9: Optimal IIR Design

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We actually want to minimize E_S but E_E is easier because it gives rise to linear equations.

However if $W_E(\omega) = \frac{W_S(\omega)}{|A(e^{j\omega})|}$, then $|E_E(\omega)| = |E_S(\omega)|$

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Overdetermined set of equations $\mathbf{Ax} = \mathbf{b}$ (#equations > #unknowns)

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This is a very widely used technique.

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[K equations, $M + N + 1$ unkowns]

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We want to force $\mathbf{a}$ and $\mathbf{b}$ to be real

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Choose K values of ω, $\left\{\begin{array}{ccc} \omega_1 & \cdots & \omega_K \end{array}\right\}$

$$\begin{pmatrix} \mathbf{U}^T & \mathbf{V}^T \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \mathbf{d}$$

[K equations, $M + N + 1$ unkowns]

where $\mathbf{U} = \left[\begin{array}{ccc} \mathbf{u}(\omega_1) & \cdots & \mathbf{u}(\omega_K) \end{array}\right]$,

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We want to force $\mathbf{a}$ and $\mathbf{b}$ to be real; find least squares solution to

$$\begin{pmatrix} \Re\left(\mathbf{U}^T\right) & \Re\left(\mathbf{V}^T\right) \\ \Im\left(\mathbf{U}^T\right) & \Im\left(\mathbf{V}^T\right) \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \Re\left(\mathbf{d}\right) \\ \Im\left(\mathbf{d}\right) \end{pmatrix}$$

Frequency Sampling

For every ω we want: $0 = W(\omega)\left(B(e^{j\omega}) - D(\omega)A(e^{j\omega})\right)$

$$= W(\omega)\left(\sum_{m=0}^{M} b[m]e^{-jm\omega} - D(\omega)\left(1 + \sum_{n=1}^{N} a[n]e^{-jn\omega}\right)\right)$$

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Replace pole p_i by $(p_i^*)^{-1}$ whenever $|p_i| \geq 1$

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But for faster convergence use Newton-Raphson . . .

Newton-Raphson

Newton: To solve $f(x) = 0$ given an initial guess x_0, we write

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So for each ω_k, we can write (omitting the ω and $e^{j\omega}$ arguments)

$$E_S \approx W_S \left(\frac{B_0}{A_0} - D \right) + \frac{W_S}{A_0} (B - B_0) - \frac{W_S B_0}{A_0^2} (A - A_0)$$

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At each iteration, calculate $A_0(e^{j\omega_k})$ and $B_0(e^{j\omega_k})$ based on $\mathbf{a}$ and $\mathbf{b}$ from the previous iteration.

Then use linear least squares to minimize the linearized E_S using the above equation replicated for each of the ω_k.

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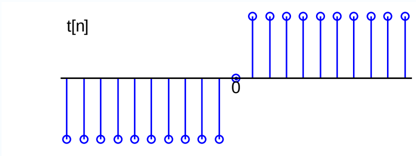
This result is a consequence of the Hilbert Relations.

Hilbert Relations

We define $t[n] = u[n-1] - u[-1-n]$



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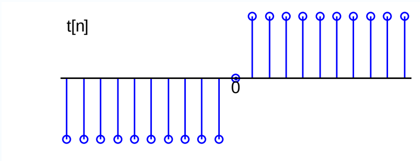

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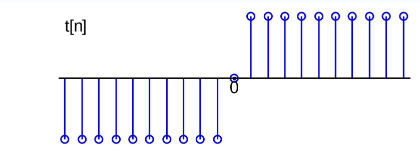

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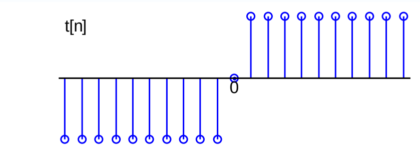

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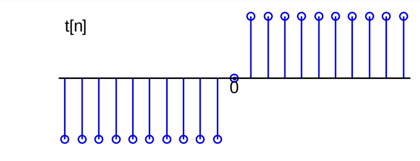
$$T(e^{j\omega}) = \frac{1+e^{-j\omega}}{1-e^{-j\omega}} = \frac{e^{j\frac{\omega}{2}}+e^{-j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}}$$



