11: Multirate Systems

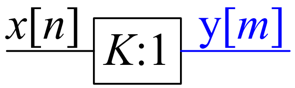
Multirate systems include more than one sample rate

Why bother?:

- May need to change the sample rate
  e.g. Audio sample rates include 32, 44.1, 48, 96 kHz
- Can relax analog or digital filter requirements
  e.g. Audio DAC increases sample rate so that the reconstruction filter can have a more gradual cutoff
- Reduce computational complexity
  FIR filter length $\propto \frac{f_s}{\Delta f}$ where Δf is width of transition band
  Lower $f_s \Rightarrow$ shorter filter + fewer samples $\Rightarrow$computation $\propto f_s^2$

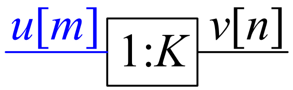
Building blocks

Downsample



$$y[m] = x[Km]$$

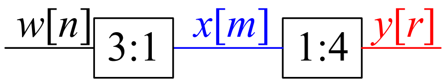
Upsample

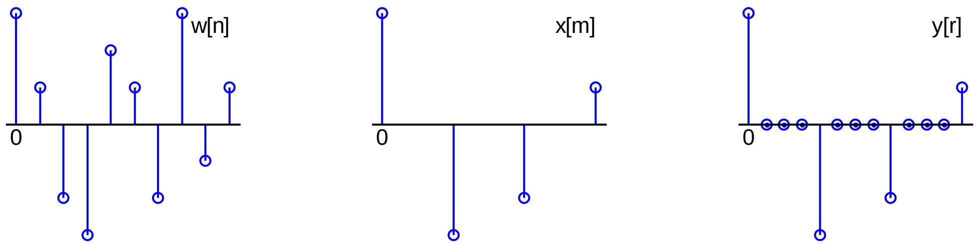


$$v[n] = \begin{cases} u\left[\frac{n}{K}\right] & K \mid n \\ 0 & \text{else} \end{cases}$$

Example:

Downsample by 3 then upsample by 4





- We use different index variables (n, m, r) for different sample rates
- Use different colours for signals at different rates (sometimes)
- Synchronization: all signals have a sample at $n = 0$.

Resampling Cascades

Successive downsamplers or upsamplers can be combined

$$\to \boxed{P{:}1} \to \boxed{Q{:}1} \to \;=\; \to \boxed{PQ{:}1} \to$$

$$\to \boxed{1{:}P} \to \boxed{1{:}Q} \to \;=\; \to \boxed{1{:}PQ} \to$$

Upsampling can be exactly inverted

$$\to \boxed{1{:}P} \to \boxed{P{:}1} \to \;=\; \longrightarrow$$

Downsampling destroys information permanently $\Rightarrow$ uninvertible

$$\to \boxed{P{:}1} \to \boxed{1{:}P} \to \;\neq\; \longrightarrow$$

Resampling can be interchanged iff P and Q are coprime (surprising!)

$$\xrightarrow{x} \boxed{P{:}1} \xrightarrow{w} \boxed{1{:}Q} \xrightarrow{y} \;=\; \xrightarrow{x} \boxed{1{:}Q} \xrightarrow{u} \boxed{P{:}1} \xrightarrow{v}$$

Proof: Left side: $y[n] = w\left[\frac{1}{Q}n\right] = x\left[\frac{P}{Q}n\right]$ if $Q \mid n$ else $y[n] = 0$.

Right side: $v[n] = u\left[Pn\right] = x\left[\frac{P}{Q}n\right]$ if $Q \mid Pn$.

But $\{Q \mid Pn \;\Rightarrow\; Q \mid n\}$ iff P and Q are coprime.

