8: IIR Filter Transformations

Continuous Time Filters

Classical continuous-time filters optimize tradeoff:
passband ripple v stopband ripple v transition width

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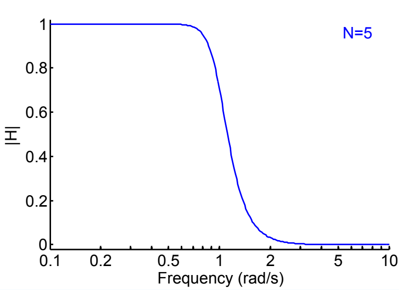

Butterworth: $\widetilde{G}^2(\Omega) = \left|\widetilde{H}(j\Omega)\right|^2 = \frac{1}{1+\Omega^{2N}}$

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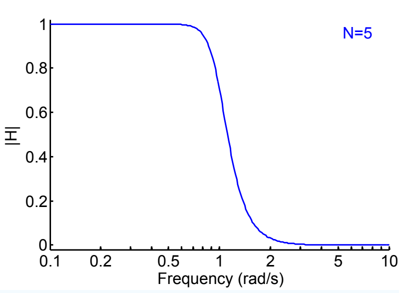

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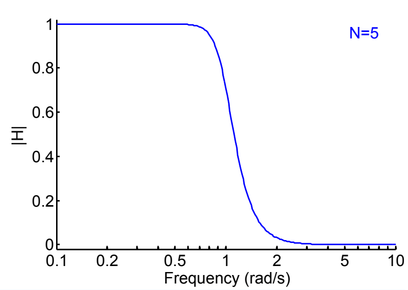

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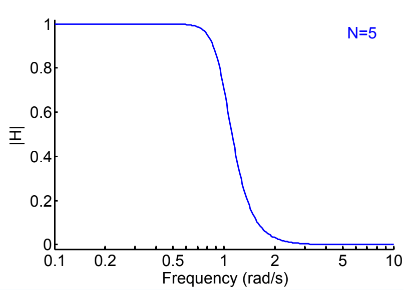

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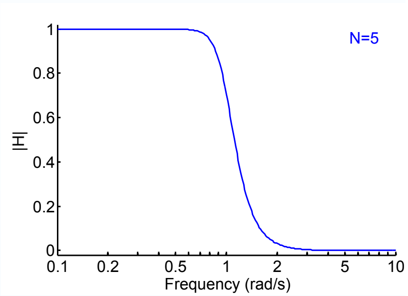

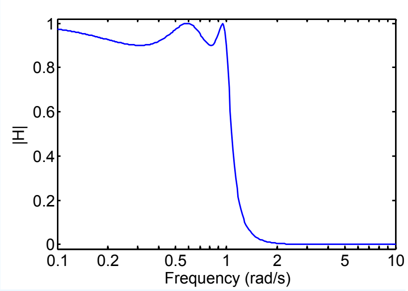

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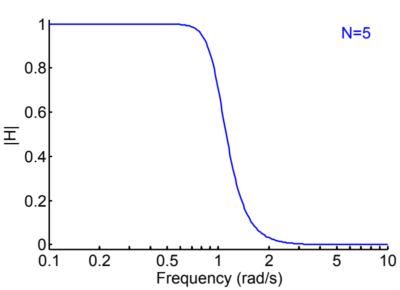

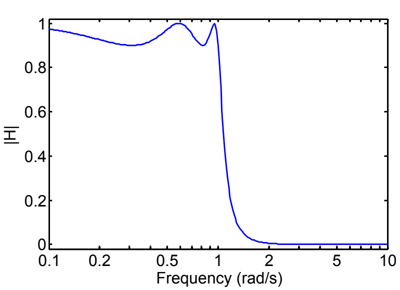

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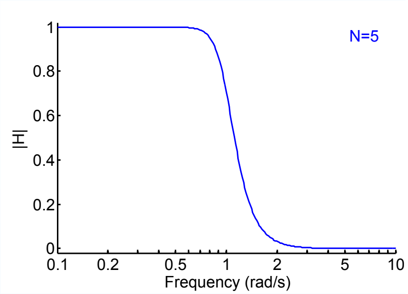

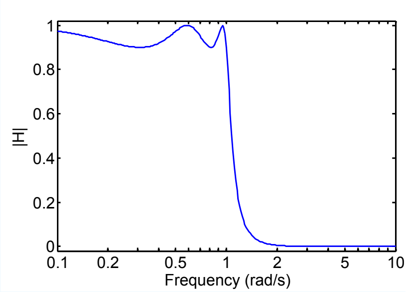

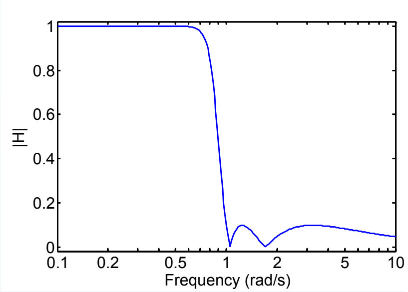

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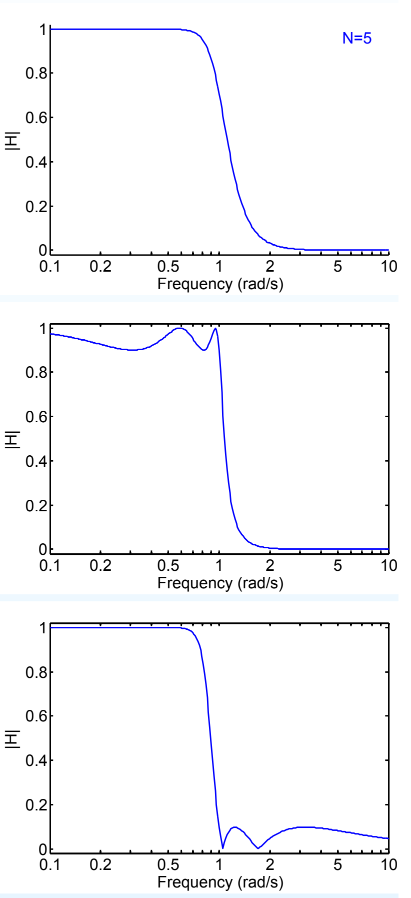

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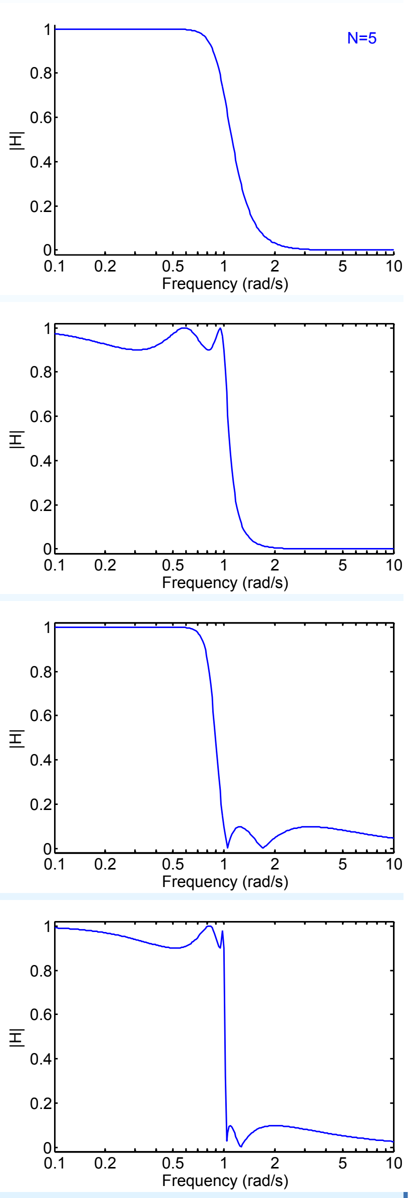
Elliptic: [no nice formula]



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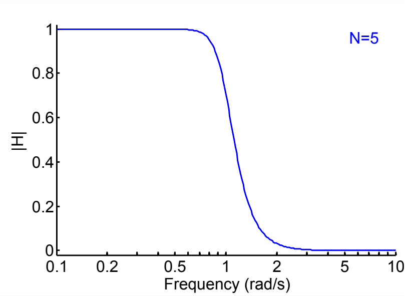

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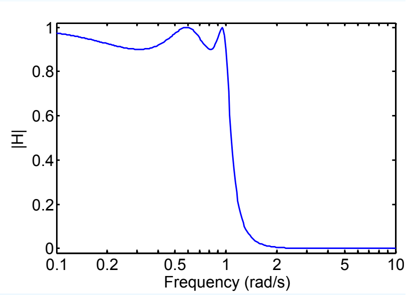

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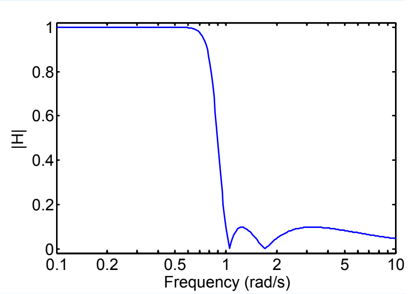

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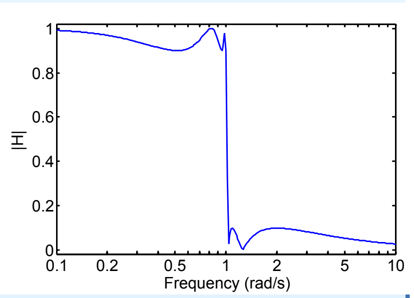

Elliptic: [no nice formula]

- Very steep + equiripple in pass and stop bands

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There are explicit formulae for pole/zero positions.

