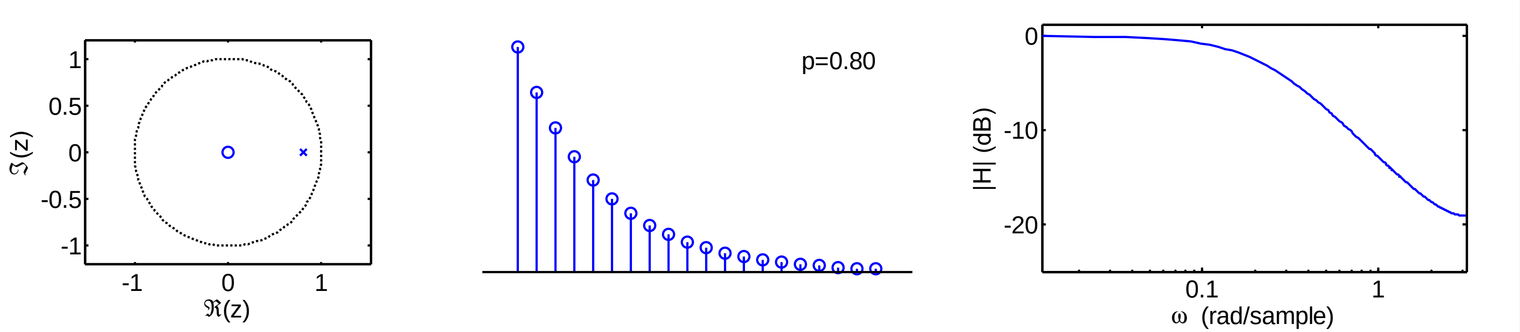

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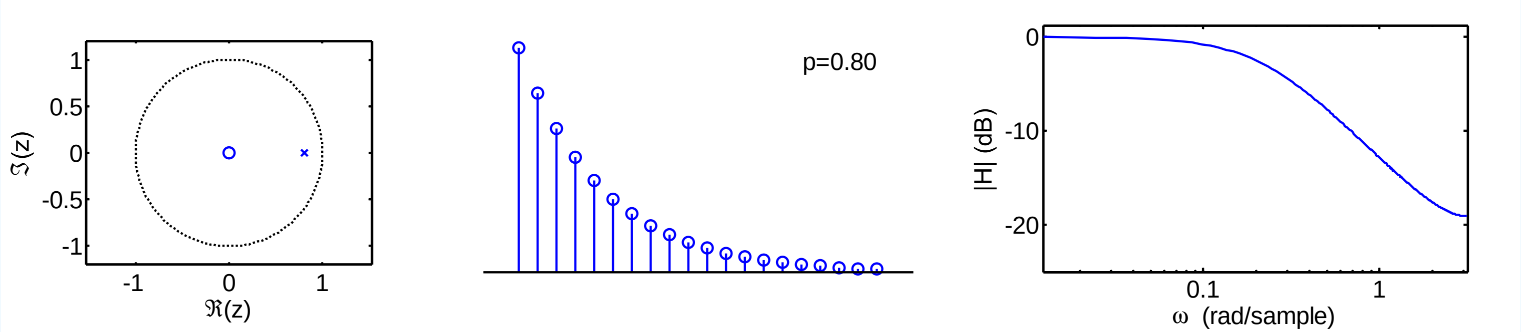
Low-pass filter with DC gain of unity.



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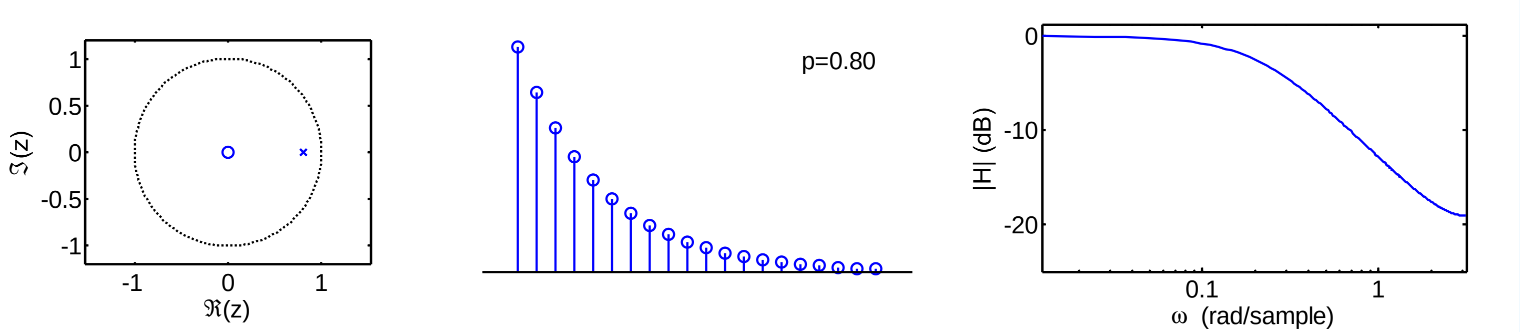
Low-pass filter with DC gain of unity.

3 dB frequency is $\omega_{3dB} = \cos^{-1}\left(1 - \frac{(1-p)^2}{2p}\right)$



Low-pass filter

1st order low pass filter: extremely common

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Impulse response:

$$h[n] = (1-p)p^n = (1-p)e^{-\frac{n}{\tau}}$$

where $\tau = \frac{1}{-\ln p}$ is the time constant in samples.

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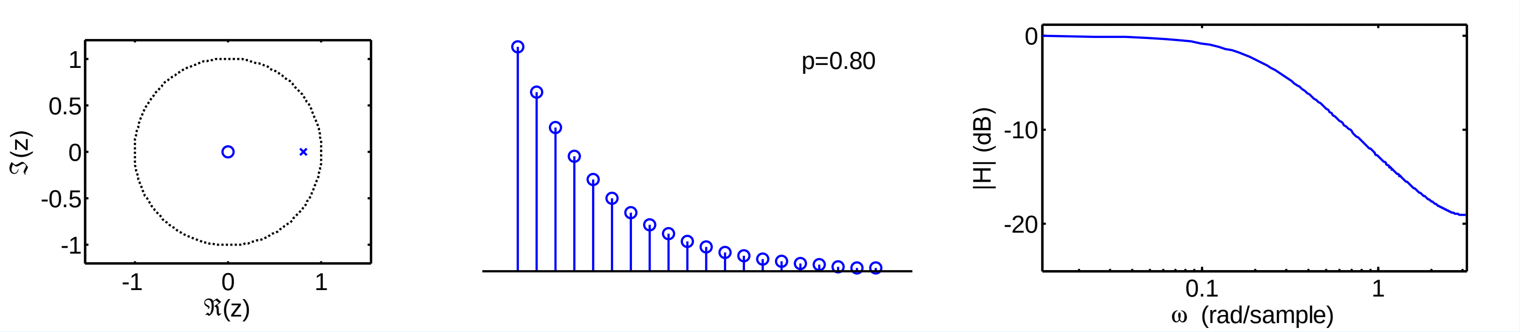

