7: Optimal FIR filters

- Optimal Filters
- Alternation Theorem
- Chebyshev Polynomials
- Maximal Error Locations
- Remez Exchange Algorithm
- Determine Polynomial
- Example Design
- FIR Pros and Cons
- Summary
- MATLAB routines

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**Example:** lowpass filter

$$d(\omega) = \begin{cases} 1 & 0 \leq \omega \leq \omega_1 \\ 0 & \omega_2 \leq \omega \leq \pi \end{cases}$$

[Diagram of frequency response with $|H(e^{j\omega})|$ and $\omega_1$, $\omega_2$, and $\pi$ marked.]
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$e(\omega) = \pm 1$ when $\overline{H}(\omega)$ lies at the edge of the specification.

**Minimax criterion:** $h[n] = \arg \min_{h[n]} \max_{\omega} |e(\omega)|$: minimize max error
Want to find the best fit line: with the smallest maximal error.
Alternation Theorem

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Assume the first maximal deviation from the line is negative as shown.
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A polynomial fit of degree $n$ to a bounded set of points is minimax if and only if it attains its maximal error at $n + 2$ points with alternating signs.
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**Alternation Theorem:**

A polynomial fit of degree $n$ to a bounded set of points is minimax if and only if it attains its maximal error at $n + 2$ points with alternating signs. There may be additional maximal error points. Fitting to a continuous function is the same as to an infinite number of points.
Chebyshev Polynomials

\[ \overline{H}(\omega) = H(e^{j\omega}) = h[0] + 2 \sum_{1}^{M} h[n] \cos n\omega \]
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\[ \overline{H}(\omega) = H(e^{j\omega}) = h[0] + 2 \sum_{1}^{M} h[n] \cos n\omega \]

But \(\cos n\omega = T_n(\cos \omega)\): Chebyshev polynomial of 1st kind

\[ \cos 2\omega = 2 \cos^2 \omega - 1 = T_2(\cos \omega) \]
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\[ T_2(x) = 2x^2 - 1 \]
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Recurrence Relation:

\[ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \text{ with } T_0(x) = 1, T_1(x) = x \]
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Proof: \( \cos (n\omega + \omega) + \cos (n\omega - \omega) = 2 \cos \omega \cos n\omega \)
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So \( \overline{H}(\omega) \) is an \( \frac{M}{2} \) order polynomial in \( \cos \omega \): alternation theorem applies.
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So $\overline{H}(\omega)$ is an $\frac{M}{2}$ order polynomial in $\cos \omega$: alternation theorem applies.

Example: Symmetric lowpass filter of order $M = 4$

$$H(z) = 0.1766z^2 + 0.4015z + 0.2124 + 0.4015z^{-1} + 0.1766z^{-2}$$
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Maximal Error Locations

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\overline{H}(\omega) = h[0] + 2 \sum_{1}^{M/2} h[n] \cos n\omega = P(\cos \omega)
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where \( P(x) \) is a polynomial of order \( \frac{M}{2} \).
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where $P(x)$ is a polynomial of order $\frac{M}{2}$.

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$= 0$ at $\omega = 0, \pi$ and at most $\frac{M}{2} - 1$ zeros of polynomial $P'(x)$. 

\[ M=18 \]
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\[\therefore\] With two bands, we have at most \( \frac{M}{2} + 3 \) maximal error frequencies.
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\[ \therefore \text{ With two bands, we have at most } \frac{M}{2} + 3 \text{ maximal error frequencies.} \]

We require \( \frac{M}{2} + 2 \) of alternating signs for the optimal fit.
Maximal error locations occur either at band edges or when $\frac{dH}{d\omega} = 0$

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We require $\frac{M}{2} + 2$ of alternating signs for the optimal fit.

Only three possibilities exist (try them all):

(a) $\omega = 0 + \text{two band edges} + \text{all} \left(\frac{M}{2} - 1\right) \text{zeros of } P'(x)$.

(b) $\omega = \pi + \text{two band edges} + \text{all} \left(\frac{M}{2} - 1\right) \text{zeros of } P'(x)$. 
Maximal Error Locations

Maximal error locations occur either at band edges or when $\frac{dH}{d\omega} = 0$

$$H(\omega) = h[0] + 2 \sum_{1}^{M} h[n] \cos n\omega = P(\cos \omega)$$

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$$\frac{dH}{d\omega} = -P'(\cos \omega) \sin \omega$$

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$\therefore$ With two bands, we have at most $\frac{M}{2} + 3$ maximal error frequencies. We require $\frac{M}{2} + 2$ of alternating signs for the optimal fit.

Only three possibilities exist (try them all):

(a) $\omega = 0 + \text{two band edges} + \text{all} \left( \frac{M}{2} - 1 \right) \text{ zeros of } P'(x)$.
(b) $\omega = \pi + \text{two band edges} + \text{all} \left( \frac{M}{2} - 1 \right) \text{ zeros of } P'(x)$.
(c) $\omega = \{0 \text{ and } \pi \} + \text{two band edges} + \left( \frac{M}{2} - 2 \right) \text{ zeros of } P'(x)$. 
Remez Exchange Algorithm

1. **Guess** the positions of the $\frac{M}{2} + 2$ maximal error frequencies and give alternating signs to the errors (e.g. choose evenly spaced $\omega$).
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2. **Determine** the error magnitude, $\epsilon$, and the $\frac{M}{2} + 1$ coefficients of the polynomial that passes through the maximal error locations.
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3. **Find the local maxima** of the error function by evaluating $e(\omega) = s(\omega) \left( H(\omega) - d(\omega) \right)$ on a dense set of $\omega$. 