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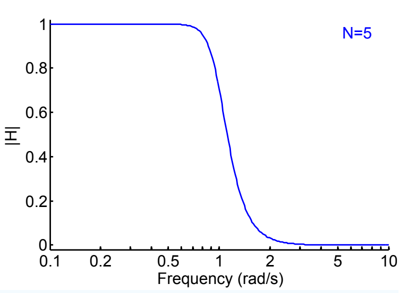

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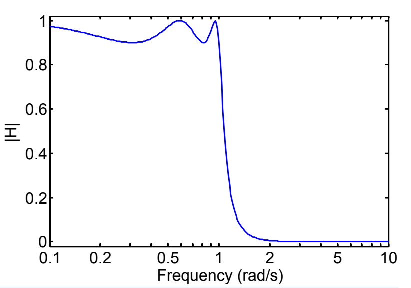

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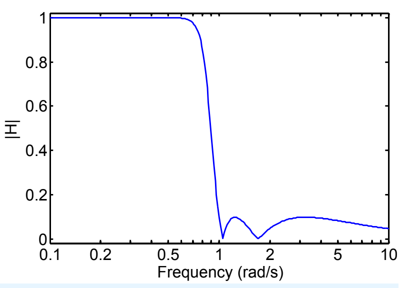

- stopband equiripple + very flat at 0

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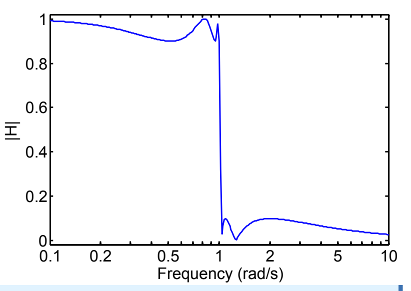

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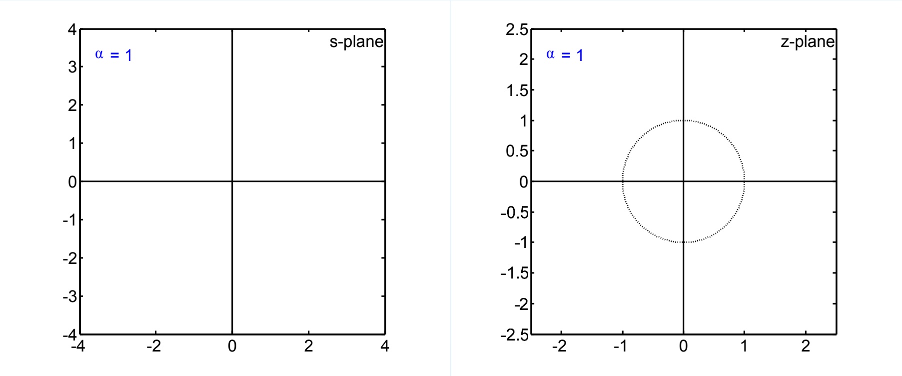
Bilinear Mapping

Change variable: $z = \frac{\alpha + s}{\alpha - s}$



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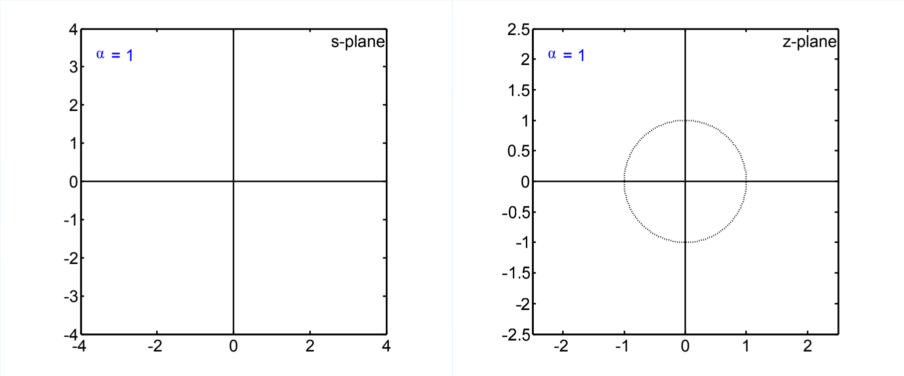

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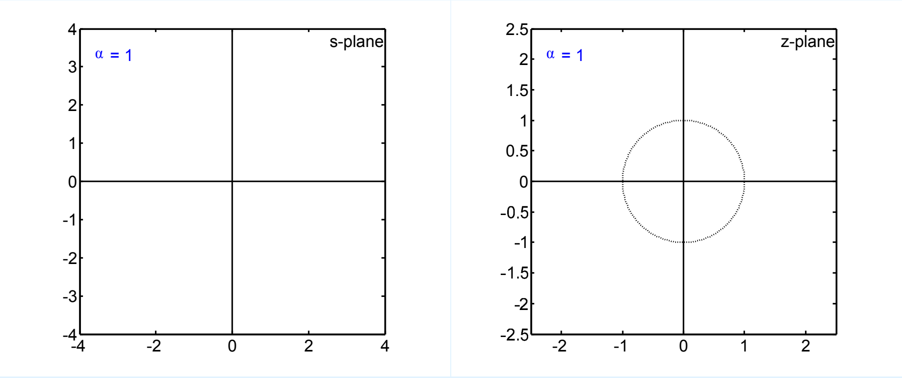
Change variable: $z = \frac{\alpha + s}{\alpha - s} \Leftrightarrow s = \alpha \frac{z-1}{z+1}$: a one-to-one invertible mapping



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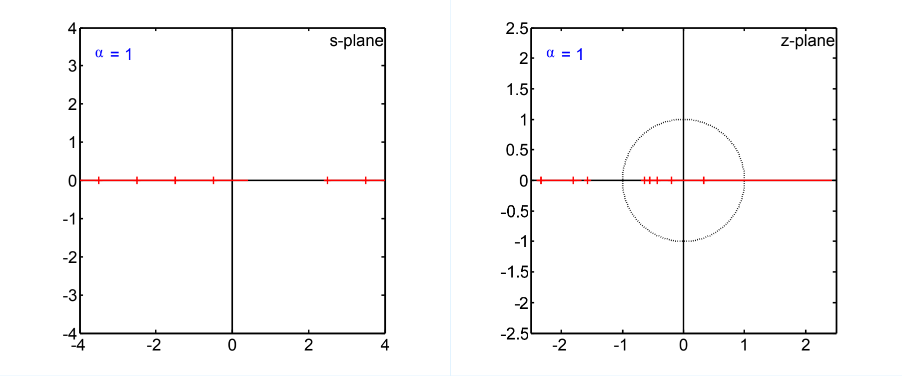

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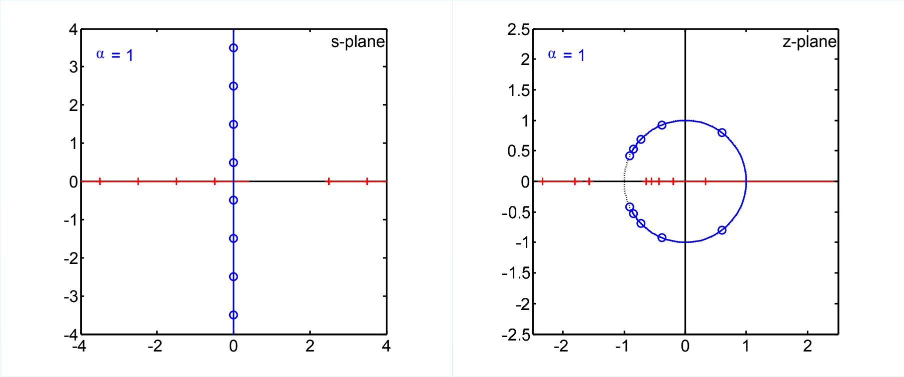

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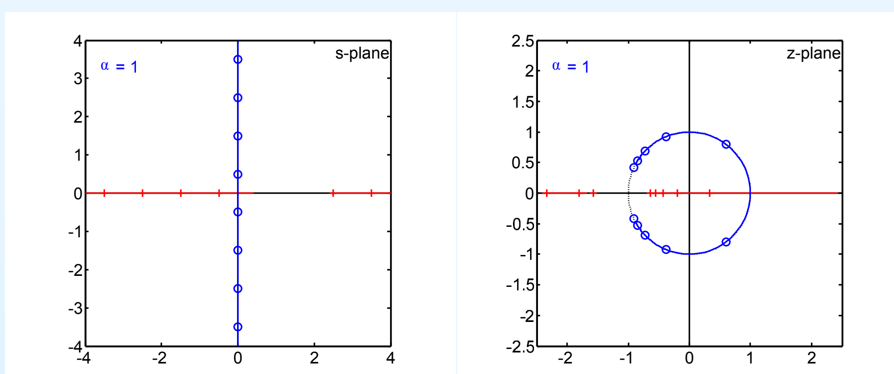
  Proof: $z = e^{j\omega} \Leftrightarrow s = \alpha\frac{e^{j\omega}-1}{e^{j\omega}+1}$



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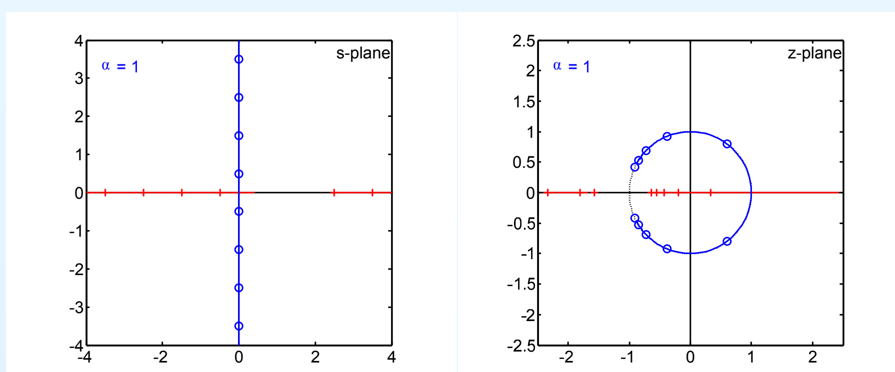

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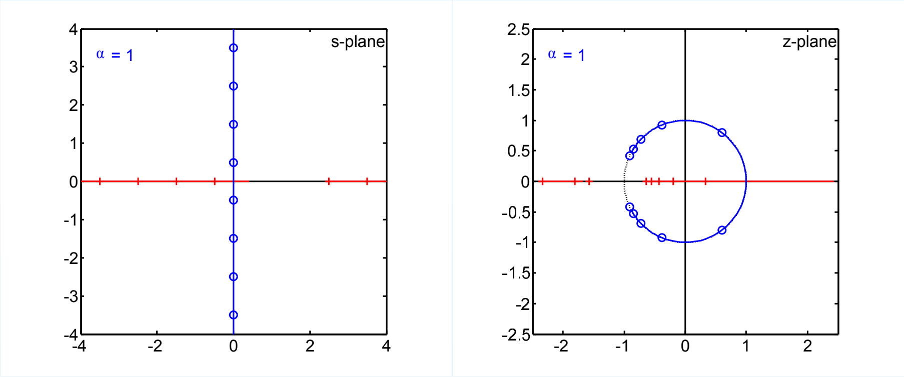