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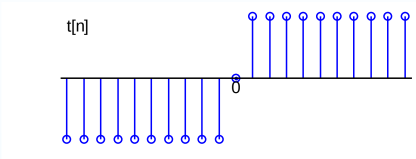

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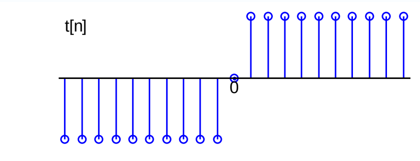

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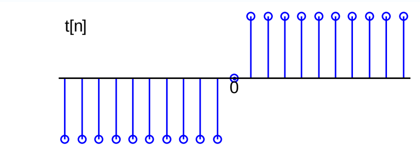

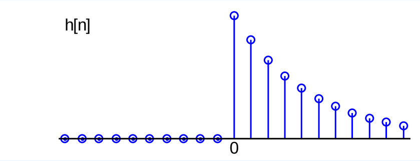

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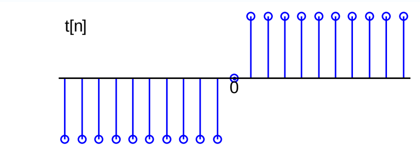

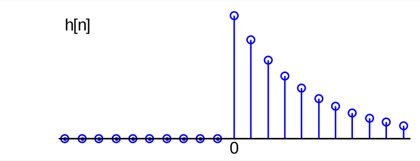

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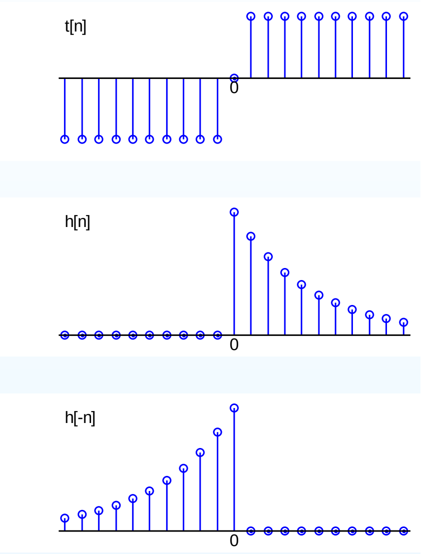

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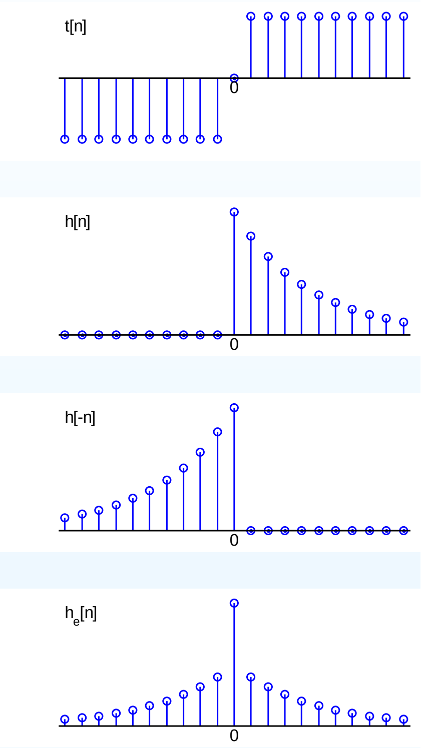

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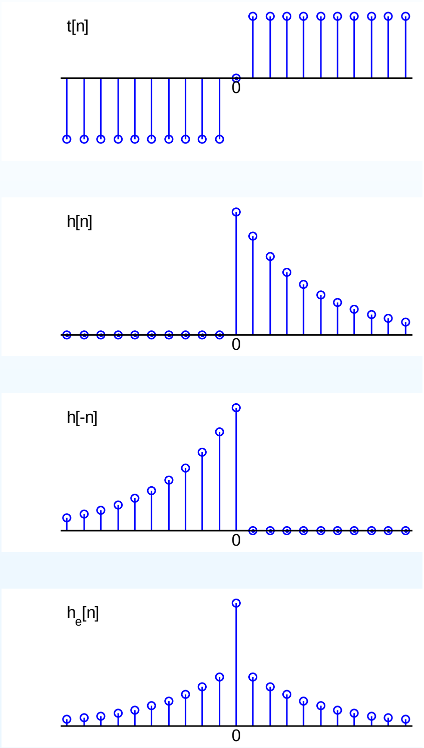