[Note: $a \mid b$ means "a divides into b exactly"]

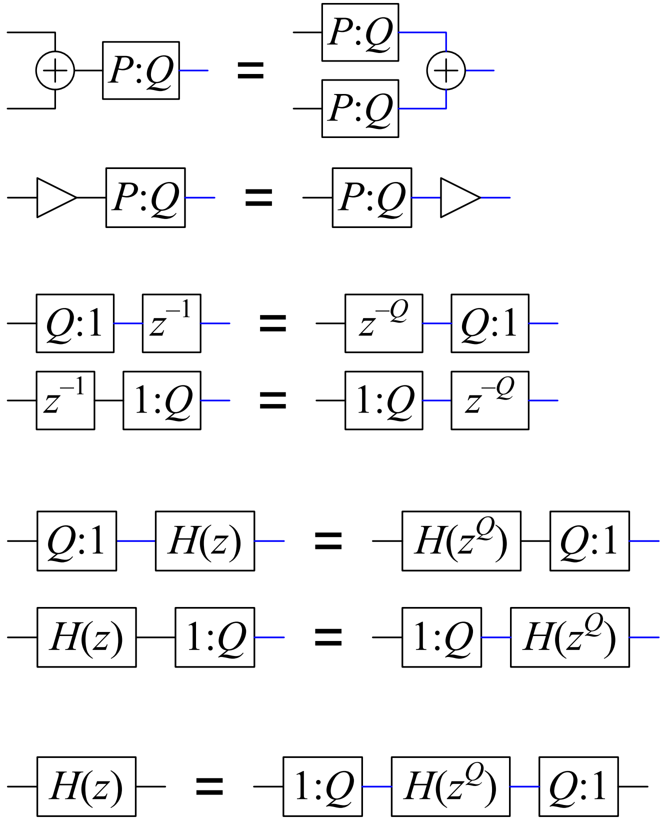
Noble Identities

Resamplers commute with addition and multiplication

Delays must be multiplied by the resampling ratio

Noble identities:
Exchange resamplers and filters

Corrollary



Example: $H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + \cdots$
$H(z^3) = h[0] + h[1]z^{-3} + h[2]z^{-6} + \cdots$

Noble Identities Proof

Define $h_Q[n]$ to be the impulse response of $H(z^Q)$.

$$x[n] \rightarrow \boxed{Q{:}1} \xrightarrow{u[r]} \boxed{H(z)} \xrightarrow{y[r]} \quad = \quad x[n] \rightarrow \boxed{H(z^Q)} \xrightarrow{v[n]} \boxed{Q{:}1} \xrightarrow{w[r]}$$

Assume that $h[r]$ is of length $M+1$ so that $h_Q[n]$ is of length $QM+1$.
We know that $h_Q[n]=0$ except when $Q \mid n$ and that $h[r]=h_Q[Qr]$.

$$\begin{aligned} w[r] = v[Qr] &= \sum_{s=0}^{QM} h_Q[s]x[Qr-s] \\ &= \sum_{m=0}^{M} h_Q[Qm]x[Qr-Qm] = \sum_{m=0}^{M} h[m]x[Q(r-m)] \\ &= \sum_{m=0}^{M} h[m]u[r-m] = y[r] \end{aligned}$$

☺

Upsampled Noble Identity:

$$x[r] \rightarrow \boxed{H(z)} \xrightarrow{u[r]} \boxed{1{:}Q} \xrightarrow{y[n]} \quad = \quad x[r] \rightarrow \boxed{1{:}Q} \xrightarrow{v[n]} \boxed{H(z^Q)} \xrightarrow{w[n]}$$

We know that $v[n]=0$ except when $Q \mid n$ and that $v[Qr]=x[r]$.

$$\begin{aligned} w[n] &= \sum_{s=0}^{QM} h_Q[s]v[n-s] = \sum_{m=0}^{M} h_Q[Qm]v[n-Qm] \\ &= \sum_{m=0}^{M} h[m]v[n-Qm] \end{aligned}$$

If $Q \nmid n$, then $v[n-Qm]=0 \ \forall m$ so $w[n]=0=y[n]$

If $Q \mid n = Qr$, then $w[Qr] = \sum_{m=0}^{M} h[m]v[Qr-Qm]$
$= \sum_{m=0}^{M} h[m]x[r-m] = u[r] = y[Qr]$

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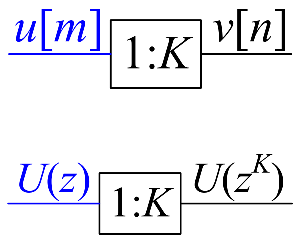
Upsampled z-transform

$$V(z) = \sum_n v[n] z^{-n} = \sum_{n \text{ s.t. } K|n} u[\tfrac{n}{K}] z^{-n}$$
$$= \sum_m u[m] z^{-Km} = U(z^K)$$



Spectrum: $V(e^{j\omega}) = U(e^{jK\omega})$

Spectrum is horizontally shrunk and replicated K times.