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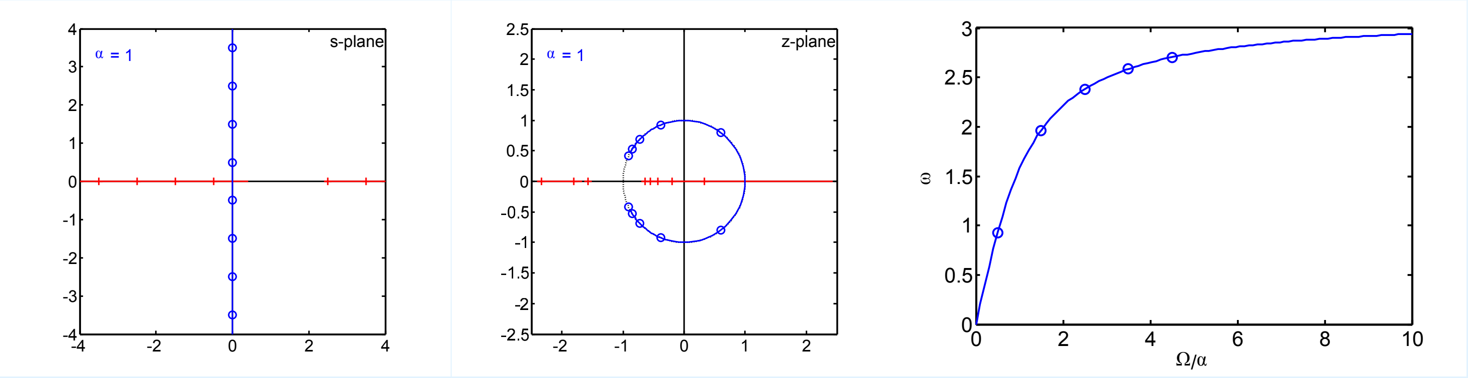

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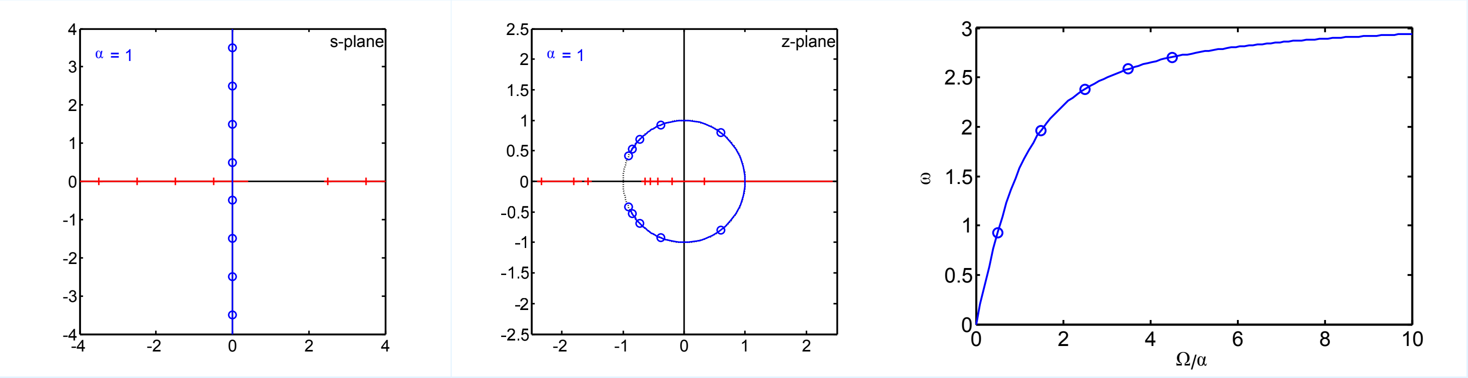
- Left half plane(s) $\leftrightarrow$ inside of unit circle (z)



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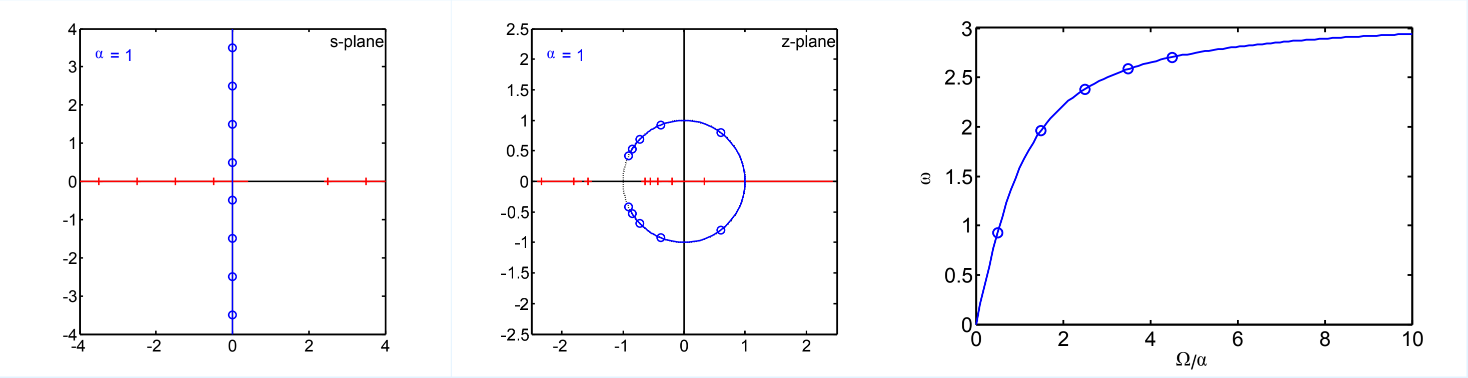

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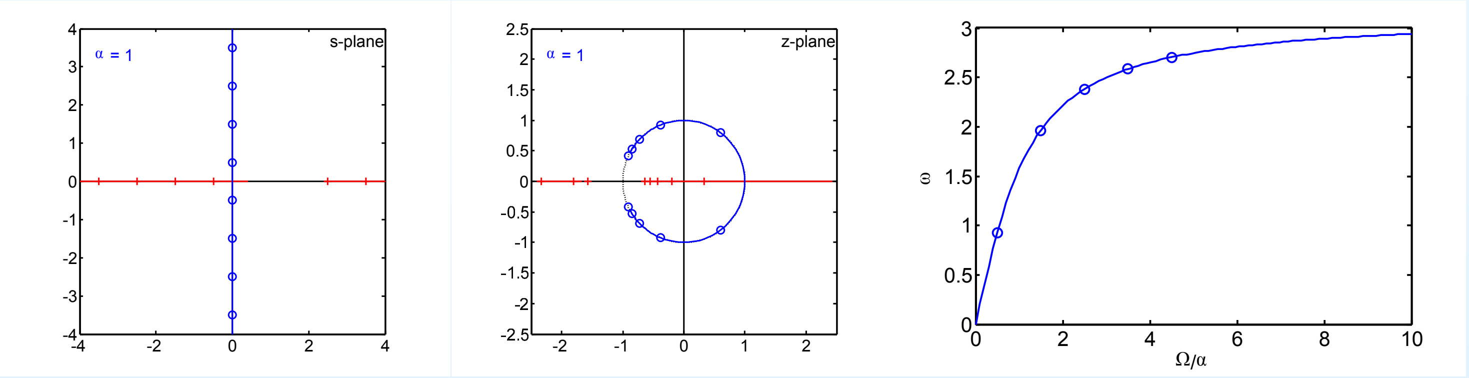

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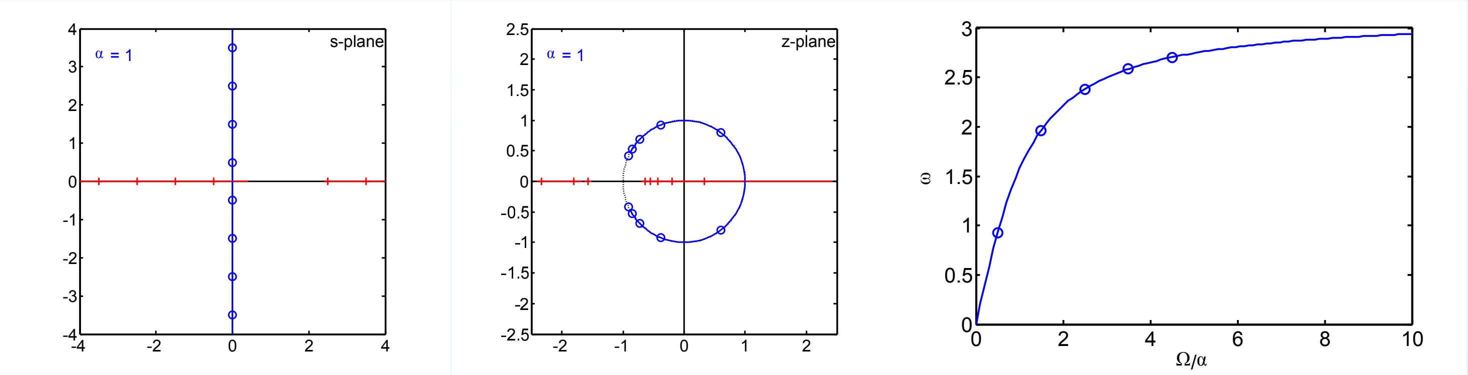

