7: Optimal FIR filters
Optimal Filters

We restrict ourselves to zero-phase filters of odd length \( M + 1 \), symmetric around \( h[0] \), i.e. \( h[-n] = h[n] \).

\[
\overline{H}(\omega) = H(e^{j\omega}) = \sum_{-\frac{M}{2}}^{\frac{M}{2}} h[n] e^{-jn\omega} = h[0] + 2 \sum_{1}^{\frac{M}{2}} h[n] \cos n\omega
\]

\( \overline{H}(\omega) \) is real but not necessarily positive (unlike \(|H(e^{j\omega})|\)).

**Weighted error:** \( e(\omega) = s(\omega) (\overline{H}(\omega) - d(\omega)) \) where \( d(\omega) \) is the target.

Choose \( s(\omega) \) to control the error variation with \( \omega \).

**Example:** lowpass filter

\[
d(\omega) = \begin{cases} 
1 & 0 \leq \omega \leq \omega_1 \\
0 & \omega_2 \leq \omega \leq \pi 
\end{cases}

s(\omega) = \begin{cases} 
\delta^{-1} & 0 \leq \omega \leq \omega_1 \\
\epsilon^{-1} & \omega_2 \leq \omega \leq \pi 
\end{cases}
\]

\( e(\omega) = \pm 1 \) when \( \overline{H}(\omega) \) lies at the edge of the specification.

**Minimax criterion:** \( h[n] = \arg \min h[n] \max \omega |e(\omega)| \): minimize max error
Alternation Theorem

Want to find the best fit line: with the smallest maximal error.

Best fit line always attains the maximal error three times with alternate signs

Proof:
Assume the first maximal deviation from the line is negative as shown. There must be an equally large positive deviation; or else just move the line downwards to reduce the maximal deviation. This must be followed by another maximal negative deviation; or else you can rotate the line and reduce the deviations.

Alternation Theorem:
A polynomial fit of degree \( n \) to a set of bounded points is minimax if and only if it attains its maximal error at \( n + 2 \) points with alternating signs. There may be additional maximal error points. Fitting to a continuous function is the same as to an infinite number of points.
Chebyshev Polynomials

\[ \overline{H}(\omega) = H(e^{j\omega}) = h[0] + 2 \sum_{1}^{M} h[n] \cos n\omega \]

But \( \cos n\omega = T_n(\cos \omega) \): Chebyshev polynomial of 1st kind

\[
\begin{align*}
\cos 2\omega &= 2\cos^2 \omega - 1 = T_2(\cos \omega) \\
\cos 3\omega &= 4\cos^3 \omega - 3\cos \omega = T_3(\cos \omega)
\end{align*}
\]

Recurrence Relation:

\[ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \text{ with } T_0(x) = 1, \; T_1(x) = x \]

Proof: \( \cos (n\omega + \omega) + \cos (n\omega - \omega) = 2 \cos \omega \cos n\omega \)

So \( \overline{H}(\omega) \) is an \( \frac{M}{2} \) order polynomial in \( \cos \omega \): alternation theorem applies.

Example: Symmetric lowpass filter of order \( M = 4 \)

\[ H(z) = 0.1766z^2 + 0.4015z + 0.2124 + 0.4015z^{-1} + 0.1766z^{-2} \]
Maximal Error Locations

Maximal error locations occur either at band edges or when \( \frac{dH}{d\omega} = 0 \)

\[
\overline{H}(\omega) = h[0] + 2 \sum_{n=1}^{M/2} h[n] \cos n\omega = P(\cos \omega)
\]

where \( P(x) \) is a polynomial of order \( \frac{M}{2} \).

\[
\frac{d\overline{H}}{d\omega} = -P'(\cos \omega) \sin \omega = 0 \text{ at } \omega = 0, \pi \text{ and at most } \frac{M}{2} - 1 \text{ zeros of polynomial } P'(x).
\]

∴ With two bands, we have at most \( \frac{M}{2} + 3 \) maximal error frequencies. We require \( \frac{M}{2} + 2 \) of alternating signs for the optimal fit.

Only three possibilities exist (try them all):

(a) \( \omega = 0 + \text{two band edges} + \text{all}(\frac{M}{2} - 1) \text{ zeros of } P'(x) \).
(b) \( \omega = \pi + \text{two band edges} + \text{all}(\frac{M}{2} - 1) \text{ zeros of } P'(x) \).
(c) \( \omega = \{0 \text{ and } \pi\} + \text{two band edges} + (\frac{M}{2} - 2) \text{ zeros of } P'(x) \).
Remez Exchange Algorithm

1. **Guess** the positions of the $\frac{M}{2} + 2$ maximal error frequencies and give alternating signs to the errors (e.g. choose evenly spaced $\omega$).

2. **Determine** the error magnitude, $\epsilon$, and the $\frac{M}{2} + 1$ coefficients of the polynomial that passes through the maximal error locations.

3. **Find the local maxima** of the error function by evaluating $e(\omega) = s(\omega) (\overline{H}(\omega) - d(\omega))$ on a dense set of $\omega$.

