10: Digital Filter Structures

Filter: $H(z) = \frac{B(z)}{A(z)}$ with input $x[n]$ and output $y[n]$

$$y[n] = \sum_{k=0}^{M} b[k]x[n-k] - \sum_{k=1}^{N} a[k]y[n-k]$$

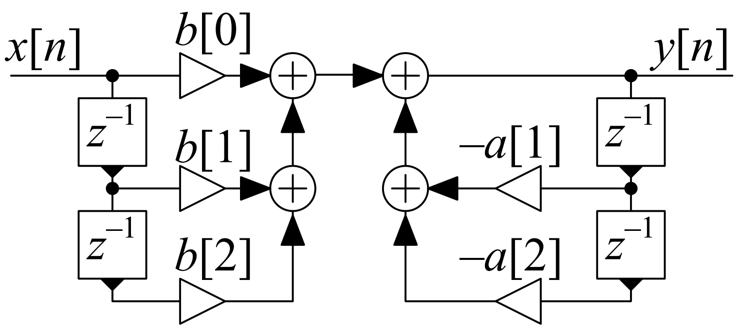
Direct forms use coefficients $a[k]$ and $b[k]$ directly

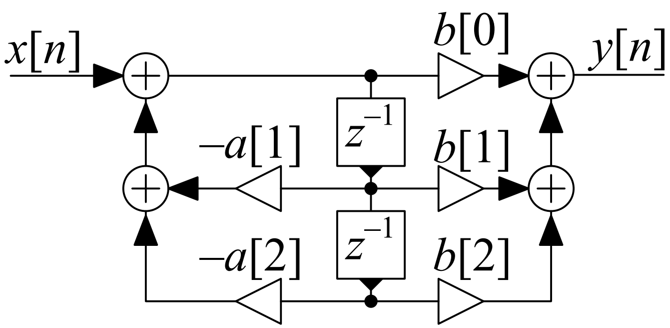
Direct Form 1:

- Direct implementation of difference equation
- Can view as $B(z)$ followed by $\frac{1}{A(z)}$

Direct Form II:

- Implements $\frac{1}{A(z)}$ followed by $B(z)$
- Saves on delays (= storage)





Transposition

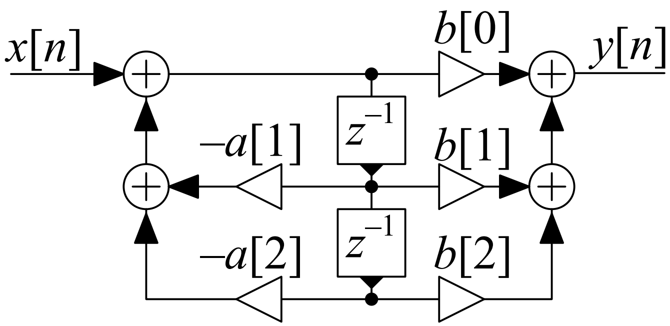
Can convert any block diagram into an equivalent transposed form:

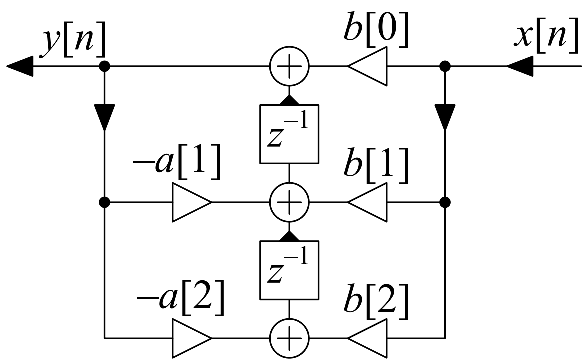
- Reverse direction of each interconnection
- Reverse direction of each multiplier
- Change junctions to adders and vice-versa
- Interchange the input and output signals

Example:

Direct form II $\rightarrow$ Direct Form II$_t$
Would normally be drawn with input on the left

Note: A valid block diagram must never have any feedback loops that don't go through a delay (z^{-1} block).