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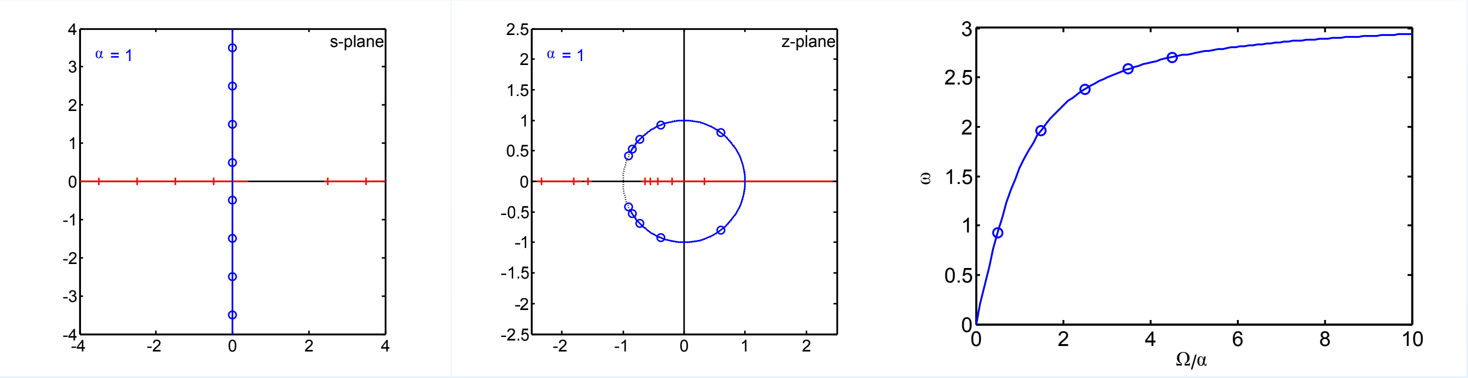
  $$= \frac{\alpha^2+2\alpha x+x^2+y^2}{\alpha^2-2\alpha x+x^2+y^2} = 1 + \frac{4\alpha x}{(\alpha-x)^2+y^2}$$

  $$x < 0 \Leftrightarrow |z| < 1$$



Bilinear Mapping

Change variable: $z = \frac{\alpha+s}{\alpha-s} \Leftrightarrow s = \alpha\frac{z-1}{z+1}$: a one-to-one invertible mapping

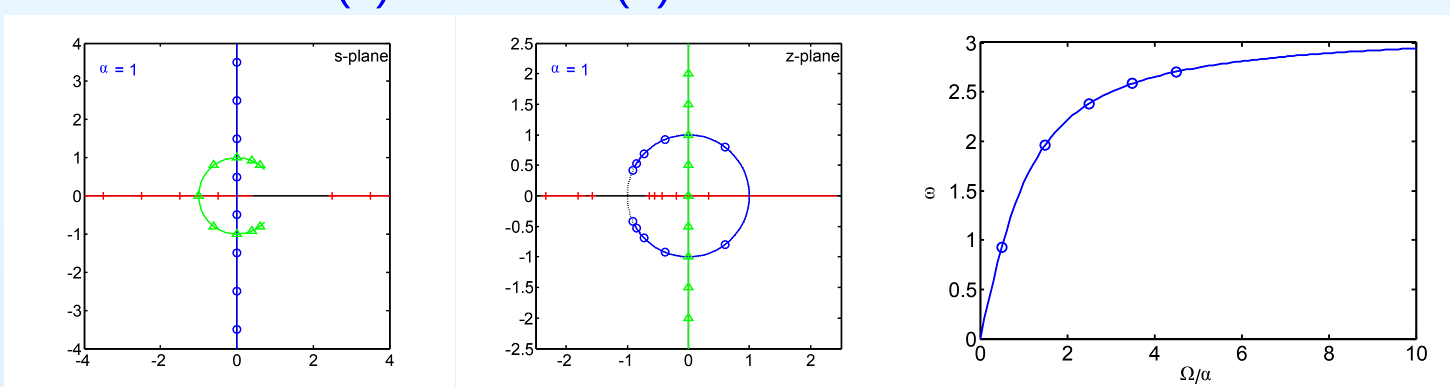
- $\Re$ axis $(s) \leftrightarrow \Re$ axis (z)
- $\Im$ axis $(s) \leftrightarrow$ Unit circle (z)

  Proof: $z = e^{j\omega} \Leftrightarrow s = \alpha\frac{e^{j\omega}-1}{e^{j\omega}+1} = \alpha\frac{e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}}+e^{-j\frac{\omega}{2}}} = j\alpha\tan\frac{\omega}{2} = j\Omega$
- Left half plane$(s) \leftrightarrow$inside of unit circle (z)

  Proof: $s = x + jy \Leftrightarrow |z|^2 = \frac{|(\alpha+x)+jy|^2}{|(\alpha-x)-jy|^2}$

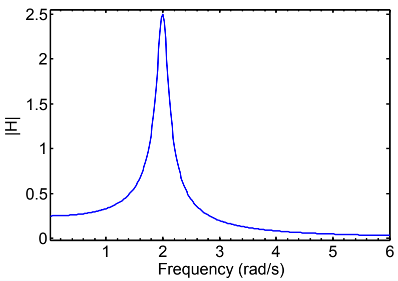
  $$= \frac{\alpha^2+2\alpha x+x^2+y^2}{\alpha^2-2\alpha x+x^2+y^2} = 1 + \frac{4\alpha x}{(\alpha-x)^2+y^2}$$

  $$x < 0 \Leftrightarrow |z| < 1$$
- Unit circle $(s) \leftrightarrow \Im$ axis (z)



Continuous Time Filters

Take $\widetilde{H}(s) = \frac{1}{s^2+0.2s+4}$ and choose $\alpha = 1$



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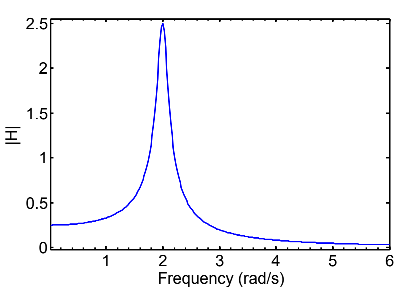

Substitute: $s = \alpha \frac{z-1}{z+1}$

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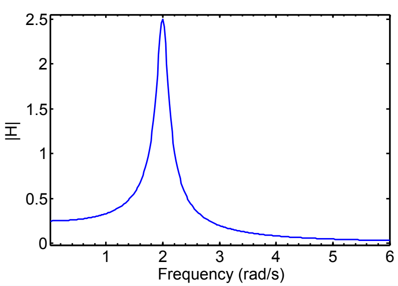

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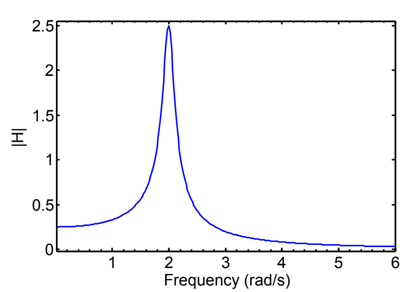

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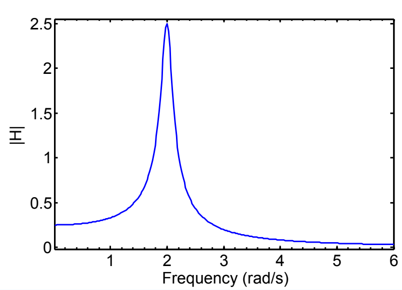

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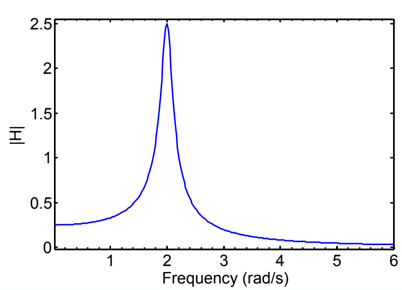

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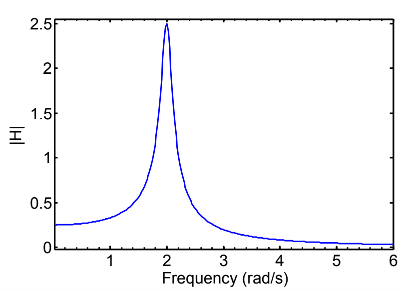

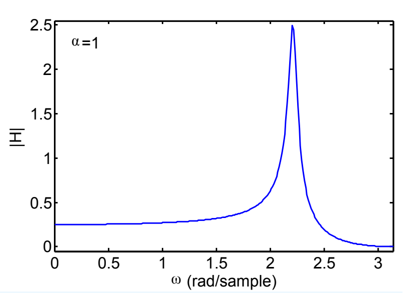

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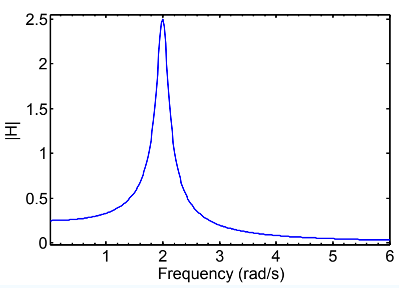

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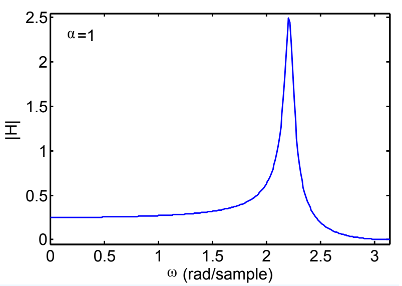

Frequency response is identical (both magnitude and phase) but with a distorted frequency axis:

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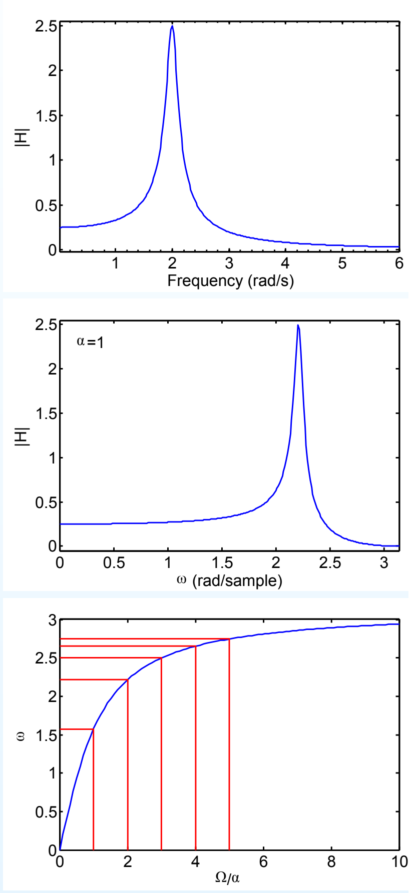