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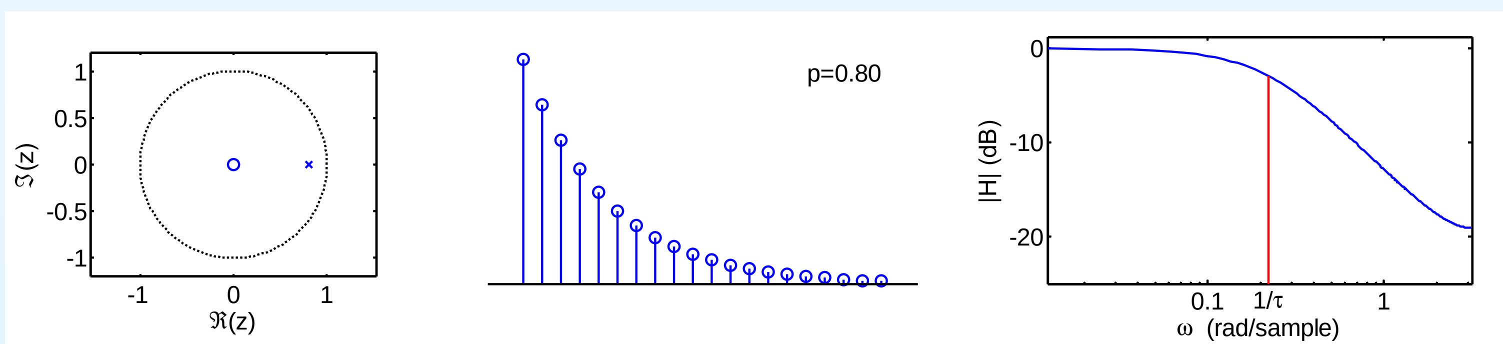

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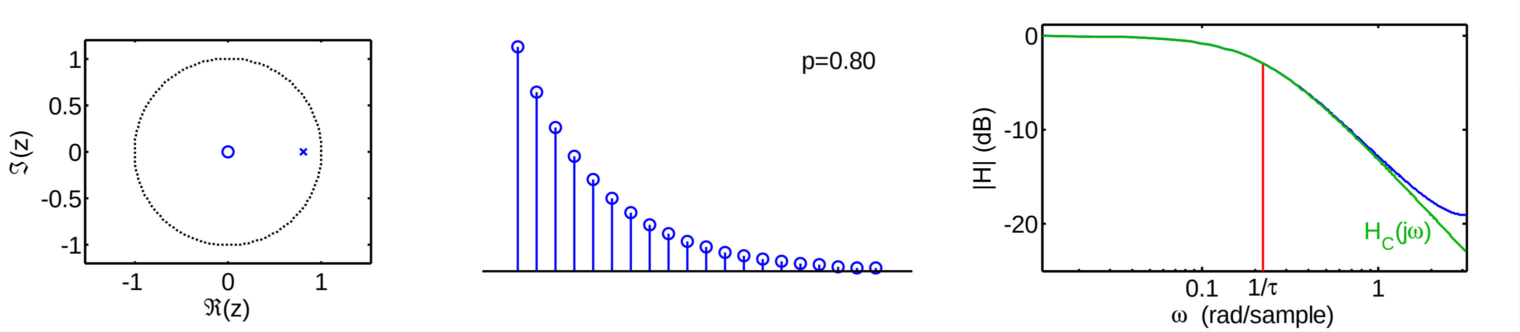

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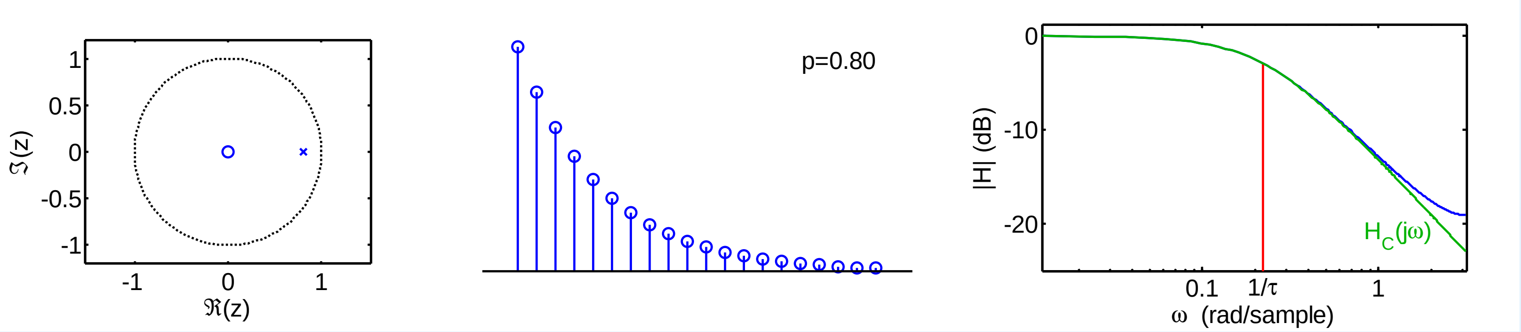

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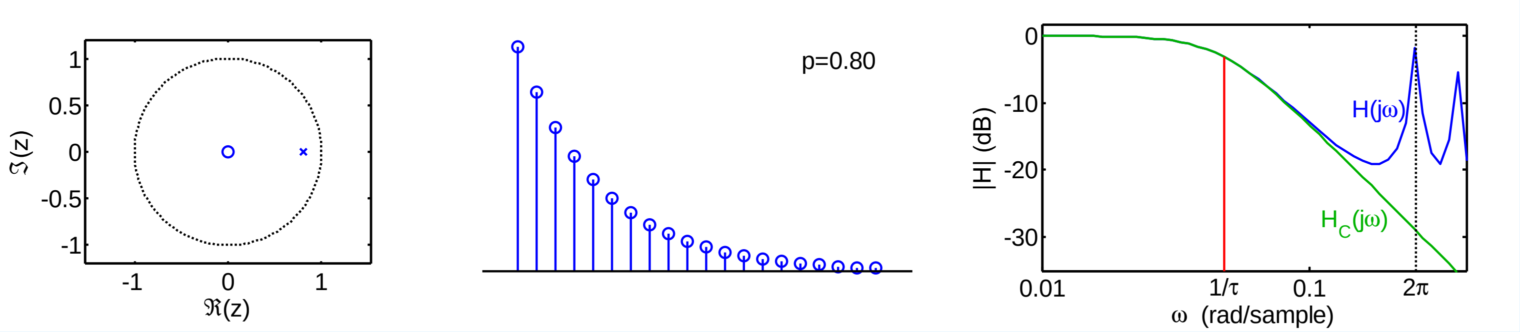

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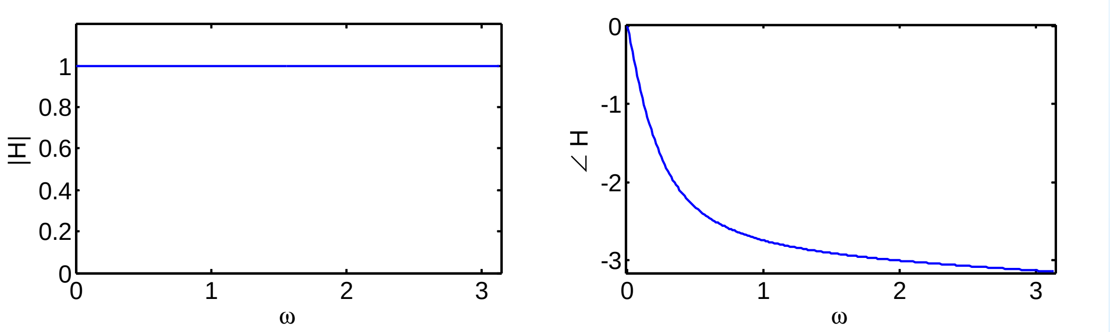