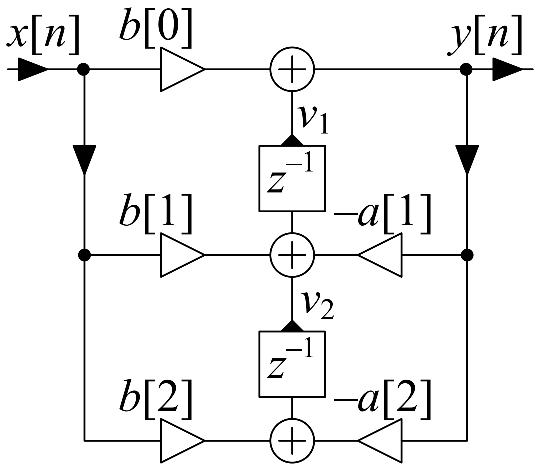




$\mathbf{v}[n]$ is a vector of delay element outputs

Can write: $\mathbf{v}[n+1] = \mathbf{P}\mathbf{v}[n] + \mathbf{q}x[n]$
$y[n] = \mathbf{r}^T\mathbf{v}[n] + sx[n]$

$\{\mathbf{P}, \mathbf{q}, \mathbf{r}^T, s\}$ is the state-space representation of the filter structure.



The transfer function is given by:

$$H(z) = \frac{B(z)}{A(z)} = \frac{\det\left(z\mathbf{I}-\mathbf{P}+\mathbf{q}\mathbf{r}^T\right)}{\det(z\mathbf{I}-\mathbf{P})} + s - 1$$

The transposed form has $\mathbf{P} \to \mathbf{P}^T$ and $\mathbf{q} \leftrightarrow \mathbf{r}$ $\Rightarrow$ same $H(z)$

Example: Direct Form II$_t$

$$\mathbf{P} = \begin{pmatrix} -a[1] & 1 \\ -a[2] & 0 \end{pmatrix} \qquad \mathbf{q} = \begin{pmatrix} b[1]-b[0]a[1] \\ b[2]-b[0]a[2] \end{pmatrix}$$

$$\mathbf{r}^T = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad s = b[0]$$

From which $H(z) = \frac{b[0]z^2+b[1]z+b[2]}{z^2+a[1]z+a[2]}$

[State-Space $\rightarrow$ Transfer Function]

[This is not examinable]

We start by proving a useful formula which shows how the determinant of a matrix, $\mathbf{A}$, changes when you add a rank-1 matrix, $\mathbf{q}\mathbf{r}^T$, onto it. The formula is known as the Matrix Determinant Lemma. For any nonsingular matrix $\mathbf{A}$ and column vectors $\mathbf{q}$ and $\mathbf{r}$, we can write

$$\begin{pmatrix} 1 & \mathbf{r}^T \\ \mathbf{0} & \mathbf{A} \end{pmatrix} \begin{pmatrix} 1+\mathbf{r}^T\mathbf{A}^{-1}\mathbf{q} & \mathbf{0}^T \\ -\mathbf{A}^{-1}\mathbf{q} & \mathbf{I} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}^T \\ -\mathbf{q} & \mathbf{I} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{r}^T \\ \mathbf{0} & \mathbf{A}+\mathbf{q}\mathbf{r}^T \end{pmatrix}.$$

It is easy to verify this by multiplying out the matrices. We now take the determinant of both sides making use of the result that the determinant of a block triangular matrix is the product of the determinants of the blocks along the diagonal (assuming they are all square). This gives:

$$\det(\mathbf{A}) \times \left(1+\mathbf{r}^T\mathbf{A}^{-1}\mathbf{q}\right) = \det\left(\mathbf{A}+\mathbf{q}\mathbf{r}^T\right) \qquad \Rightarrow \qquad \mathbf{r}^T\mathbf{A}^{-1}\mathbf{q} = \frac{\det\left(\mathbf{A}+\mathbf{q}\mathbf{r}^T\right)}{\det(\mathbf{A})} - 1$$

Now we take the z-transform of the state space equations

$$\begin{aligned} \mathbf{v}[n+1] &= \mathbf{P}\mathbf{v}[n] + \mathbf{q}x[n] \\ y[n] &= \mathbf{r}^T\mathbf{v}[n] + sx[n] \end{aligned} \quad \xrightarrow[z-\text{transform}]{} \quad \begin{aligned} z\mathbf{V} &= \mathbf{P}\mathbf{V} + \mathbf{q}X \\ Y &= \mathbf{r}^T\mathbf{V} + sX \end{aligned}$$

The upper equation gives $(z\mathbf{I}-\mathbf{P})\,\mathbf{V} = \mathbf{q}X$ from which $\mathbf{V} = (z\mathbf{I}-\mathbf{P})^{-1}\,\mathbf{q}X$ and by substituting this in the lower equation, we get $\frac{Y}{X} = \mathbf{r}^T\,(z\mathbf{I}-\mathbf{P})^{-1}\,\mathbf{q} + s \;=\; \frac{\det\left(z\mathbf{I}-\mathbf{P}+\mathbf{q}\mathbf{r}^T\right)}{\det(z\mathbf{I}-\mathbf{P})} + s - 1.$

If all computations were exact, it would not make any difference which of the equivalent structures was used. However ...