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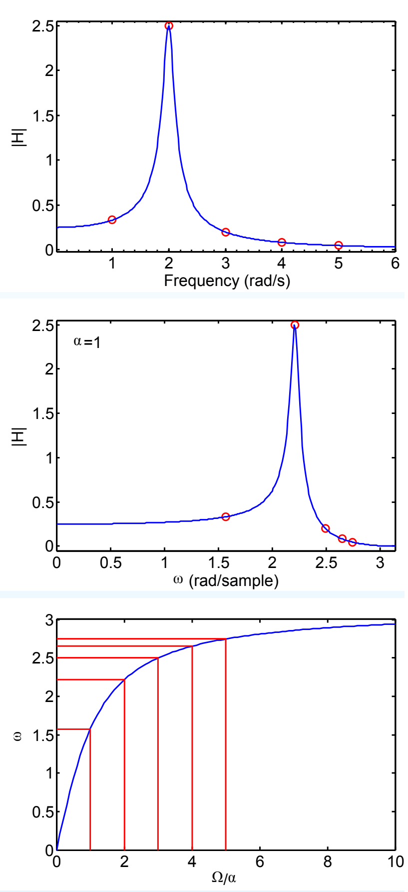
$$\Omega = \left[\begin{array}{ccccc} \alpha & 2\alpha & 3\alpha & 4\alpha & 5\alpha \end{array}\right]$$
$$\rightarrow \omega = \left[\begin{array}{ccccc} 1.6 & 2.2 & 2.5 & 2.65 & 2.75 \end{array}\right]$$



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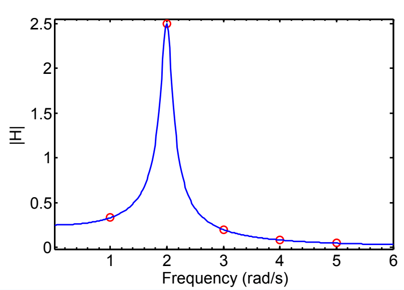
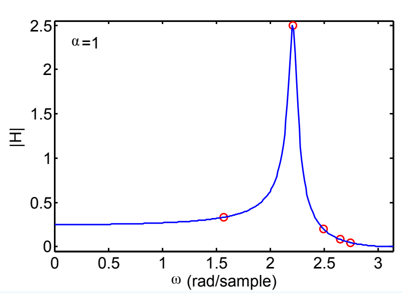
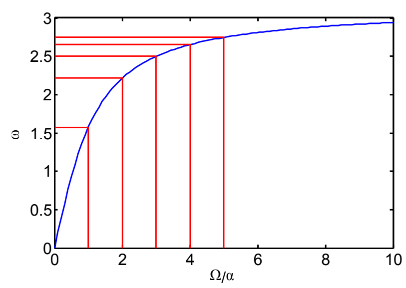
Choosing α: Set $\alpha = \frac{\Omega_0}{\tan\frac{1}{2}\omega_0}$ to map $\Omega_0 \rightarrow \omega_0$







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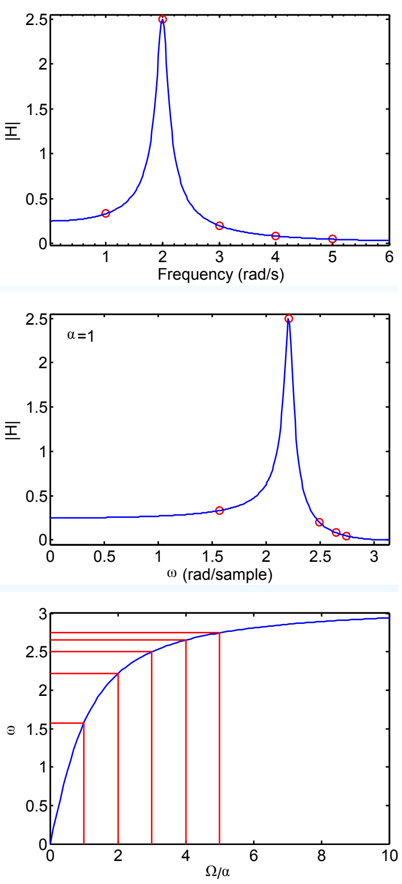

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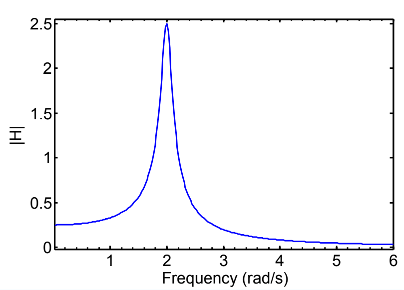
Set $\alpha = 2f_s = \frac{2}{T}$ to map low frequencies to themselves

Mapping Poles and Zeros

Alternative method: $\widetilde{H}(s) = \frac{1}{s^2+0.2s+4}$



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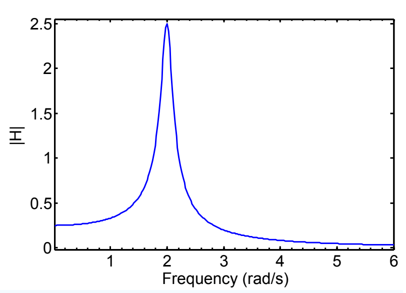
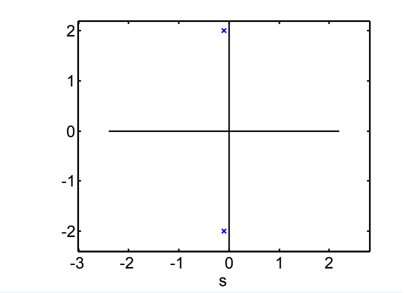
Find the poles and zeros: $p_s = -0.1 \pm 2j$





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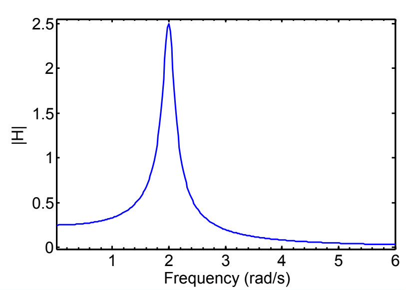

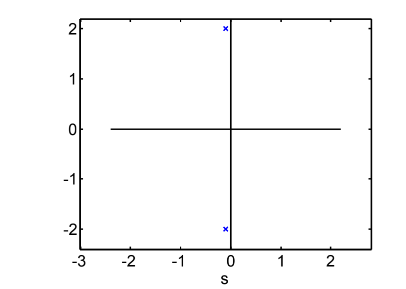

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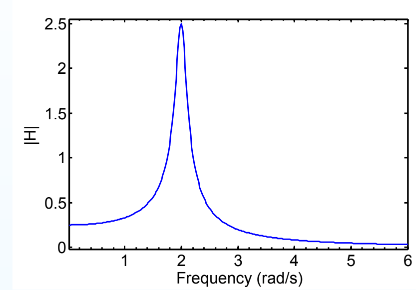

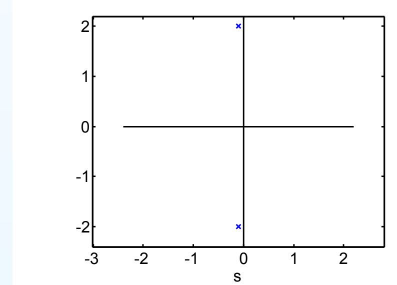

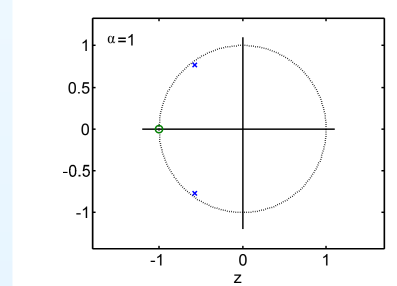

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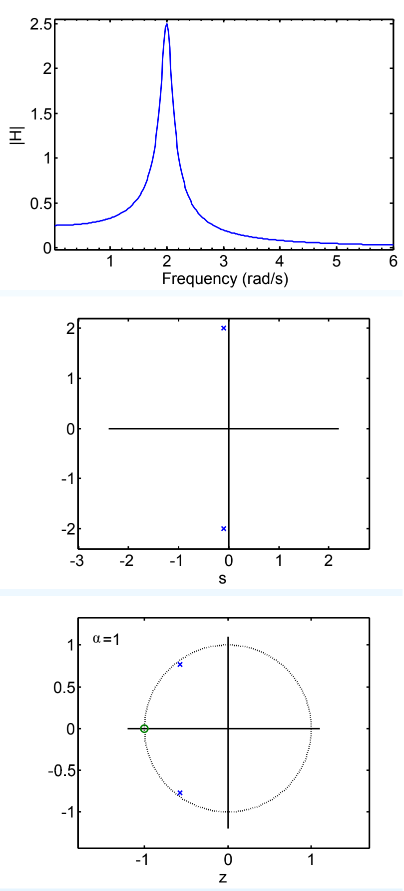
After the transformation we will always end up with the same number of poles as zeros:



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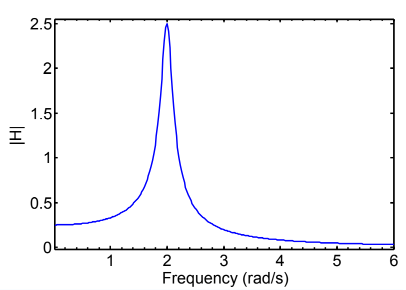

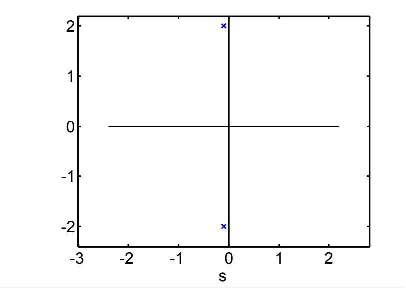

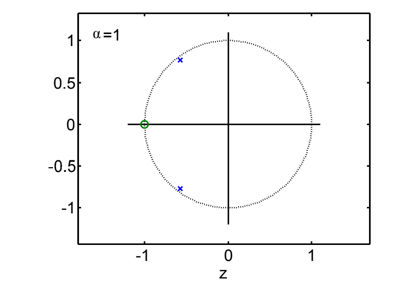

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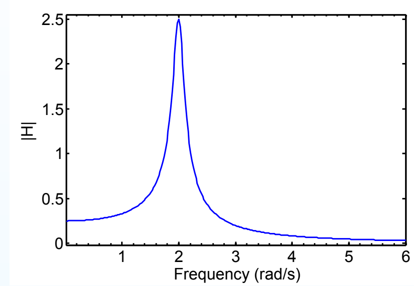