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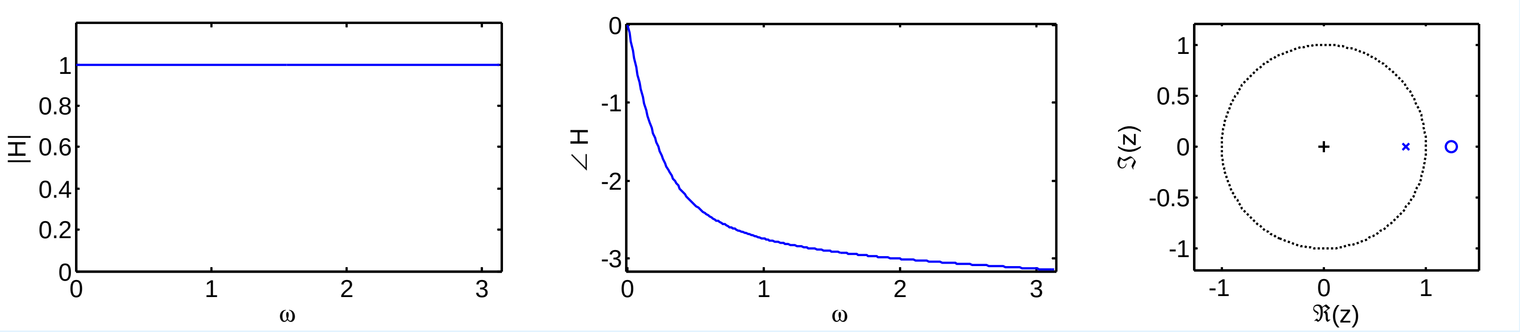

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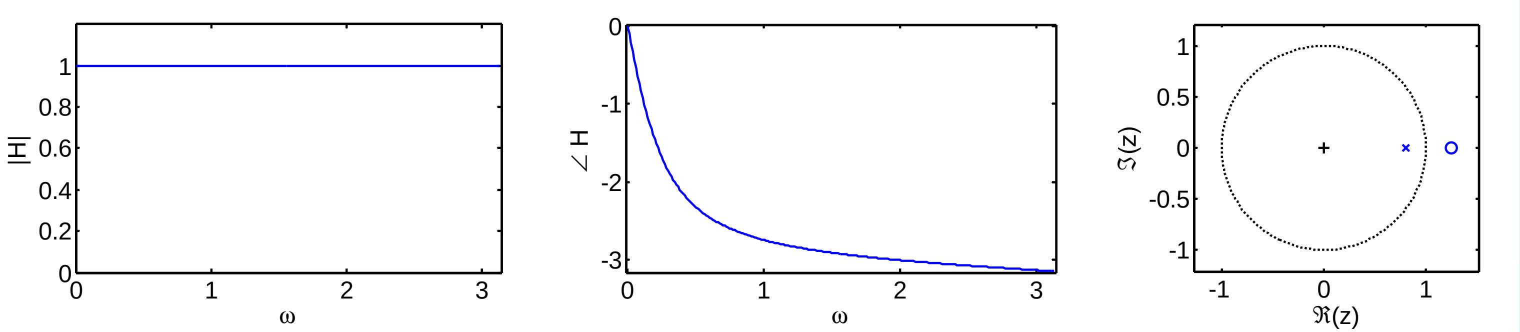

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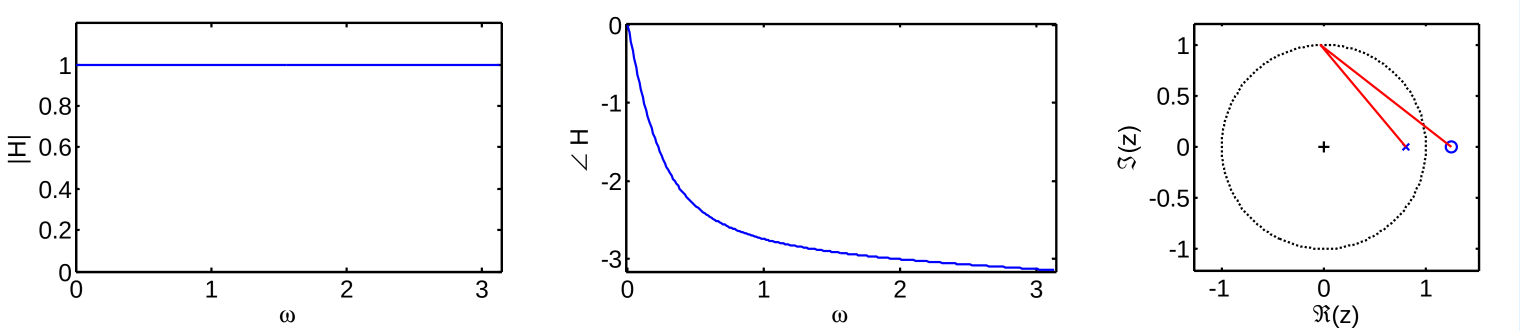

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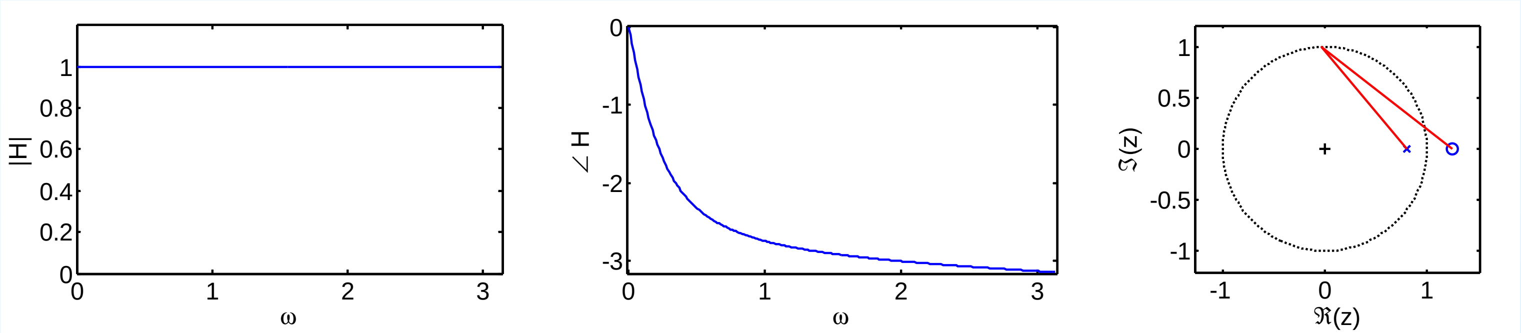

In an allpass filter, the zeros are the poles reflected in the unit circle.

Group Delay

Group delay: $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega}$ = delay of the modulation envelope.