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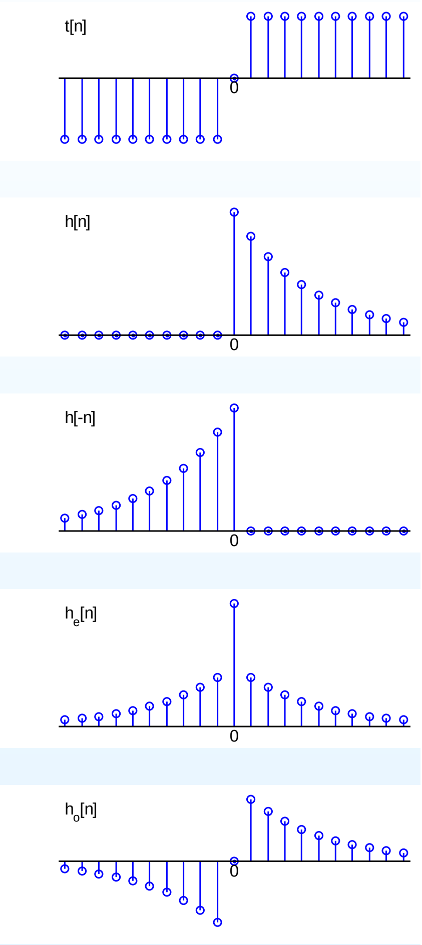

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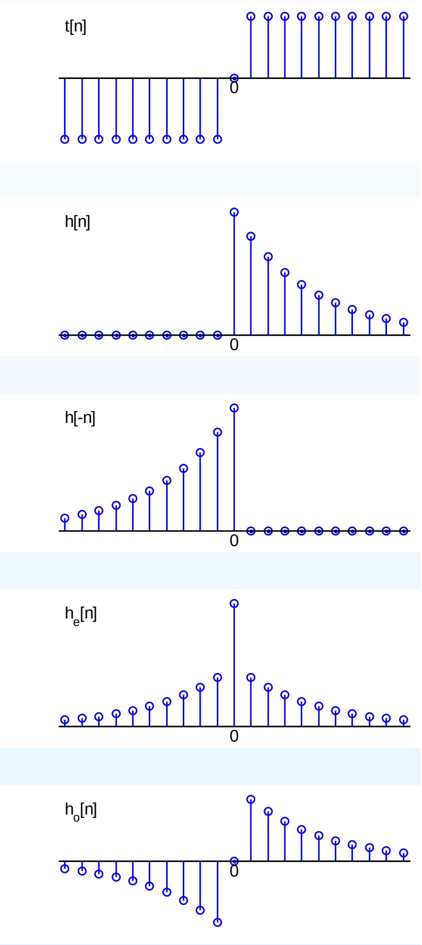

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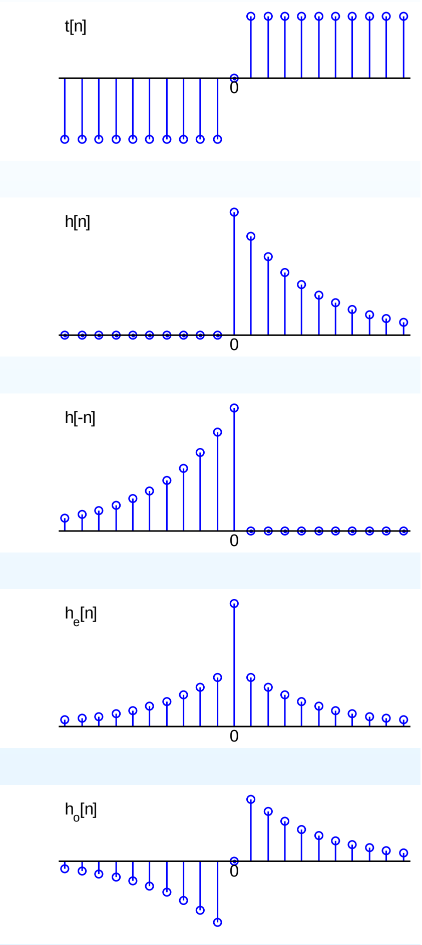