- Coefficient precision
  Coefficients are stored to finite precision and so are not exact. The filter actually implemented is therefore incorrect.
- Arithmetic precision
  Arithmetic calculations are not exact.
  - Worst case for arithmetic errors is when calculating the difference between two similar values:
    $1.23456789 - 1.23455678 = 0.00001111$: 9 s.f. $\rightarrow$ 4 s.f.

  Arithmetic errors introduce noise that is then filtered by the transfer function between the point of noise creation and the output.

The roots of high order polynomials can be very sensitive to small changes in coefficient values.

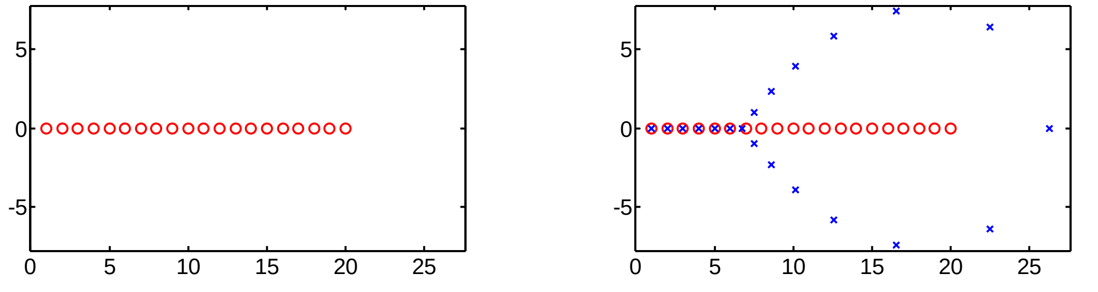
Wilkinson's polynomial: (famous example)

$$f(x) = \prod_{n=1}^{20}(x-n) = x^{20} - 210x^{19} + 20615x^{18} - \ldots$$

has roots well separated on the real axis.

Multiplying the coefficient of x^{19} by 1.000001 moves the roots a lot.

"Speaking for myself I regard it as the most traumatic experience in my career as a numerical analyst", James Wilkinson 1984



Moral: Avoid using direct form for filters orders over about 10.

Avoid high order polynomials by factorizing into quadratic terms:

$$\frac{B(z)}{A(z)} = g\frac{\prod\left(1+b_{k,1}z^{-1}+b_{k,2}z^{-2}\right)}{\prod\left(1+a_{k,1}z^{-1}+a_{k,2}z^{-2}\right)} = g\prod_{k=1}^{K}\frac{1+b_{k,1}z^{-1}+b_{k,2}z^{-2}}{1+a_{k,1}z^{-1}+a_{k,2}z^{-2}}$$

where $K = \max\left(\left\lceil \frac{M}{2}\right\rceil, \left\lceil \frac{N}{2}\right\rceil\right)$.

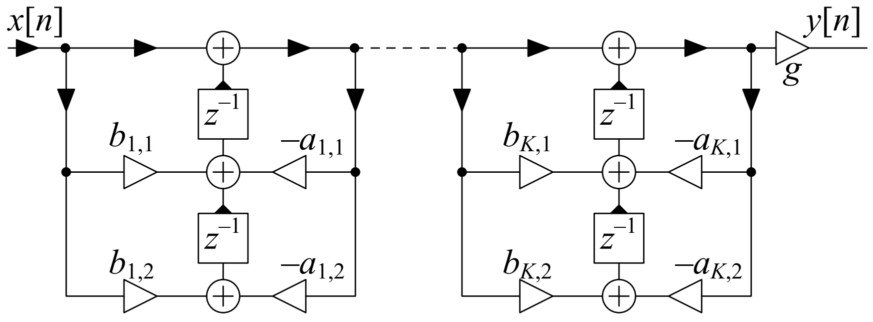
The term $\frac{1+b_{k,1}z^{-1}+b_{k,2}z^{-2}}{1+a_{k,1}z^{-1}+a_{k,2}z^{-2}}$ is a biquad (bi-quadratic section).