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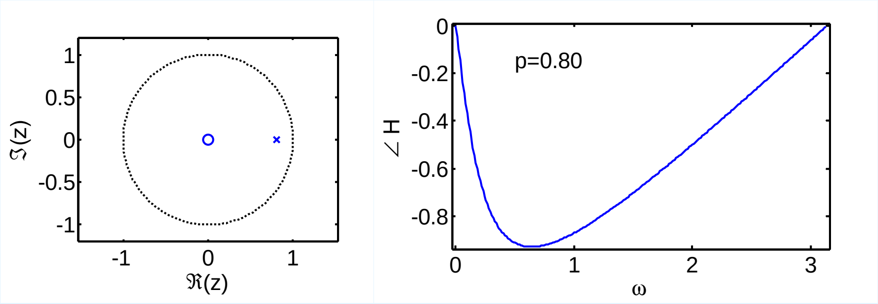

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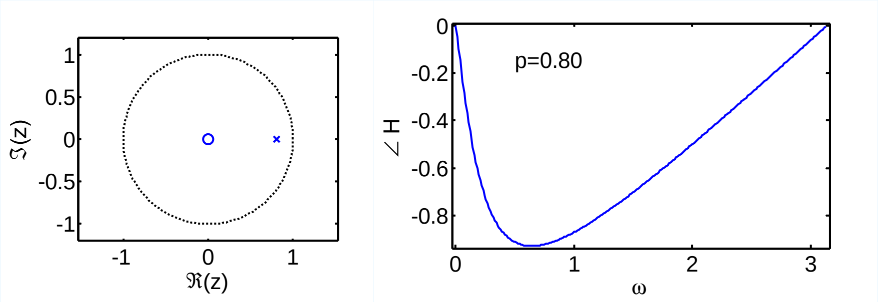

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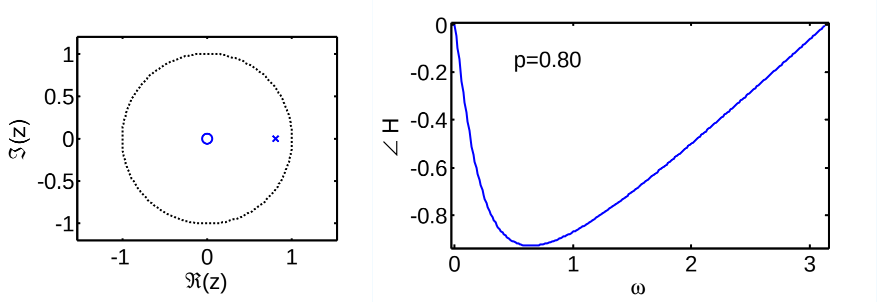

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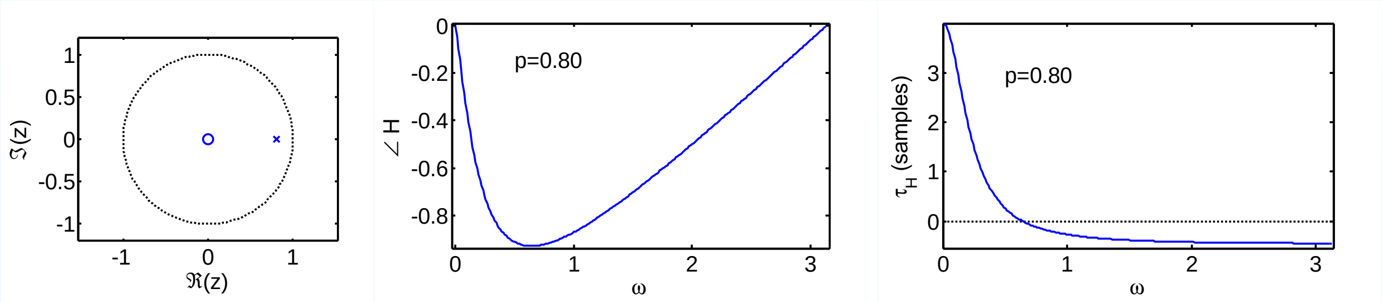

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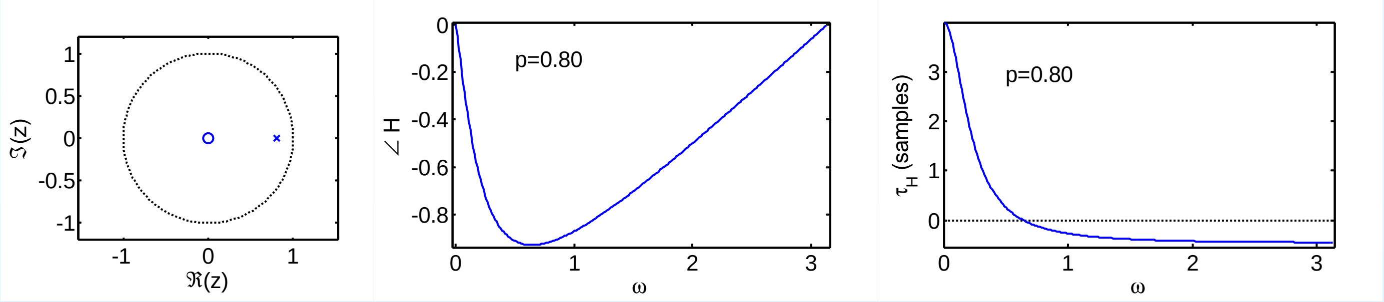

Average group delay (over ω) = (# poles – # zeros) within the unit circle

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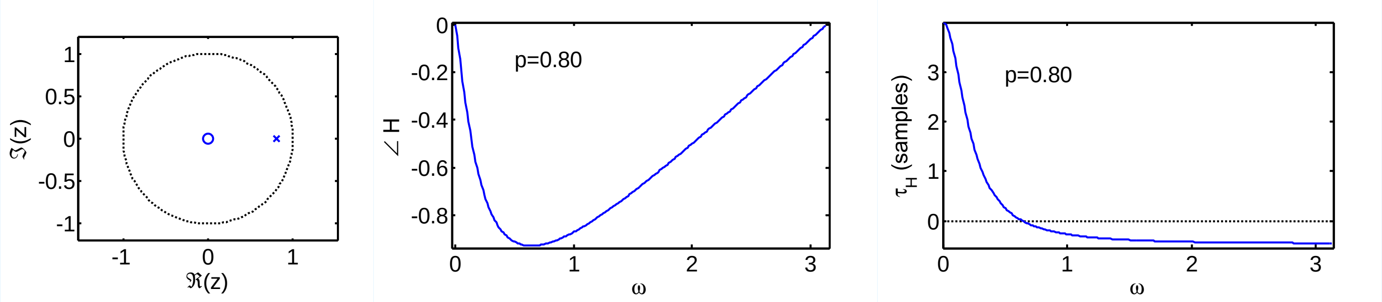
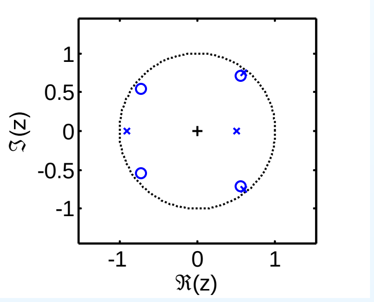