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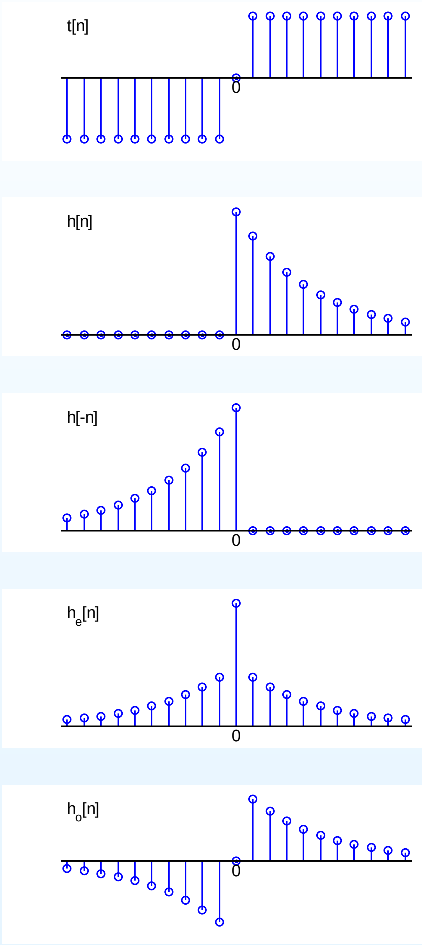

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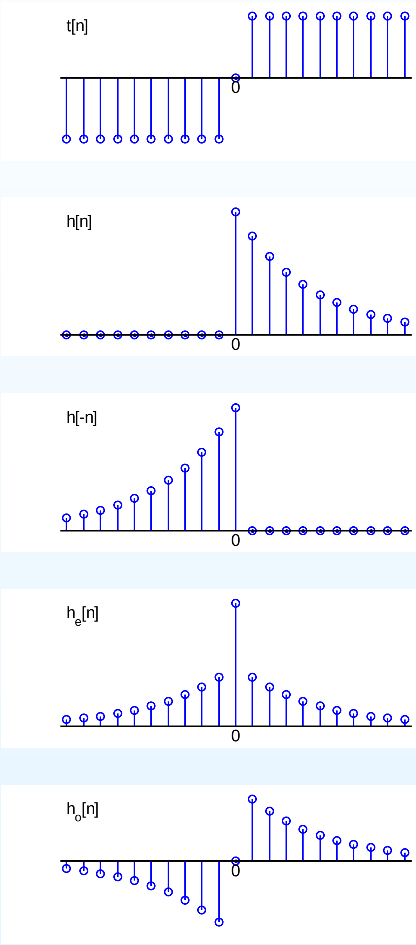

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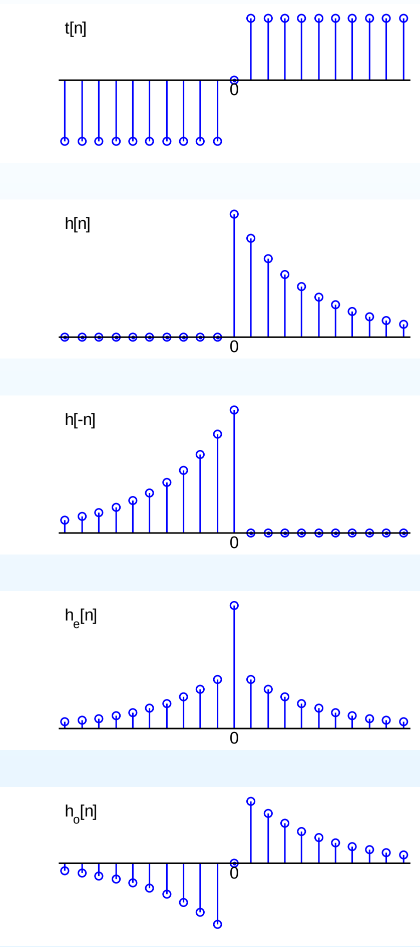