We need to choose:

(a) which poles to pair with which zeros in each biquad

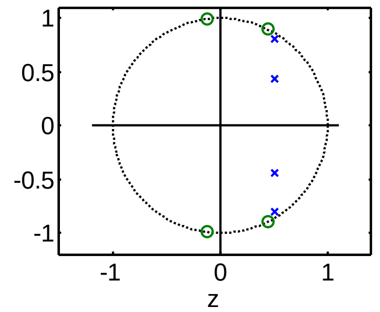
(b) how to order the biquads

Direct Form II Transposed



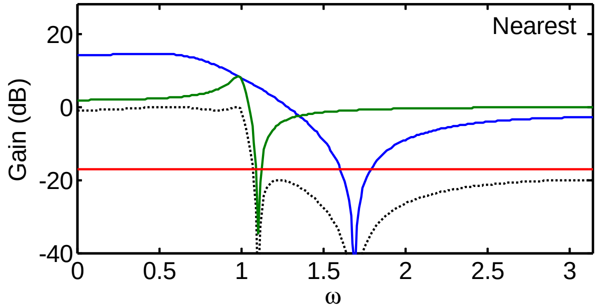
Example: Elliptic lowpass filter

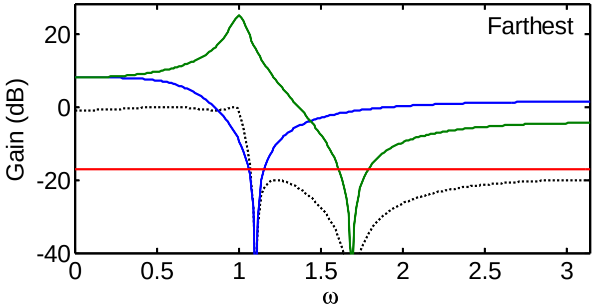
2 pole pairs and 2 zero pairs
need 2 biquads



Noise introduced in one biquad is amplified by all the subsequent ones:

- Make the peak gain of each biquad as small as possible
  - Pair poles with nearest zeros to get lowest peak gain
    begin with the pole nearest the unit circle
  - Pairing with farthest zeros gives higher peak biquad gain
- Poles near the unit circle have the highest peaks and introduce most noise so place them last in the chain





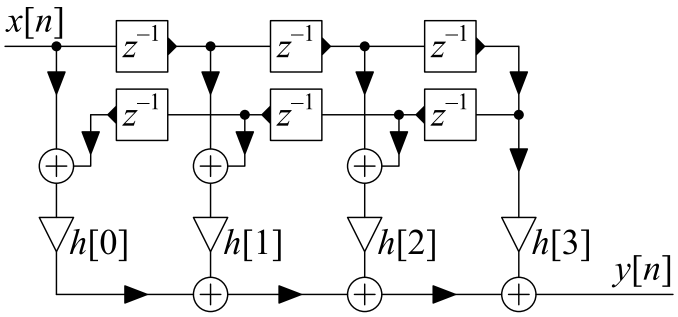
Implementation can take advantage of any symmetry in the coefficients.

Linear phase filters are always FIR and have symmetric (or, more rarely, antisymmetric) coefficients.

$$H(z) = \sum_{m=0}^{M} h[m]z^{-m} \qquad h[M-m] = h[m]$$

$$= h\left[\frac{M}{2}\right] z^{-\frac{M}{2}} + \sum_{m=0}^{\frac{M}{2}-1} h[m]\left(z^{-m} + z^{m-M}\right) \qquad \text{[m even]}$$

For M even, we only need $\frac{M}{2}+1$ multiplies instead of $M+1$.
We still need M additions and M delays.

$M = 6$: 

For M odd (no central coefficient), we only need $\frac{M+1}{2}$ multiplies.

Software Implementation:

All that matters is the total number of multiplies and adds

Hardware Implementation:

Delay elements (z^{-1}) represent storage registers

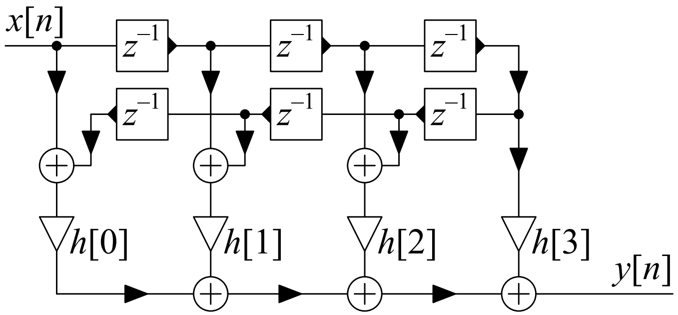
The maximum clock speed is limited by the number of sequential operations between registers