Average group delay (over ω) = (# poles – # zeros) within the unit circle

Zeros on the unit circle count –½

Minimum Phase

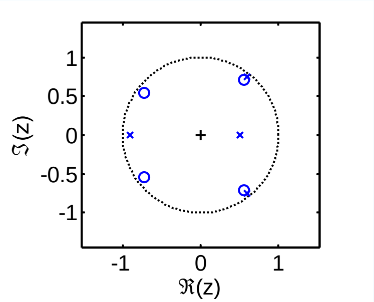
Average group delay (over ω) = (# poles – # zeros) within the unit circle



Minimum Phase

Average group delay (over ω) = (# poles – # zeros) within the unit circle

- zeros on the unit circle count –½

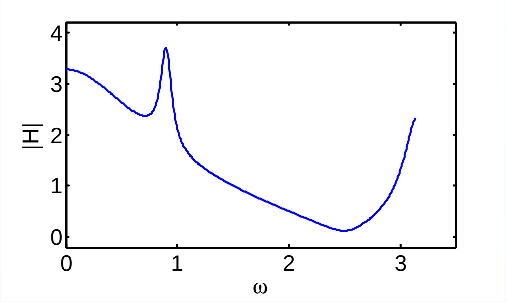


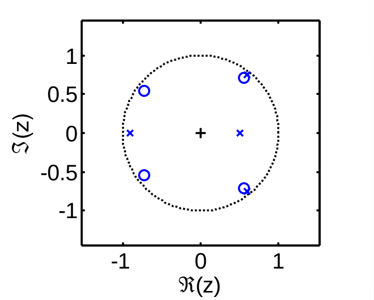
Minimum Phase

+

Average group delay (over ω) = (# poles – # zeros) within the unit circle

- zeros on the unit circle count $-\frac{1}{2}$

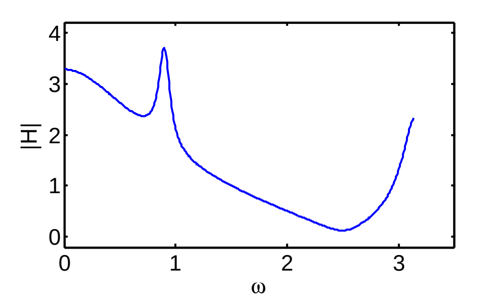


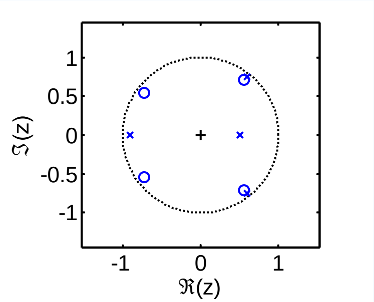


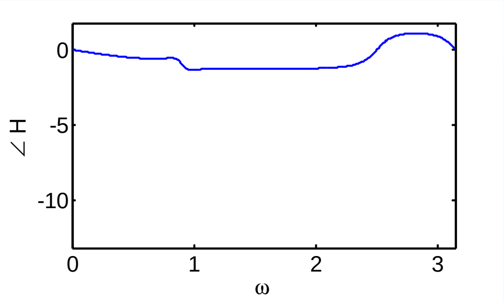
Minimum Phase

Average group delay (over ω) = (# poles – # zeros) within the unit circle

- zeros <u>on</u> the unit circle count –½







Minimum Phase

+

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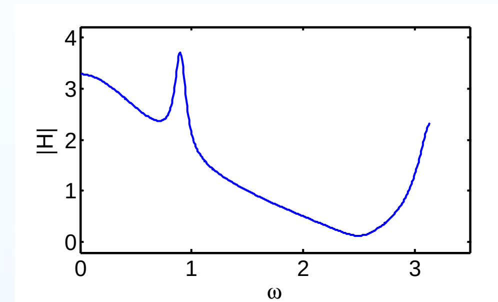

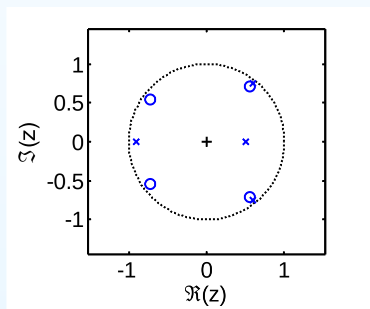

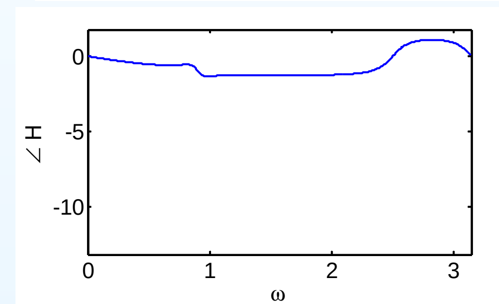

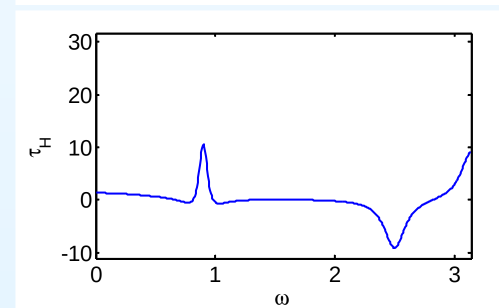

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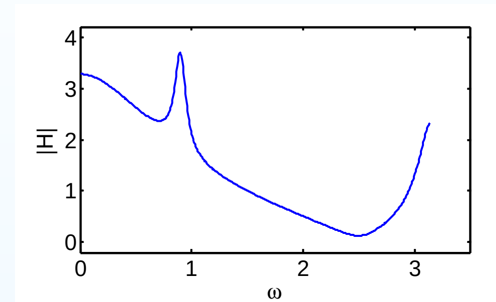

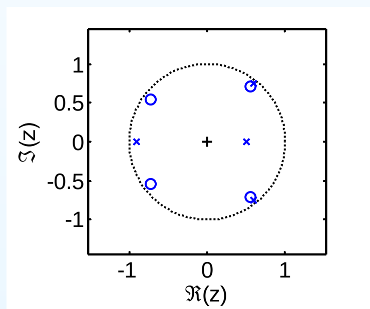

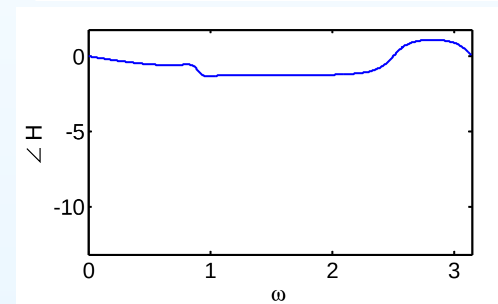

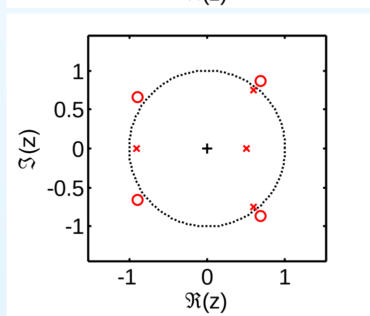

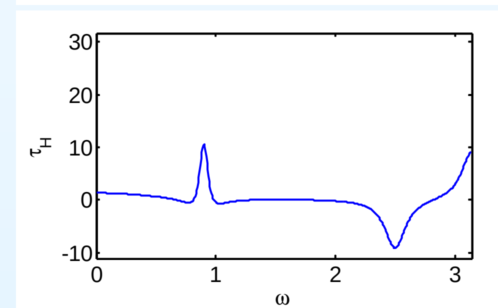

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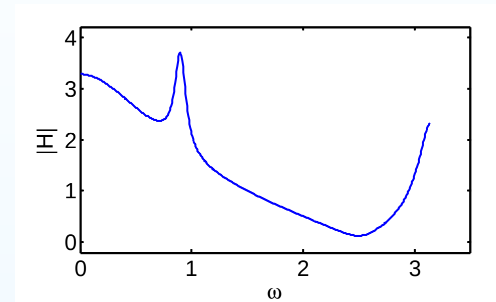