![Diagram](image-url)
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3. **Find the local maxima** of the error function by evaluating $e(\omega) = s(\omega) \left( \overline{H(\omega)} - d(\omega) \right)$ on a dense set of $\omega$.

4. **Update the maximal error frequencies** to be an alternating subset of the local maxima + band edges + $\{0 \text{ and/or } \pi\}$. 

![Graph showing iteration 1 for M = 4]
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   If maximum error is $> \epsilon$, go back to step 2.
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Remez Exchange Algorithm

1. **Guess** the positions of the \( M/2 + 2 \) maximal error frequencies and give alternating signs to the errors (e.g. choose evenly spaced \( \omega \)).

2. **Determine** the error magnitude, \( \epsilon \), and the \( M/2 + 1 \) coefficients of the polynomial that passes through the maximal error locations.

3. **Find the local maxima** of the error function by evaluating \( e(\omega) = s(\omega) (\overline{H}(\omega) - d(\omega)) \) on a dense set of \( \omega \).

4. **Update the maximal error frequencies** to be an alternating subset of the local maxima + band edges + \( \{0 \text{ and/or } \pi\} \).
   
   If maximum error is \( \geq \epsilon \), go back to step 2. (typically 15 iterations)
Remez Exchange Algorithm

1. **Guess** the positions of the $\frac{M}{2} + 2$ maximal error frequencies and give alternating signs to the errors (e.g. choose evenly spaced $\omega$).

2. **Determine** the error magnitude, $\epsilon$, and the $\frac{M}{2} + 1$ coefficients of the polynomial that passes through the maximal error locations.

3. **Find the local maxima** of the error function by evaluating $e(\omega) = s(\omega) \left( H(\omega) - d(\omega) \right)$ on a dense set of $\omega$.

4. **Update the maximal error frequencies** to be an alternating subset of the local maxima + band edges + $\{0 \text{ and/or } \pi\}$.
   
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5. **Evaluate** $\overline{H}(\omega)$ on $M + 1$ evenly spaced $\omega$ and do an IDFT to get $h[n]$. 
For each extremal frequency, \( \omega_i \) for \( 1 \leq i \leq \frac{M}{2} + 2 \)

\[ d(\omega_i) = \overline{H}(\omega_i) + \frac{(-1)^i \epsilon}{s(\omega_i)} \]
For each extremal frequency, $\omega_i$ for $1 \leq i \leq \frac{M}{2} + 2$

$$d(\omega_i) = \overline{H}(\omega_i) + \frac{(-1)^i \epsilon}{s(\omega_i)} = h[0] + 2 \sum_{n=1}^{\frac{M}{2}} h[n] \cos n\omega_i + \frac{(-1)^i \epsilon}{s(\omega_i)}$$
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Method 2: Don’t calculate $h[n]$ explicitly

Multiply equations by $c_i = \prod_{j \neq i} \frac{1}{\cos \omega_i - \cos \omega_j}$ and add:

$$\sum_{i=1}^{\frac{M}{2} + 2} c_i \left( h[0] + 2 \sum_{n=1}^{\frac{M}{2}} h[n] \cos n\omega + \frac{(-1)^i \epsilon}{s(\omega_i)} \right) = \sum_{i=1}^{\frac{M}{2} + 2} c_i d(\omega_i)$$
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$\left(\frac{M}{2} + 1\right)$-polynomial going through all the $\overline{H}(\omega_i)$ [actually order $\frac{M}{2}$]
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Bandpass $\omega = [0.5, 1],\ M=36$
Example Design

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![Filter Design Graph](image)
Example Design

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Reciprocal pairs give a linear phase shift
FIR Pros and Cons

- Can have **linear phase**
  - no envelope distortion, all frequencies have the same delay 😊
  - symmetric or antisymmetric: $h[n] = h[-n] \forall n$ or $-h[-n] \forall n$
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    \( \propto f_s^2 \) for a given specification in unscaled \( \Omega \) units.
Summary

Optimal Filters: minimax error criterion
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- very robust, works for filters with \( M > 1000 \)
- Efficient: computation \( \propto M^2 \)
- can go mad in the transition regions
Summary

Optimal Filters: minimax error criterion

- use weight function, $s(\omega)$, to allow different errors in different frequency bands
- symmetric filter has zeros on unit circle or in reciprocal pairs
- Response of symmetric filter is a polynomial in $\cos \omega$
- Alternation Theorem: $M_2 + 2$ maximal errors with alternating signs

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For further details see Mitra: 10.
## MATLAB routines

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>firpm</td>
<td>optimal FIR filter design</td>
</tr>
<tr>
<td>firpmord</td>
<td>estimate require order for firpm</td>
</tr>
<tr>
<td>cfirpm</td>
<td>arbitrary-response filter design</td>
</tr>
<tr>
<td>remez</td>
<td>[obsolete] optimal FIR filter design</td>
</tr>
</tbody>
</table>