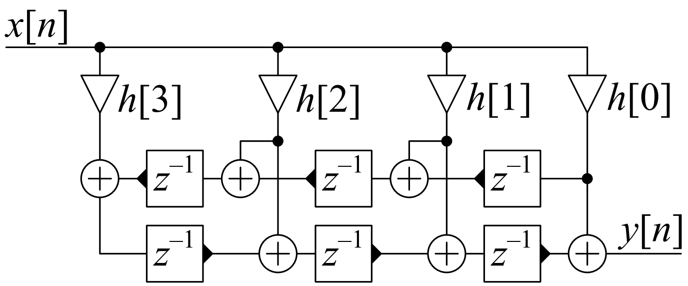
Example: Symmetric Linear Phase Filter

Direct form: Maximum sequential delay $= 4a + m$

Transpose form: Maximum sequential delay $= a + m$ ☺

a and m are the delays of adder and multiplier respectively





Allpass Filters

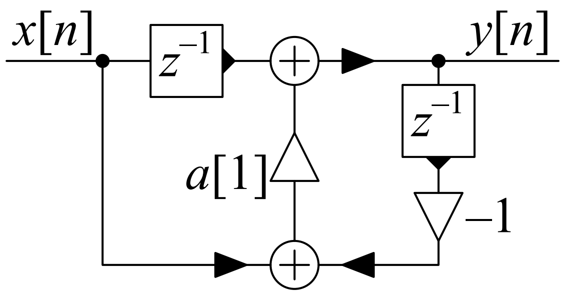
Allpass filters have mirror image numerator and denominator coefficients:

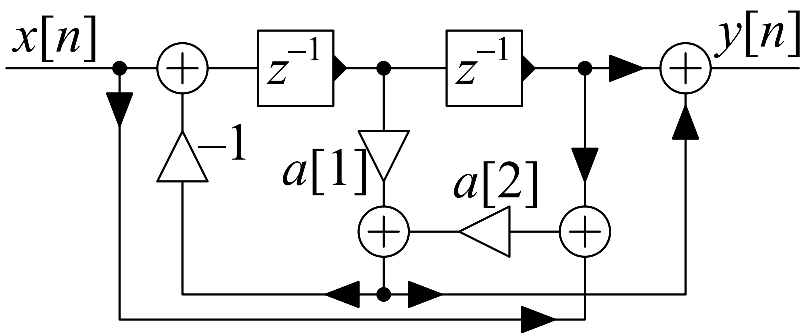
$$b[n] = a[N-n] \qquad \Leftrightarrow \qquad B(z) = z^{-N}A(z^{-1})$$

$$\Rightarrow \left|H(e^{j\omega})\right| \equiv 1 \forall \omega$$

There are several efficient structures, e.g.

- First Order: $H(z) = \frac{a[1]+z^{-1}}{1+a[1]z^{-1}}$
- Second Order: $H(z) = \frac{a[2]+a[1]z^{-1}+z^{-2}}{1+a[1]z^{-1}+a[2]z^{-2}}$

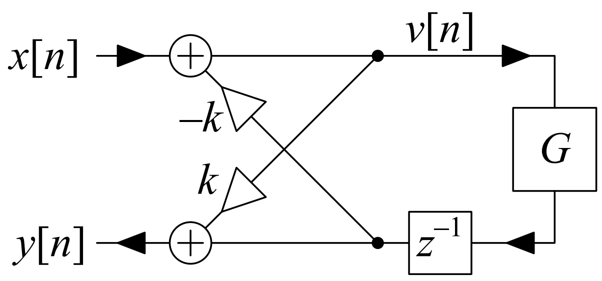




Allpass filters have a gain magnitude of 1 even with coefficient errors.

Lattice Stage

Suppose G is allpass: $G(z) = \frac{z^{-N}A(z^{-1})}{A(z)}$



$$V(z) = X(z) - kGz^{-1}V(z)$$

$$\Rightarrow V(z) = \frac{1}{1+kGz^{-1}}X(z)$$

$$Y(z) = kV(z) + Gz^{-1}V(z) = \frac{k+z^{-1}G}{1+kGz^{-1}}X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{kA(z)+z^{-N-1}A(z^{-1})}{A(z)+kz^{-N-1}A(z^{-1})} \triangleq \frac{z^{-(N+1)}D(z^{-1})}{D(z)}$$

Obtaining $\{d[n]\}$ from $\{a[n]\}$:

$$d[n] = \begin{cases} 1 & n = 0 \\ a[n] + ka[N+1-n] & 1 \le n \le N \\ k & n = N+1 \end{cases}$$

Obtaining $\{a[n]\}$ from $\{d[n]\}$:

$$k = d[N+1] \qquad a[n] = \frac{d[n]-kd[N+1-n]}{1-k^2}$$

If $G(z)$ is stable then $\frac{Y(z)}{X(z)}$ is stable if and only if $|k| < 1$ (see note)

[Proof of Stability Criterion]

We want to show that if $G(z)$ is a stable allpass filter then $\frac{Y(z)}{X(z)} = \frac{k+z^{-1}G(z)}{1+kz^{-1}G(z)}$ is stable if and only if $|k| < 1$.

