7: Optimal FIR filters

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[Diagram of frequency response and error]
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**Minimax criterion:** $h[n] = \arg \min_{h[n]} \max_{\omega} |e(\omega)|$: minimize max error
Alternation Theorem

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A polynomial fit of degree $n$ to a set of bounded points is minimax if and only if it attains its maximal error at $n + 2$ points with alternating signs. There may be additional maximal error points. Fitting to a continuous function is the same as to an infinite number of points.
Chebyshev Polynomials

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\[ H(z) = 0.1766z^2 + 0.4015z + 0.2124 + 0.4015z^{-1} + 0.1766z^{-2} \]
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$$= P(\cos \omega)$$

where $P(x)$ is a polynomial of order $\frac{M}{2}$. 

![Graph of H(\omega) for M=18](image)
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Only three possibilities exist (try them all):

(a) \( \omega = 0 + \text{two band edges} + \text{all}\left(\frac{M}{2} - 1\right) \text{ zeros of } P'(x) \).
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(c) \( \omega = \{0 \text{ and } \pi\} + \text{two band edges} + \left( \frac{M}{2} - 2 \right) \text{ zeros of } P'(x). \)
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1. **Guess** the positions of the $\frac{M}{2} + 2$ maximal error frequencies and give alternating signs to the errors (e.g. choose evenly spaced $\omega$).
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3. **Find the local maxima** of the error function by evaluating $e(\omega) = s(\omega) \left( \overline{H}(\omega) - d(\omega) \right)$ on a dense set of $\omega$. 

![Diagram](image)
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3. **Find the local maxima** of the error function by evaluating $e(\omega) = s(\omega) \left( H(\omega) - d(\omega) \right)$ on a dense set of $\omega$.

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5. **Evaluate** $\overline{H}(\omega)$ on $M + 1$ evenly spaced $\omega$ and do an IDFT to get $h[n]$. 

![Graphs of Remez Exchange Algorithm iterations](image)
Remex Step 2: Determine Polynomial

For each extremal frequency, $\omega_i$ for $1 \leq i \leq \frac{M}{2} + 2$

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Reciprocal pairs give a linear phase shift
FIR Pros and Cons

- Can have **linear phase**
  - no envelope distortion, all frequencies have the same delay 😊
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- Efficient: computation $\propto M^2$
Summary

**Optimal Filters:** minimax error criterion

- use weight function, \( s(\omega) \), to allow different errors in different frequency bands
- symmetric filter has zeros on unit circle or in reciprocal pairs
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For further details see Mitra: 10.
## MATLAB routines

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