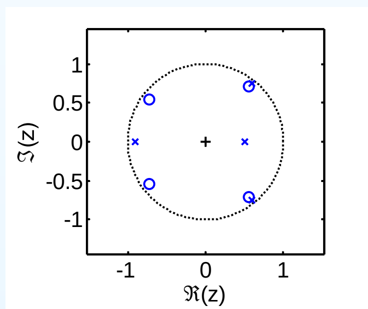

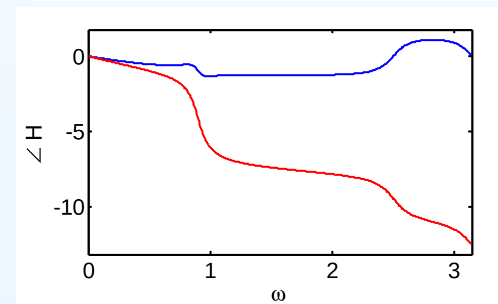

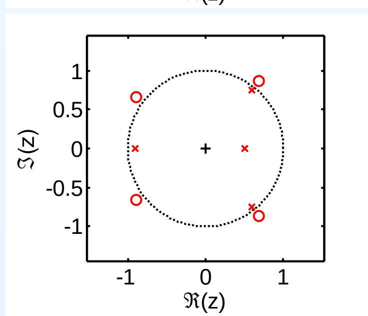

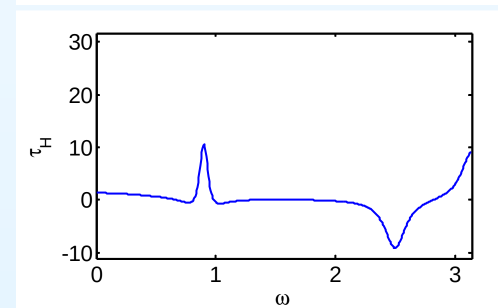

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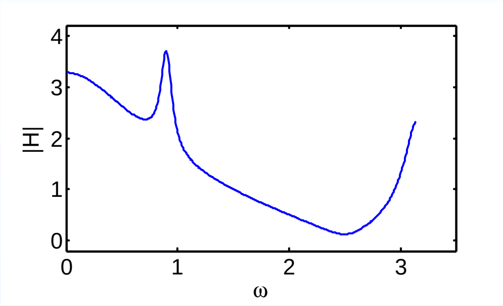

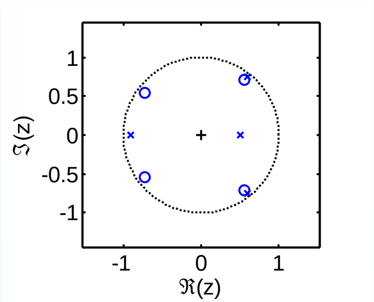

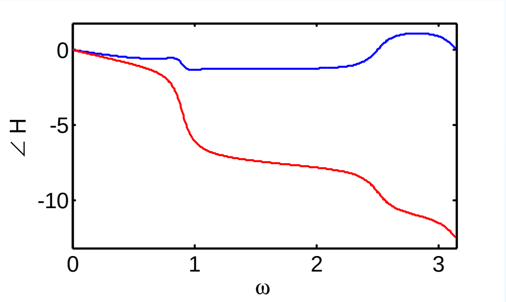

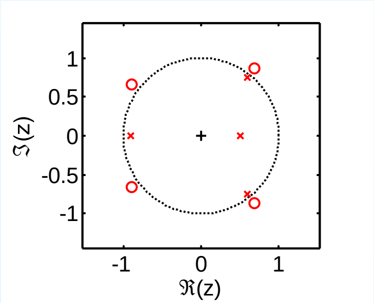

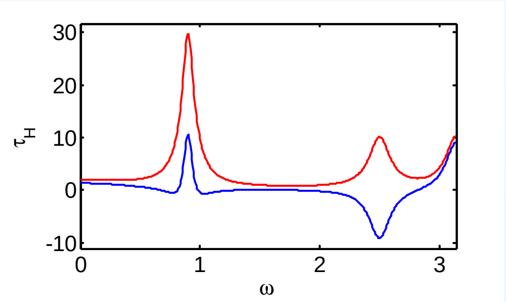

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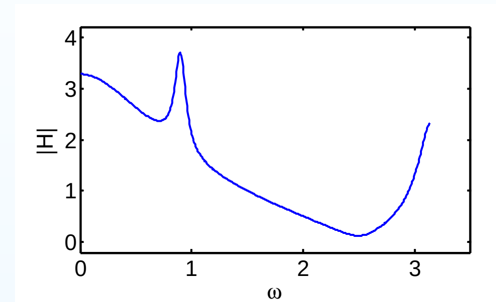
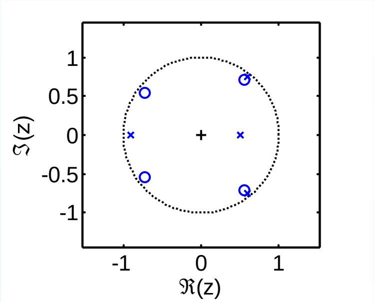
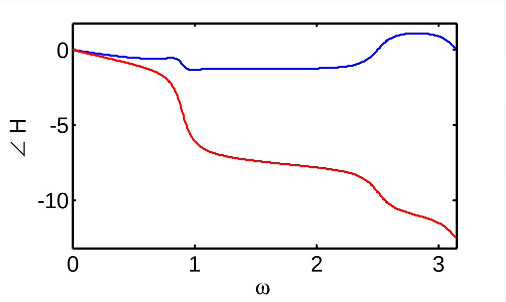
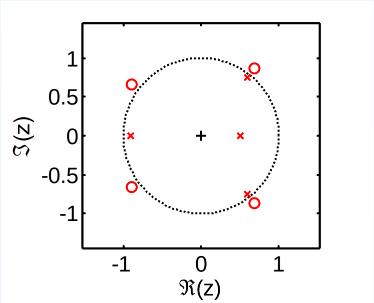
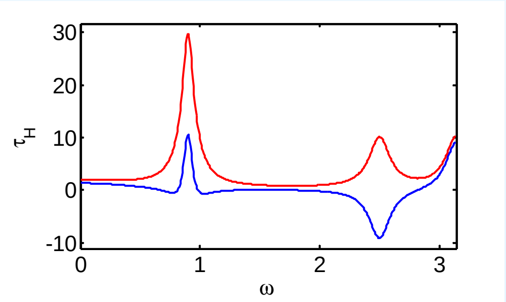


Reflecting an interior zero to the exterior multiplies $\left|H(e^{j\omega})\right|$ by a constant but increases average group delay by 1 sample.