We make use of a property of allpass filters (proved in a note in lecture 5) that if $G(z)$ is a stable allpass filter, then $|G(z)| \gtreqless 1$ according to whether $|z| \lesseqgtr 1$.

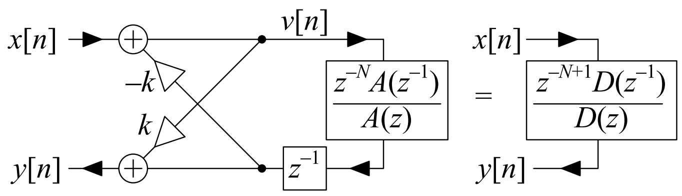
If z is a root of the denominator $1 + kz^{-1}G(z)$, then

$$\begin{aligned} kz^{-1}G(z) &= -1 \\ \Rightarrow |k| \times |z^{-1}| \times |G(z)| &= 1 \\ \Rightarrow |k| &= \frac{|z|}{|G(z)|} \end{aligned}$$

It follows from the previously stated property of $G(z)$ that $|z| \lesseqgtr 1 \Leftrightarrow \frac{|z|}{|G(z)|} \lesseqgtr 1 \Leftrightarrow |k| \lesseqgtr 1$.

Example $A(z) \leftrightarrow D(z)$

Suppose $N = 3$, $k = 0.5$ and
$A(z) = 1 + 4z^{-1} - 6z^{-2} + 10z^{-3}$



$A(z) \rightarrow D(z)$

	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
$A(z)$	1	4	-6	10	
$z^{-4}A(z^{-1})$		10	-6	4	1
$D(z) = A(z) + kz^{-4}A(z^{-1})$	1	9	-9	12	0.5

$D(z) \rightarrow A(z)$

	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
$D(z)$	1	9	-9	12	0.5
$k = d[N+1]$					0.5
$z^{-4}D(z^{-1})$	0.5	12	-9	9	1
$D(z) - kz^{-4}D(z^{-1})$	0.75	3	-4.5	7.5	0
$A(z) = \frac{D(z)-kz^{-4}D(z^{-1})}{1-k^2}$	1	4	-6	10	0