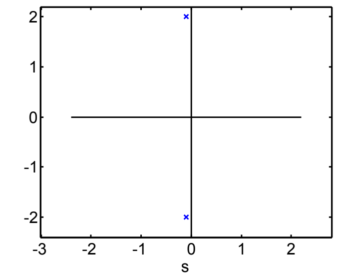

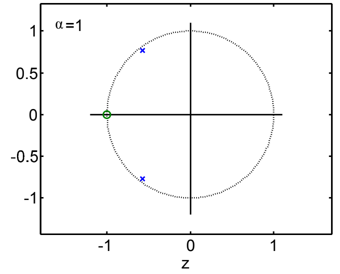

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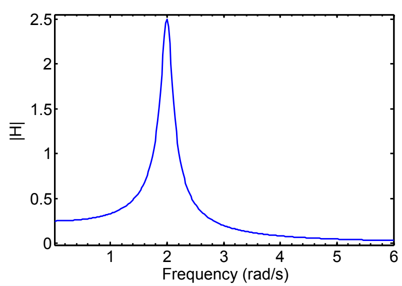

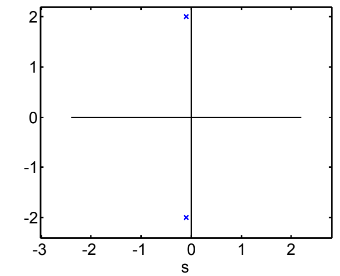

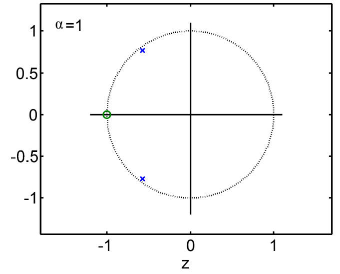

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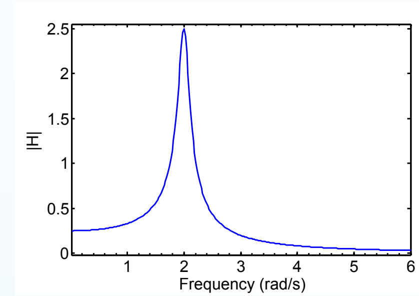
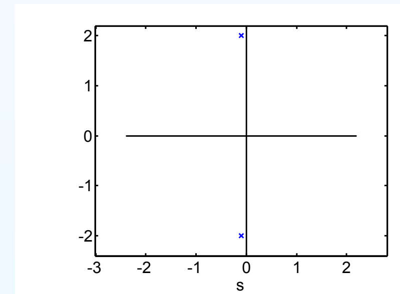
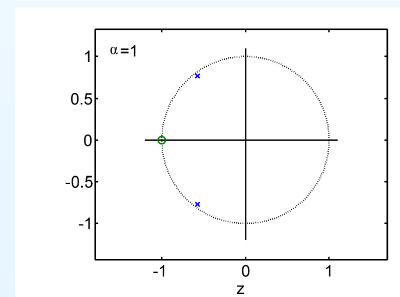
Choose overall scale factor, g, to give the same gain at any convenient pair of mapped frequencies:







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Find the poles and zeros: $p_s = -0.1 \pm 2j$

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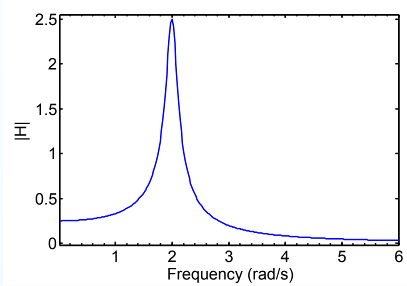

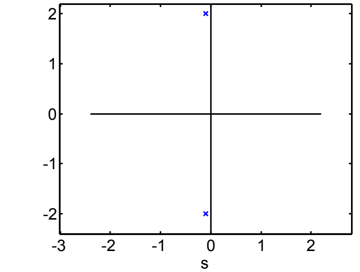

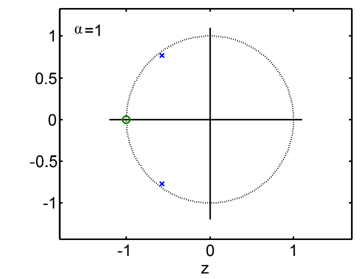

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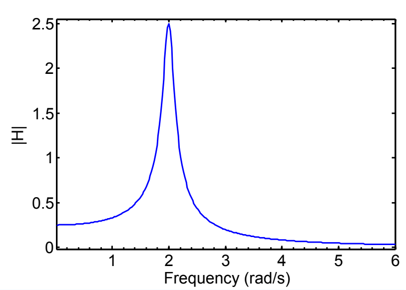

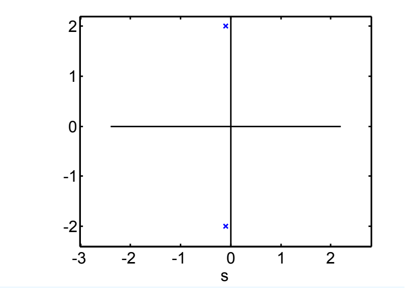

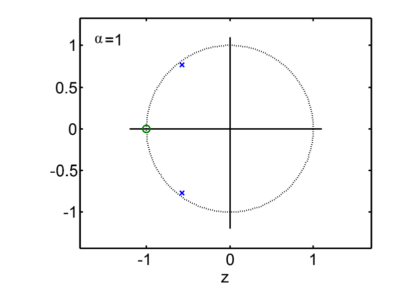

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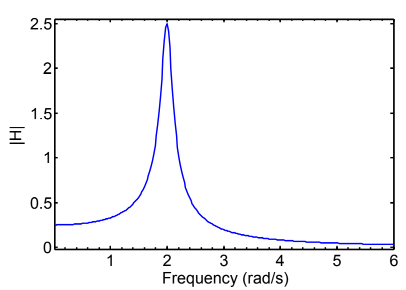

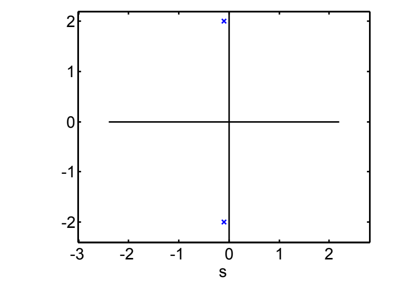

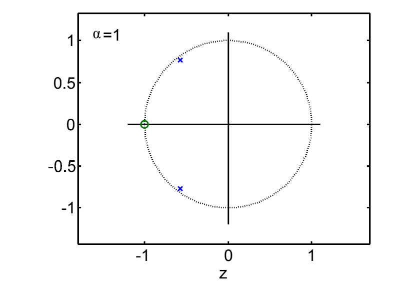

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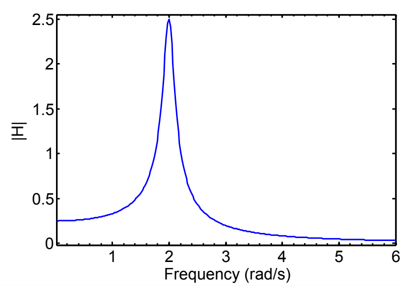

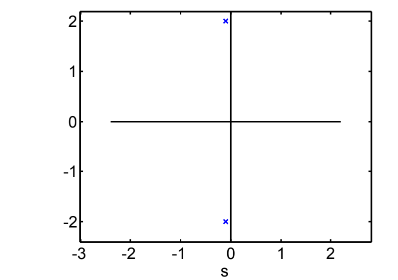

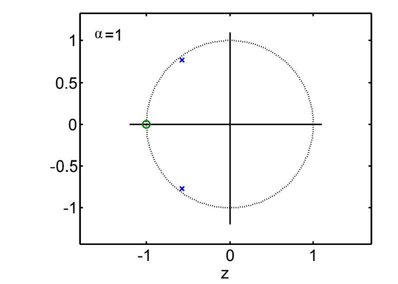

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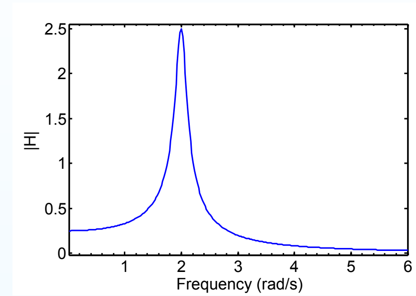

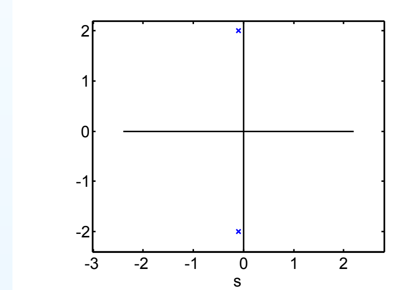

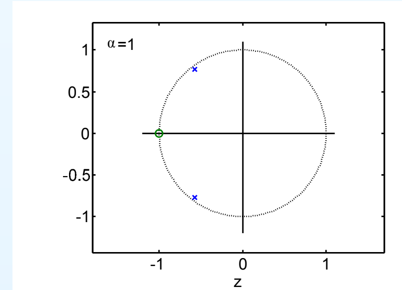

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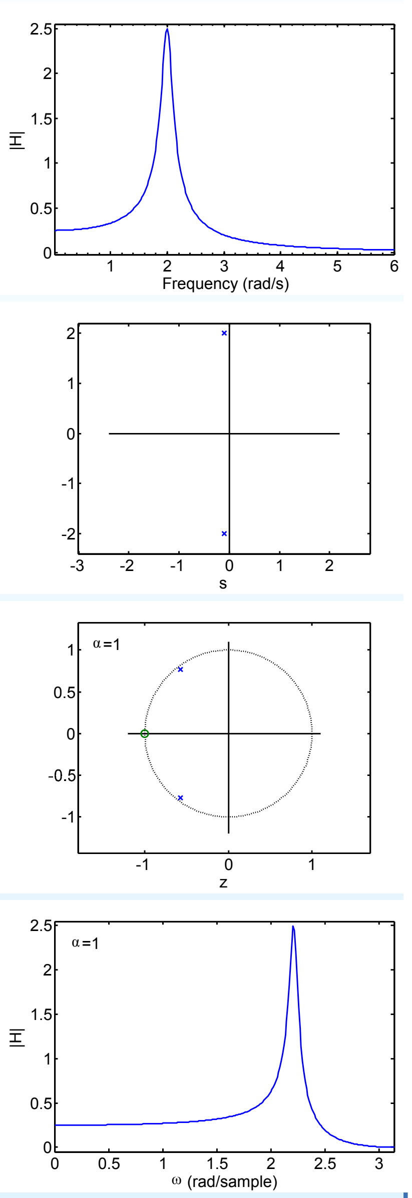