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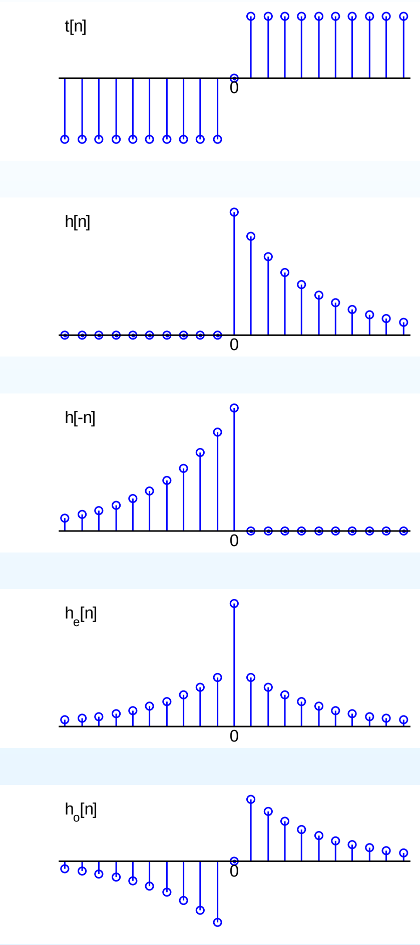

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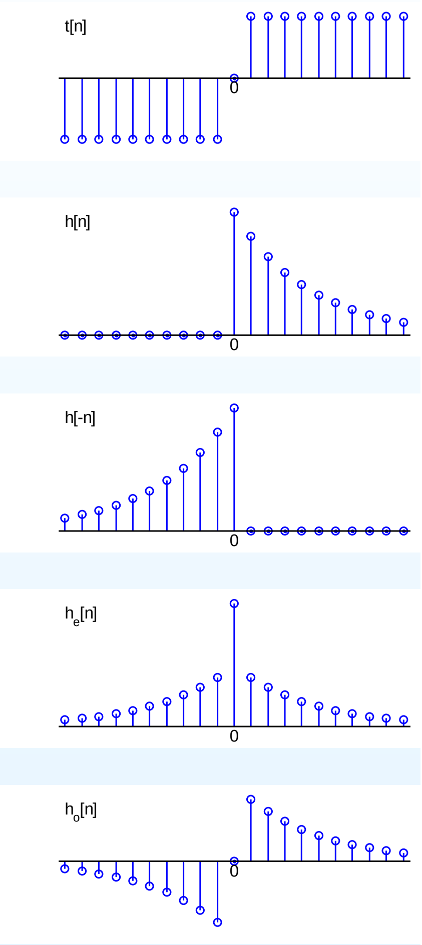

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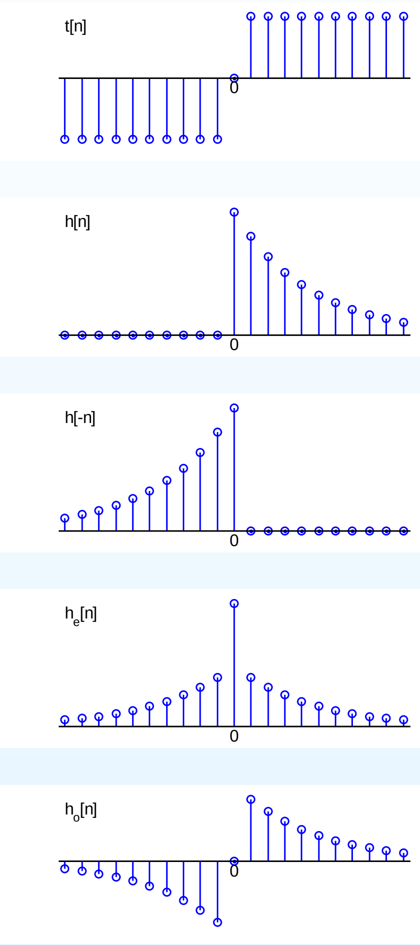