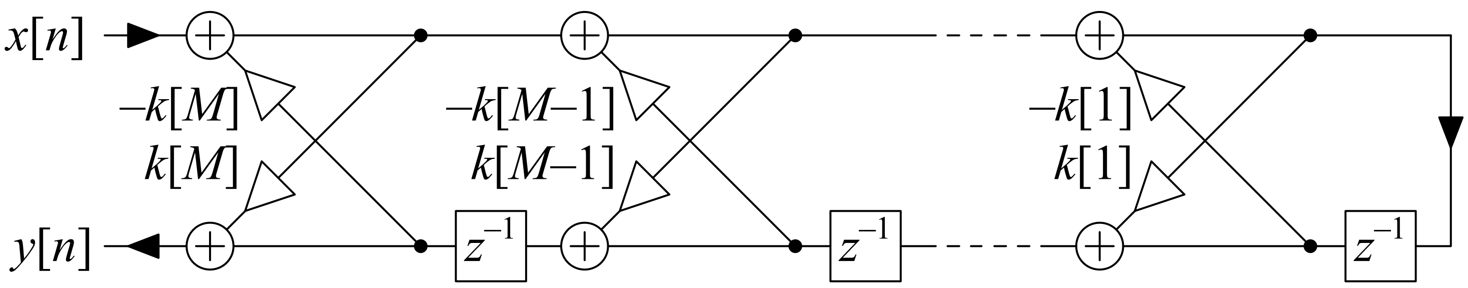
Allpass Lattice

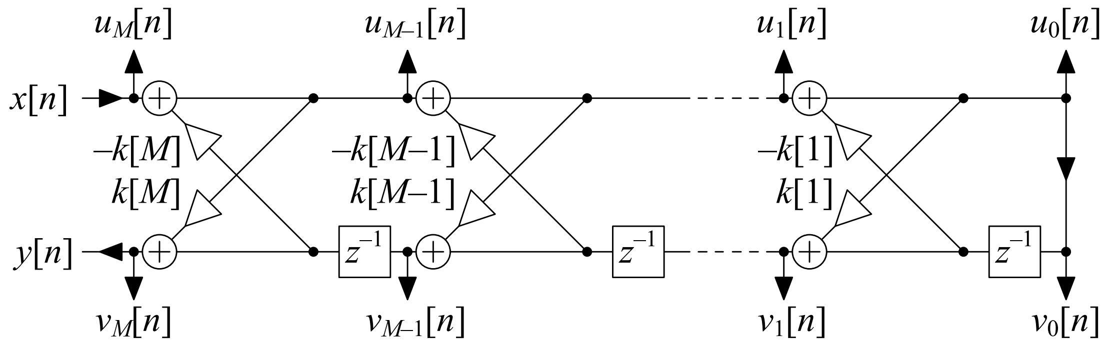
We can implement any allpass filter $H(z) = \frac{z^{-M}A(z^{-1})}{A(z)}$ as a lattice filter with M stages:

- Initialize $A_M(z) = A(z)$
- Repeat for $m = M : -1 : 1$
  - $k[m] = a_m[m]$
  - $a_{m-1}[n] = \frac{a_m[n] - k[m]a_m[m-n]}{1-k^2[m]}$ for $0 \le n \le m-1$

    equivalently $A_{m-1}(z) = \frac{A_m(z) - k[m]z^{-m}A_m(z^{-1})}{1-k^2[m]}$

$A(z)$ is stable iff $|k[m]| < 1$ for all m (good stability test)





Label outputs $u_m[n]$ and $v_m[n]$ and define $H_m(z) = \frac{V_m(z)}{U_m(z)} = \frac{z^{-m}A_m(z^{-1})}{A_m(z)}$

From earlier slide (slide 12):

$$\frac{U_{m-1}(z)}{U_m(z)} = \frac{1}{1+k[m]z^{-1}H_{m-1}(z)} = \frac{A_{m-1}(z)}{A_{m-1}(z)+k[m]z^{-m}A_{m-1}(z^{-1})} = \frac{A_{m-1}(z)}{A_m(z)}$$

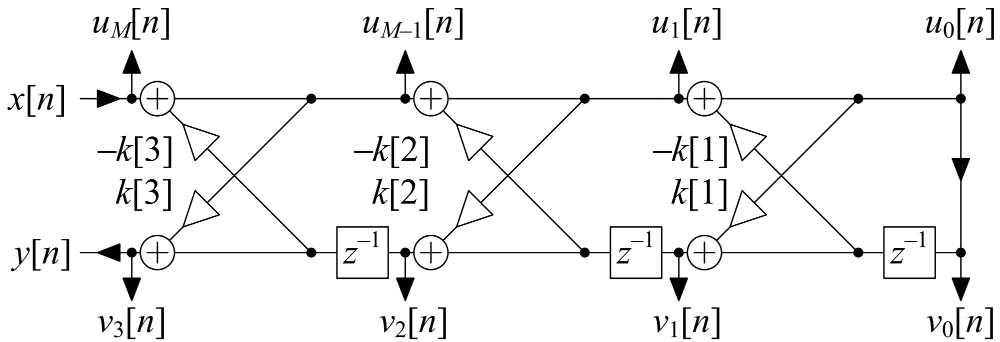
Hence:

$$\frac{U_m(z)}{X(z)} = \frac{A_m(z)}{A(z)} \qquad \text{and} \qquad \frac{V_m(z)}{X(z)} = \frac{U_m(z)}{X(z)} \times \frac{V_m(z)}{U_m(z)} = \frac{z^{-m}A_m(z^{-1})}{A(z)}$$

The numerator of $\frac{V_m(z)}{X(z)}$ is of order m so you can create any numerator of order M by summing appropriate multiples of $V_m(z)$:

$$w[n] = \sum_{m=0}^{M} c_m v_m[n] \quad \Rightarrow \quad W(z) = \frac{\sum_{m=0}^{M} c_m z^{-m} A_m(z^{-1})}{A(z)}$$

Lattice Example



$$A(z) = A_3(z) = 1 + 0.2z^{-1} - 0.23z^{-2} + 0.2z^{-3}$$

- $k[3] = 0.2 \Rightarrow a_2[\,] = \frac{[1,\ 0.2,\ -0.23] - 0.2[0.2,\ -0.23,\ 0.2]}{1-0.2^2} = [1,\ 0.256,\ -0.281]$
- $k[2] = -0.281 \Rightarrow a_1[\,] = \frac{[1,\ 0.256] + 0.281[-0.281,\ 0.256]}{1-0.281^2} = [1,\ 0.357]$
- $k[1] = 0.357 \Rightarrow a_0[\,] = 1$

$$\frac{V_0(z)}{X(z)} = \frac{1}{1+0.2z^{-1}-0.23z^{-2}+0.2z^{-3}} \qquad \frac{V_1(z)}{X(z)} = \frac{0.357+z^{-1}}{1+0.2z^{-1}-0.23z^{-2}+0.2z^{-3}}$$