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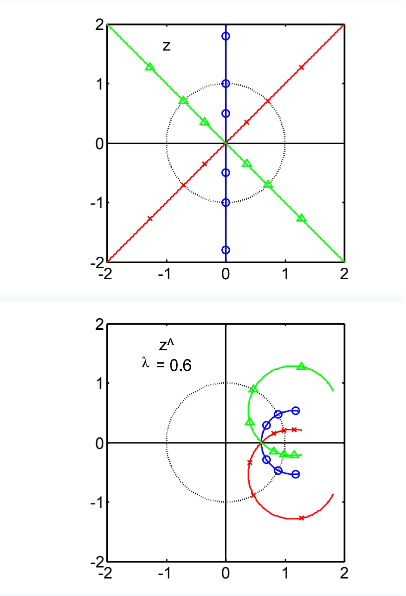

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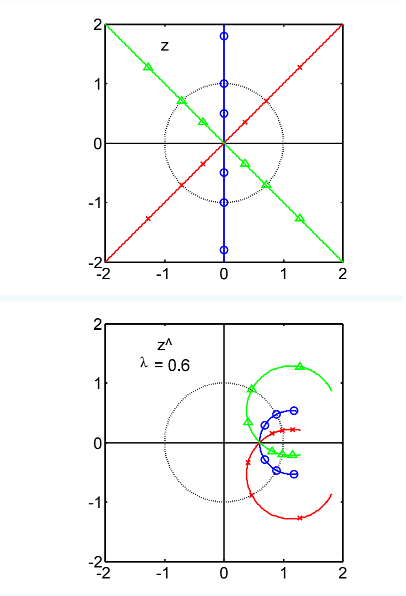

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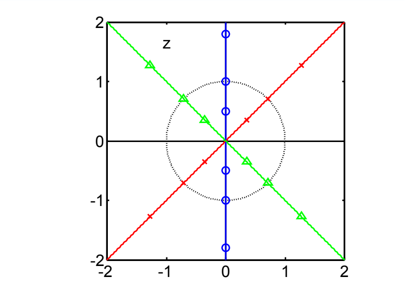

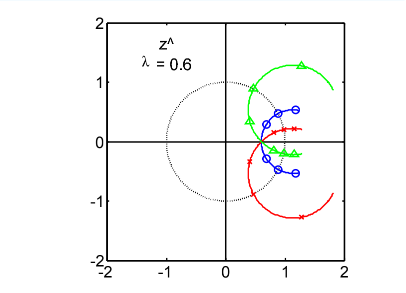

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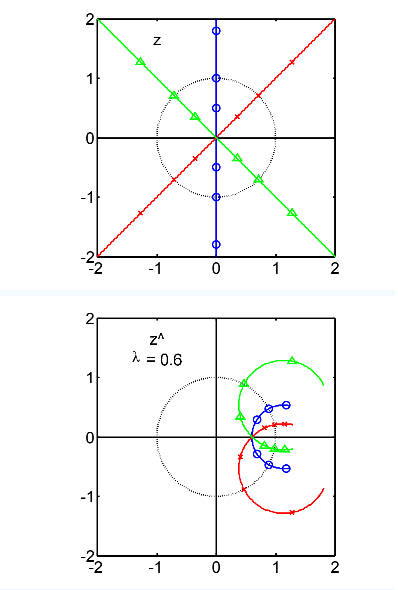

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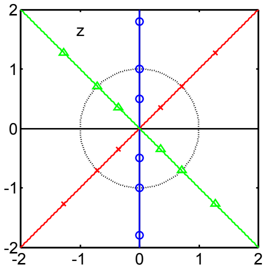

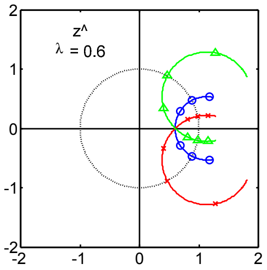

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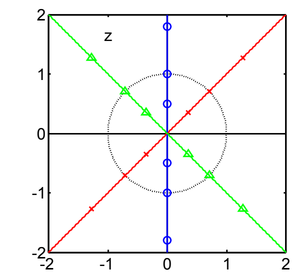

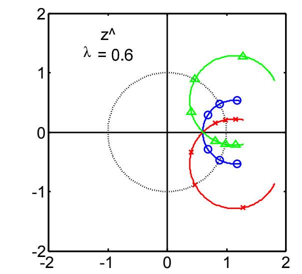

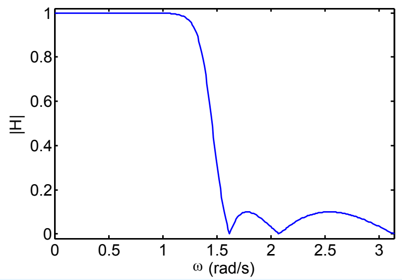

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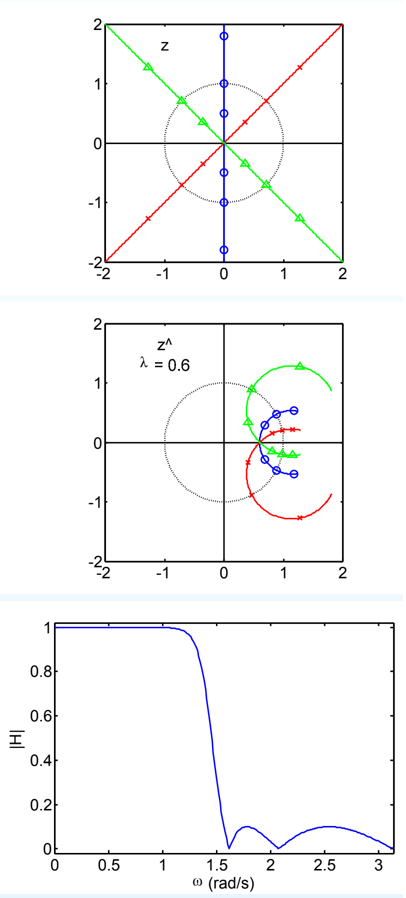

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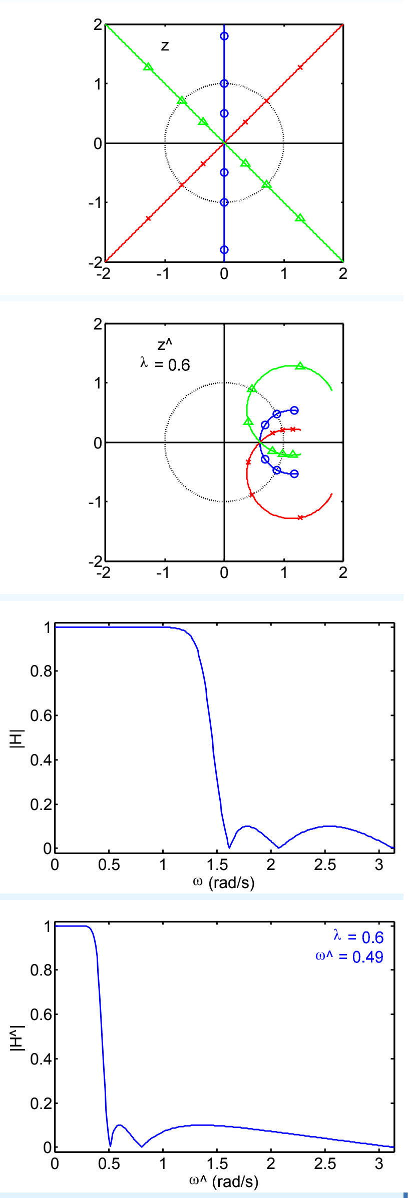

Constantinides Transformations

Transform any lowpass filter with cutoff frequency ω_0 to:

Target	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$ $\rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$ $\rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

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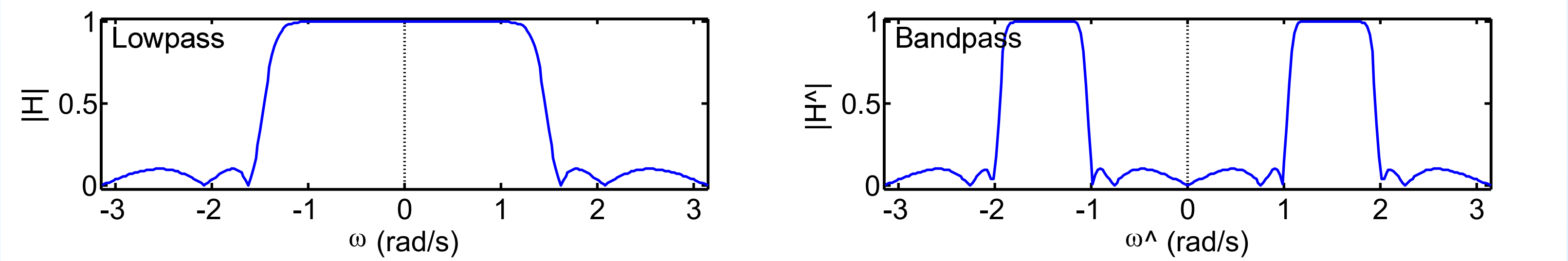
Bandstop $\hat{\omega}_1 \nless \hat{\omega} \nless \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$ $\rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Bandpass and bandstop transformations are quadratic and so will double the order:



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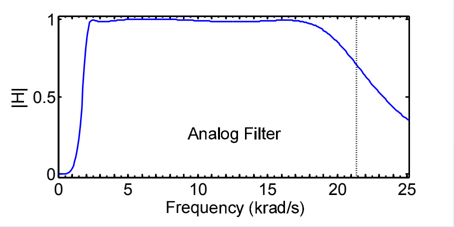