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So, if $H(z)$ is minimum phase (all p_n and q_m inside unit circle) then $\ln H(z)$ is the z-transform of a stable causal sequence and:

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$$\ln\left(1 - az^{-1}\right) = -az^{-1} - \frac{a^2}{2}z^{-2} - \frac{a^3}{3}z^{-3} - \ldots$$

causal and stable provided $|a| < 1$

So, if $H(z)$ is minimum phase (all p_n and q_m inside unit circle) then $\ln H(z)$ is the z-transform of a stable causal sequence and:

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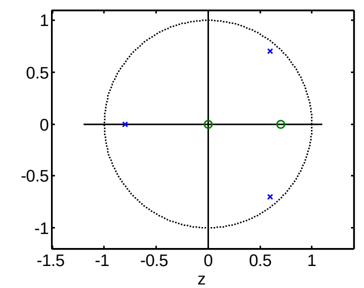

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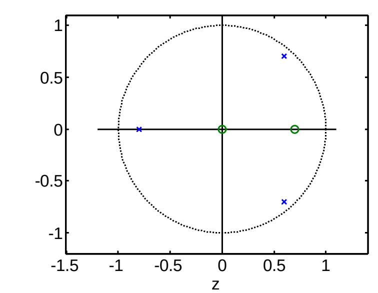

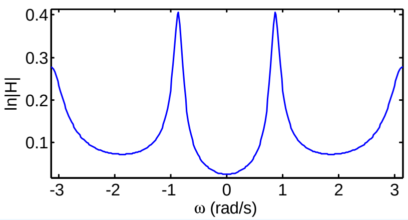

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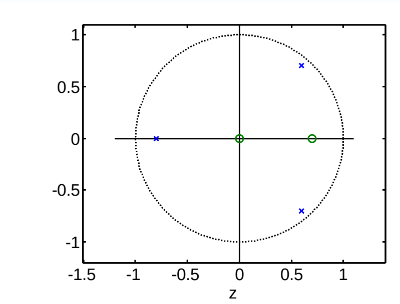

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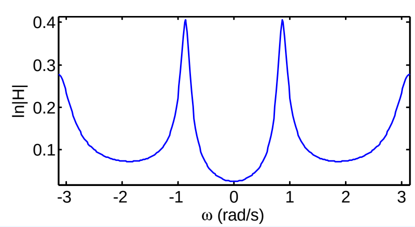

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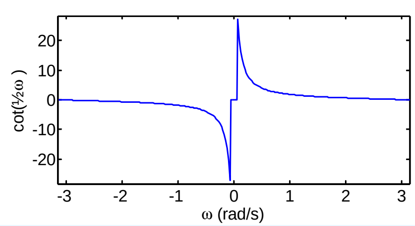

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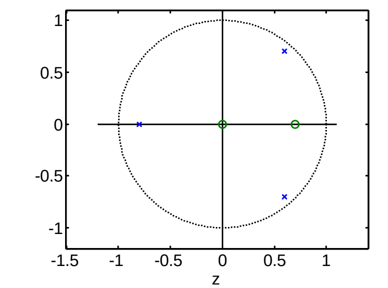

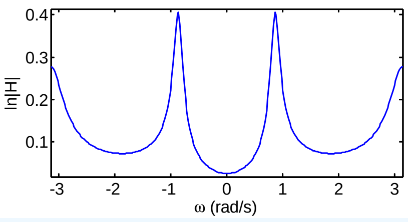

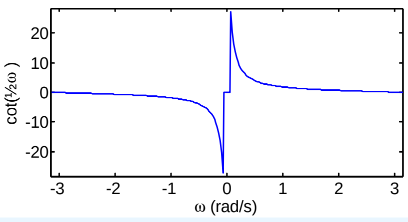

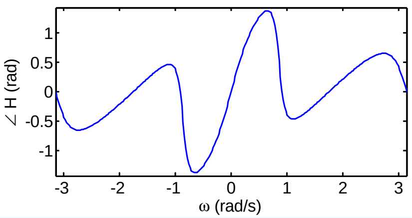

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For further details see Mitra: 9.

MATLAB routines

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