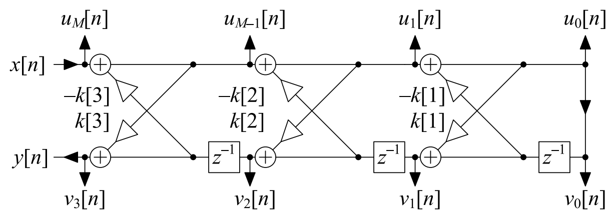
$$\frac{V_2(z)}{X(z)} = \frac{-0.281+0.256z^{-1}+z^{-2}}{1+0.2z^{-1}-0.23z^{-2}+0.2z^{-3}} \qquad \frac{V_3(z)}{X(z)} = \frac{0.2-0.23z^{-1}+0.2z^{-2}+z^{-3}}{1+0.2z^{-1}-0.23z^{-2}+0.2z^{-3}}$$

Add together multiples of $\frac{V_m(z)}{X(z)}$ to create an arbitrary $\frac{B(z)}{1+0.2z^{-1}-0.23z^{-2}+0.2z^{-3}}$

Lattice Example Numerator

Form a new output signal as $w[n] = \sum_{m=0}^{M} c_m v_m[n]$

$W(z) = \sum_{m=0}^{M} c_m V_m(z) = \frac{B(z)}{1+0.2z^{-1}-0.23z^{-2}+0.2z^{-3}} X(z)$



$$\frac{V_0(z)}{X(z)} = \frac{1}{1+0.2z^{-1}-0.23z^{-2}+0.2z^{-3}} \qquad \frac{V_1(z)}{X(z)} = \frac{0.357+z^{-1}}{1+0.2z^{-1}-0.23z^{-2}+0.2z^{-3}}$$

$$\frac{V_2(z)}{X(z)} = \frac{-0.281+0.256z^{-1}+z^{-2}}{1+0.2z^{-1}-0.23z^{-2}+0.2z^{-3}} \qquad \frac{V_3(z)}{X(z)} = \frac{0.2-0.23z^{-1}+0.2z^{-2}+z^{-3}}{1+0.2z^{-1}-0.23z^{-2}+0.2z^{-3}}$$

We have $\begin{pmatrix} b[0] \\ b[1] \\ b[2] \\ b[3] \end{pmatrix} = \begin{pmatrix} 1 & 0.357 & -0.281 & 0.2 \\ 0 & 1 & 0.256 & -0.23 \\ 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$

Hence choose c_m as $\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0.357 & -0.281 & 0.2 \\ 0 & 1 & 0.256 & -0.23 \\ 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} b[0] \\ b[1] \\ b[2] \\ b[3] \end{pmatrix}$

Summary

- Filter block diagrams
  - Direct forms
  - Transposition
  - State space representation
- Precision issues: coefficient error, arithmetic error
  - cascaded biquads
- Allpass filters
  - first and second order sections
- Lattice filters
  - Arbitrary allpass response
  - Arbitrary IIR response by summing intermediate outputs

For further details see Mitra: 8.

MATLAB routines

residuez	$\frac{b(z^{-1})}{a(z^{-1})} \to \sum_k \frac{r_k}{1-p_k z^{-1}}$
tf2sos,sos2tf	$\frac{b(z^{-1})}{a(z^{-1})} \leftrightarrow \prod_l \frac{b_{0,l}+b_{1,l}z^{-1}+b_{2,l}z^{-2}}{1+a_{1,l}z^{-1}+a_{2,l}z^{-2}}$
zp2sos,sos2zp	$\{z_m, p_k, g\} \leftrightarrow \prod_l \frac{b_{0,l}+b_{1,l}z^{-1}+b_{2,l}z^{-2}}{1+a_{\in 1,l}z^{-1}+a_{2,l}z^{-2}}$
zp2ss,ss2zp	$\{z_m, p_k, g\} \leftrightarrow \begin{cases} x' = Ax + Bu \\ y = Cx + Du \end{cases}$
tf2ss,ss2tf	$\frac{b(z^{-1})}{a(z^{-1})} \leftrightarrow \begin{cases} x' = Ax + Bu \\ y = Cx + Du \end{cases}$
poly	$\text{poly}(\mathbf{A}) = \det(z\mathbf{I}-\mathbf{A})$