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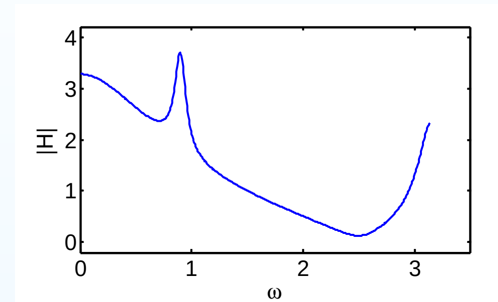

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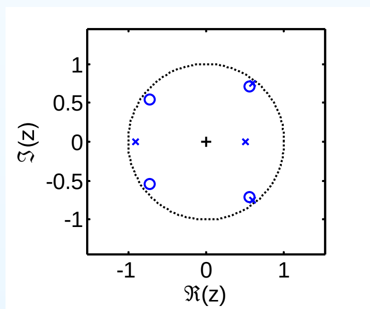

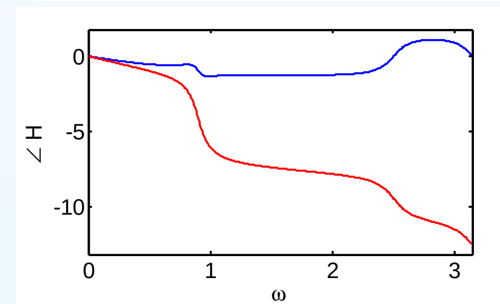

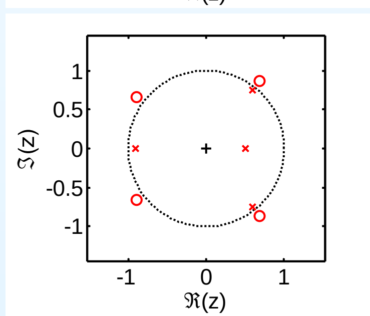

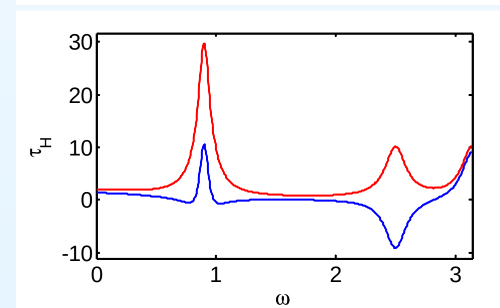

A filter with all zeros inside the unit circle is a minimum phase filter:

Minimum Phase

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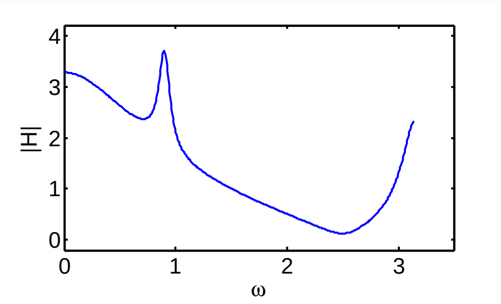
- zeros on the unit circle count –½



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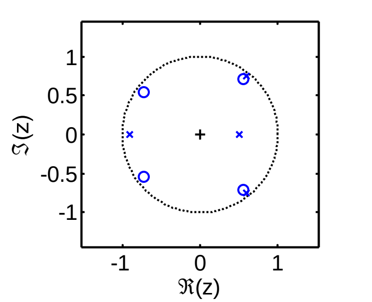

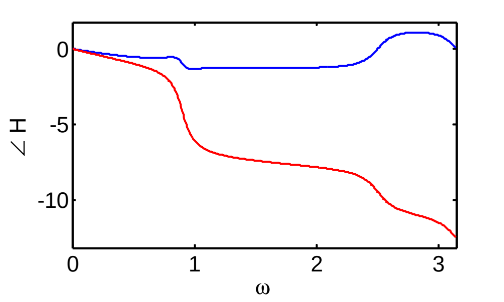

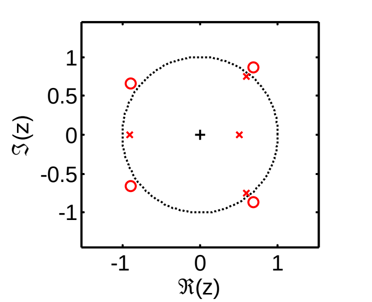

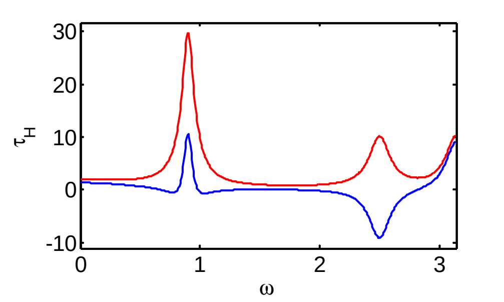

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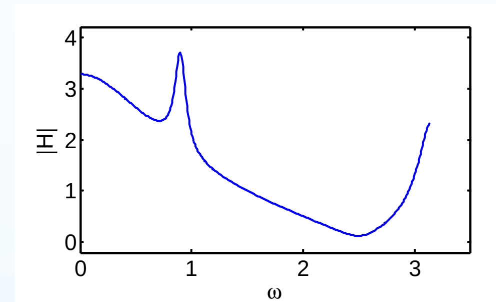

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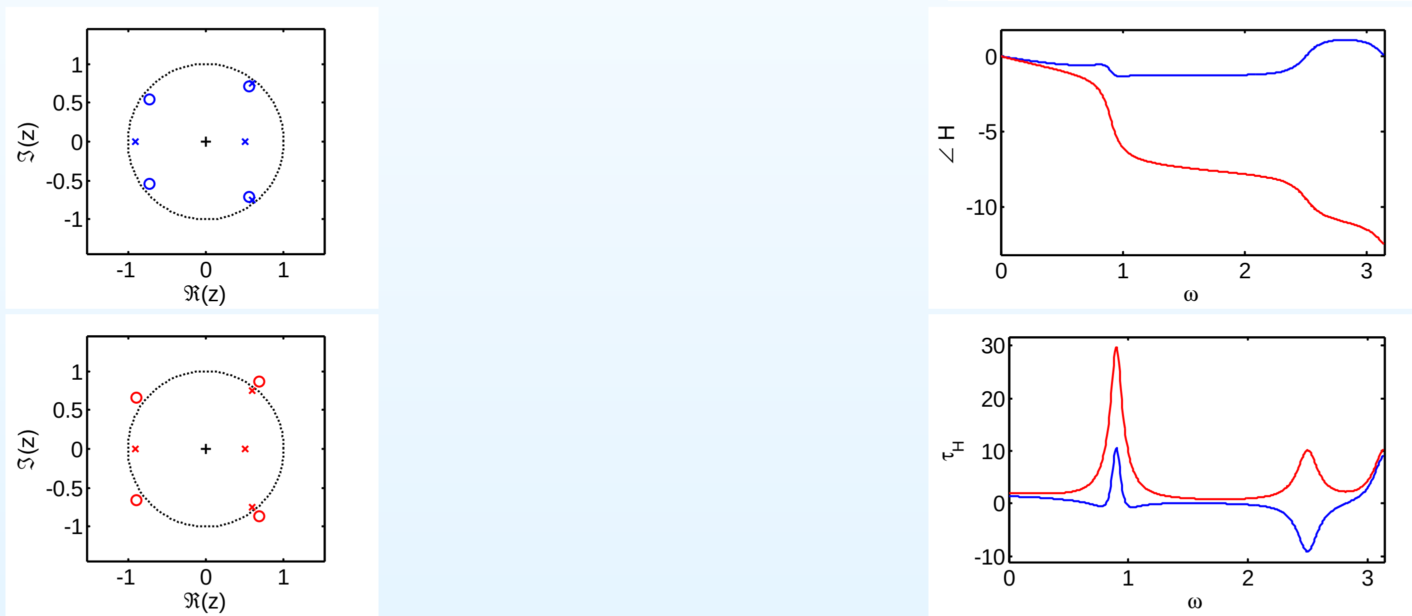