4. **Update the maximal error frequencies** to be an alternating subset of the local maxima + band edges + \{0 and/or $\pi$\}. 
   - If maximum error is $> \epsilon$, go back to step 2. (typically 15 iterations)

5. **Evaluate** $\overline{H}(\omega)$ on $M + 1$ evenly spaced $\omega$ and do an **IDFT** to get $h[n]$. 

![Graphs showing iterations of Remez Exchange Algorithm](image-url)
For each extremal frequency, $\omega_i$ for $1 \leq i \leq \frac{M}{2} + 2$

$$d(\omega_i) = \overline{H}(\omega_i) + \frac{(-1)^i \epsilon}{s(\omega_i)} = h[0] + 2 \sum_{n=1}^{\frac{M}{2}} h[n] \cos n\omega_i + \frac{(-1)^i \epsilon}{s(\omega_i)}$$

**Method 1:** (Computation time $\propto M^3$)

Solve $\frac{M}{2} + 2$ equations in $\frac{M}{2} + 2$ unknowns for $h[n] + \epsilon$.

In step 3, evaluate $\overline{H}(\omega) = h[0] + 2 \sum_{n=1}^{\frac{M}{2}} h[n] \cos n\omega_i$

**Method 2:** Don’t calculate $h[n]$ explicitly (Computation time $\propto M^2$)

Multiply the $\omega_i$ equation by $c_i = \prod_{j \neq i} \frac{1}{\cos \omega_i - \cos \omega_j}$ and add them:

$$\sum_{i=1}^{\frac{M}{2}+2} c_i \left( h[0] + 2 \sum_{n=1}^{\frac{M}{2}} h[n] \cos n\omega + \frac{(-1)^i \epsilon}{s(\omega_i)} \right) = \sum_{i=1}^{\frac{M}{2}+2} c_i d(\omega_i)$$

All terms involving $h[n]$ sum to zero leaving

$$\sum_{i=1}^{\frac{M}{2}+2} \frac{(-1)^i c_i}{s(\omega_i)} \epsilon = \sum_{i=1}^{\frac{M}{2}+2} c_i d(\omega_i)$$

Solve for $\epsilon$ then calculate the $\overline{H}(\omega_i)$ then use Lagrange interpolation:

$$\overline{H}(\omega) = P(\cos \omega) = \sum_{i=1}^{\frac{M}{2}+2} \overline{H}(\omega_i) \prod_{j \neq i} \frac{\cos \omega - \cos \omega_j}{\cos \omega_i - \cos \omega_j}$$

$(\frac{M}{2} + 1)$-polynomial going through all the $\overline{H}(\omega_i)$ [actually order $\frac{M}{2}$]
Filter Specifications:

- Bandpass $\omega = [0.5, 1]$, Transition widths: $\Delta \omega = 0.2$
- Stopband Attenuation: $-25$ dB and $-15$ dB
- Passband Ripple: $\pm 0.3$ dB

Determine gain tolerances for each band:

$-25$ dB = 0.056, $-0.3$ dB = 1 − 0.034, $-15$ dB = 0.178

Predicted order: $M = 36$

$\frac{M}{2} + 2$ extremal frequencies are distributed between the bands

Filter meets specs 😊; clearer on a decibel scale

Most zeros are on the unit circle + three reciprocal pairs

Reciprocal pairs give a linear phase shift
FIR Pros and Cons

- Can have linear phase
  - no envelope distortion, all frequencies have the same delay 😊
  - symmetric or antisymmetric: \( h[n] = h[−n] \forall n \) or \( −h[−n] \forall n \)
  - antisymmetric filters have \( H(e^{j0}) = H(e^{j\pi}) = 0 \)
  - symmetry means you only need \( \frac{M}{2} + 1 \) multiplications to implement the filter.

- Always stable 😊

- Low coefficient sensitivity 😊

- Optimal design method fast and robust 😊

- Normally needs higher order than an IIR filter 😊
  - Filter order \( M \approx \frac{\text{dBatten}}{3.5\Delta\omega} \) where \( \Delta\omega \) is the most rapid transition
  - Filtering complexity \( \propto M \times f_s \approx \frac{\text{dBatten}}{3.5\Delta\omega} f_s = \frac{\text{dBatten}}{3.5\Omega} f_s^2 \)
  - \( \propto f_s^2 \) for a given specification in unscaled \( \Omega \) units.
**Summary**

**Optimal Filters:** minimax error criterion

- use weight function, \( s(\omega) \), to allow different errors in different frequency bands
- symmetric filter has zeros on unit circle or in reciprocal pairs
- Response of symmetric filter is a polynomial in \( \cos \omega \)
- Alternation Theorem: \( \frac{M}{2} + 2 \) maximal errors with alternating signs

**Remez Exchange Algorithm** (also known as Parks-McLelllan Algorithm)

- multiple constant-gain bands separated by transition regions
- very robust, works for filters with \( M > 1000 \)
- Efficient: computation \( \propto M^2 \)
- can go mad in the transition regions

Modified version works on arbitrary gain function

- Does not always converge

For further details see Mitra: 10.
### MATLAB routines

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