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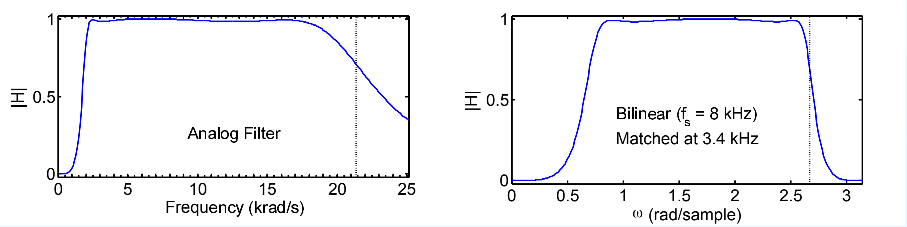

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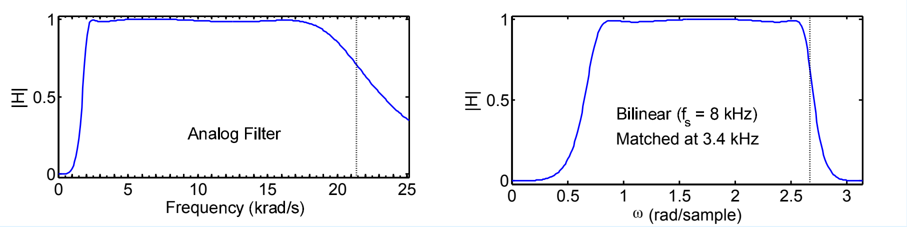

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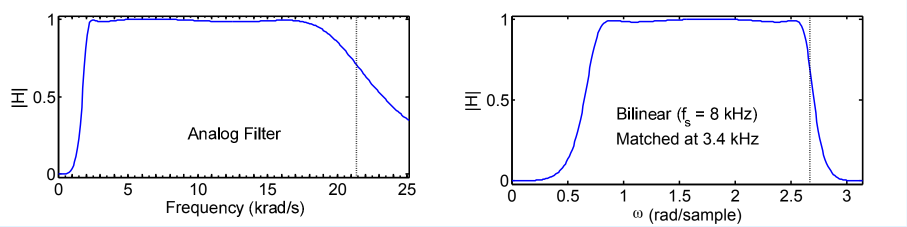

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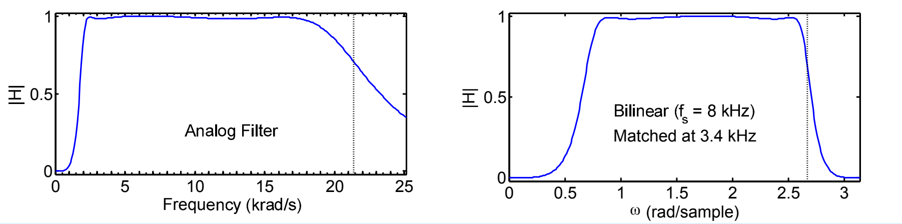

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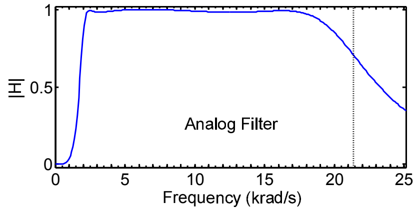

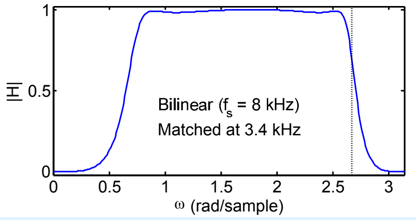

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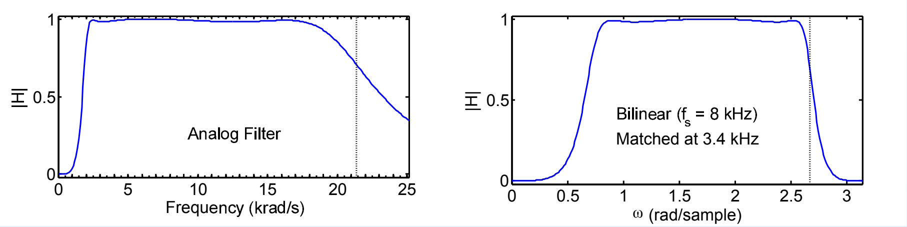

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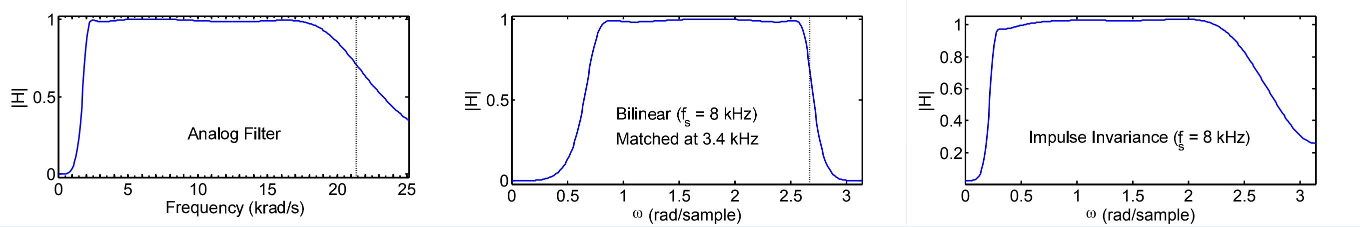

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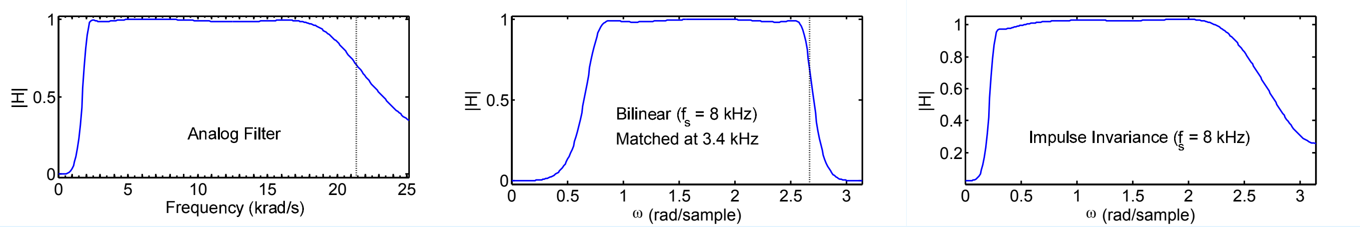

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☺ Impulse response correct.

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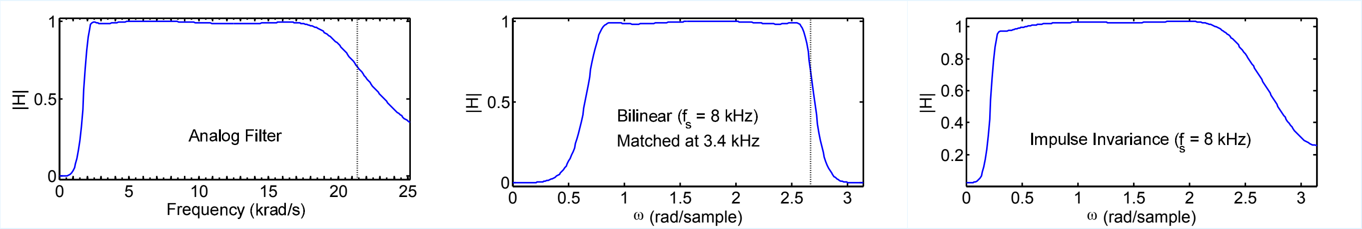

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Properties:

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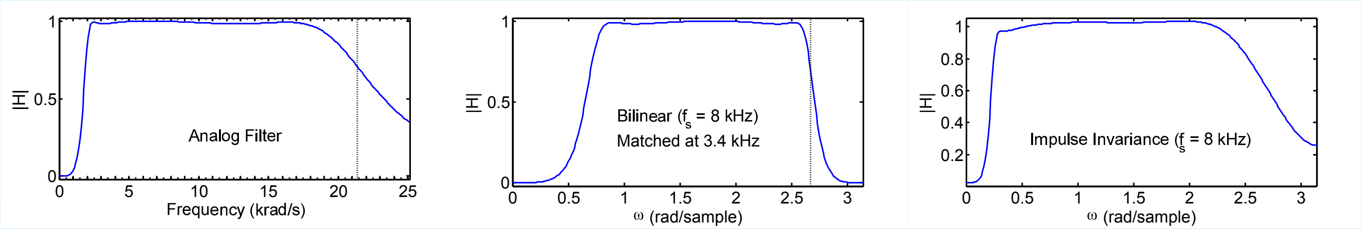

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Properties:

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☹ Frequency response is aliased.

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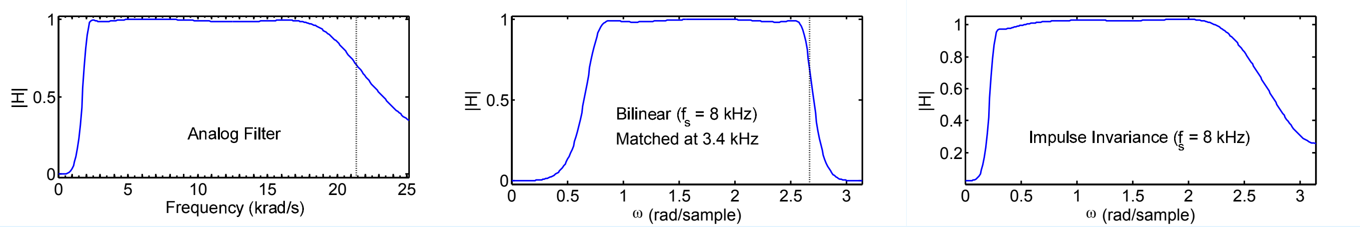

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For further details see Mitra: 9.

MATLAB routines

bilinear	Bilinear mapping
impinvar	Impulse invariance
butter butterord	Analog or digital Butterworth filter
cheby1 cheby1ord	Analog or digital Chebyshev filter
cheby2 cheby2ord	Analog or digital Inverse Chebyshev filter
ellip ellipord	Analog or digital Elliptic filter