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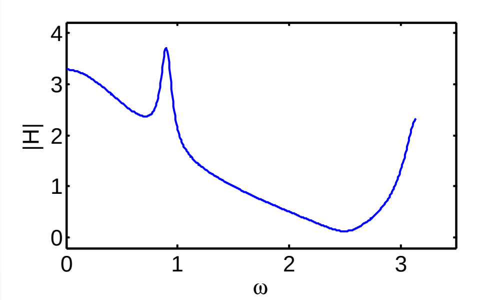

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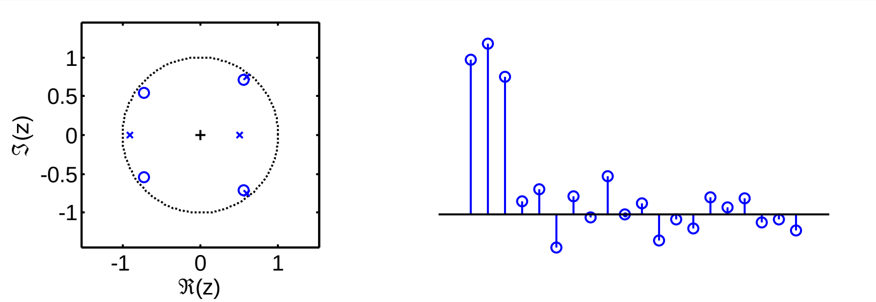

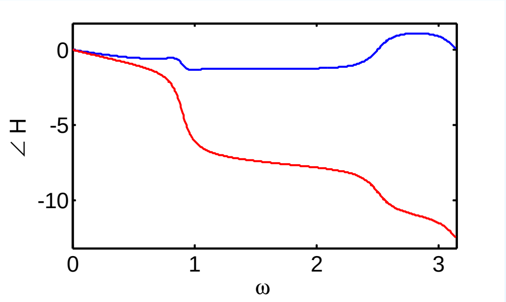

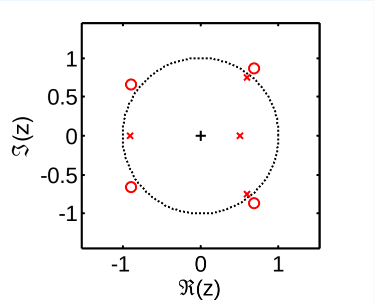

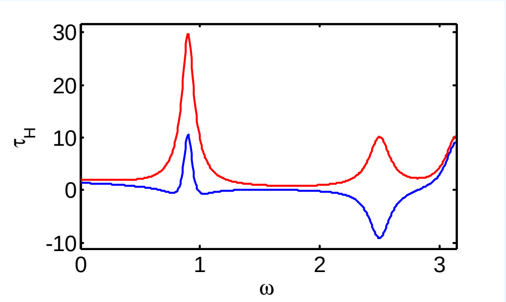

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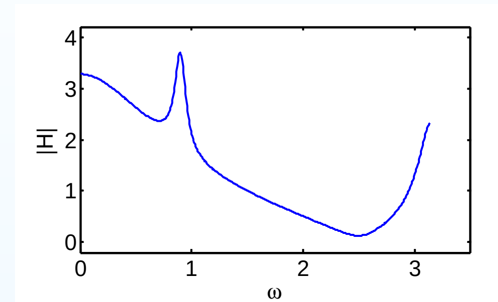

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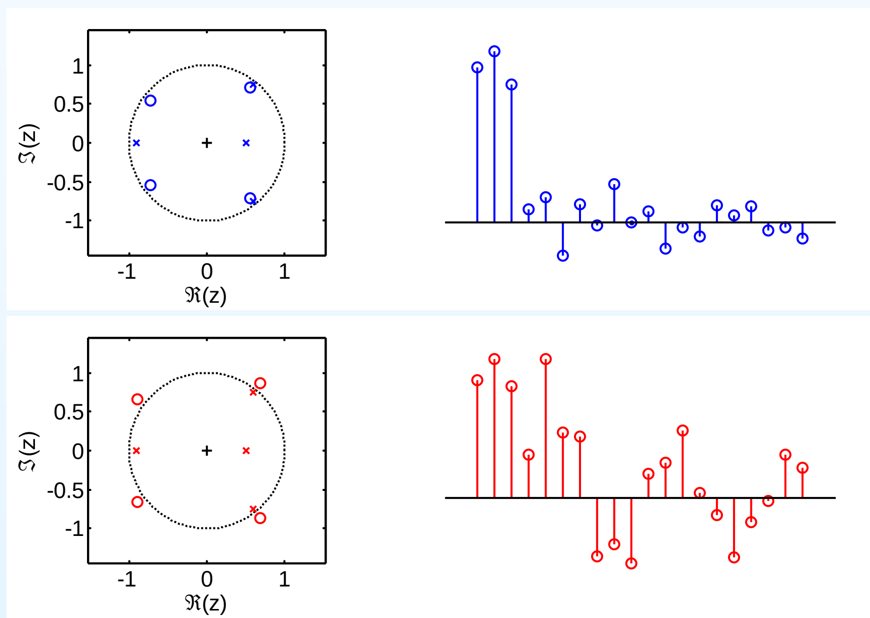

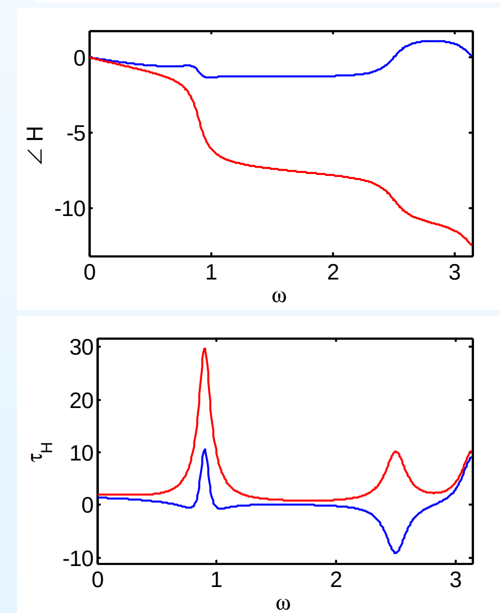

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For further details see Mitra: 6, 7.

MATLAB routines

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impz	Impulse response
residuez	partial fraction expansion
grpdelay	Group Delay
freqz	Calculate filter frequency response