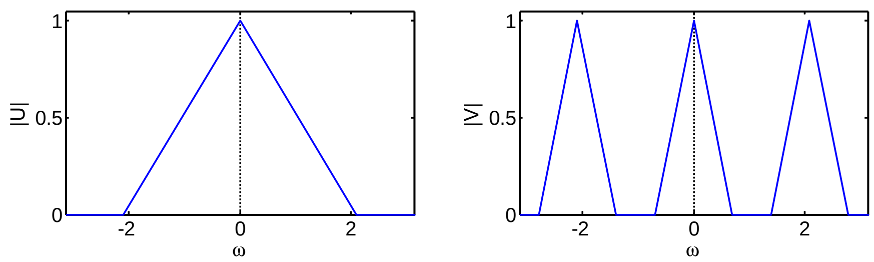
Total energy unchanged; power (= energy/sample) multiplied by $\frac{1}{K}$

Upsampling normally followed by a LP filter to remove images.

Example:

$K = 3$: three images of the original spectrum in all.

Energy unchanged: $\frac{1}{2\pi} \int \left|U(e^{j\omega})\right|^2 d\omega = \frac{1}{2\pi} \int \left|V(e^{j\omega})\right|^2 d\omega$



Downsampled z-transform

Define $c_K[n] = \delta_{K|n}[n] = \frac{1}{K}\sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}}$

$$x[n] \rightarrow \boxed{K{:}1} \xrightarrow{y[m]} \boxed{1{:}K} \rightarrow x_K[n]$$

Now define $x_K[n] = \begin{cases} x[n] & K \mid n \\ 0 & K \nmid n \end{cases} = c_K[n]x[n]$

$$X_K(z) = \sum_n x_K[n]z^{-n} = \frac{1}{K}\sum_n \sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}} x[n]z^{-n}$$
$$= \frac{1}{K}\sum_{k=0}^{K-1}\sum_n x[n]\left(e^{\frac{-j2\pi k}{K}}z\right)^{-n} = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}}z)$$

From previous slide:

$$X(z) \rightarrow \boxed{K{:}1} \rightarrow \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}}z^{\frac{1}{K}})$$

$$X_K(z) = Y(z^K)$$
$$\Rightarrow Y(z) = X_K(z^{\frac{1}{K}}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}}z^{\frac{1}{K}})$$

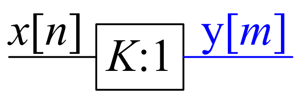
Frequency Spectrum:

$$Y(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{j(\omega-2\pi k)}{K}})$$
$$= \frac{1}{K}\left(X(e^{\frac{j\omega}{K}}) + X(e^{\frac{j\omega}{K}-\frac{2\pi}{K}}) + X(e^{\frac{j\omega}{K}-\frac{4\pi}{K}}) + \cdots\right)$$

Average of K aliased versions, each expanded in ω by a factor of K.
Downsampling is normally preceded by a LP filter to prevent aliasing.

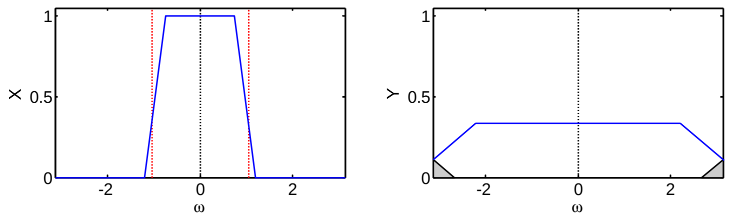
Downsampled Spectrum

$$Y(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{j(\omega-2\pi k)}{K}})$$



Example 1:

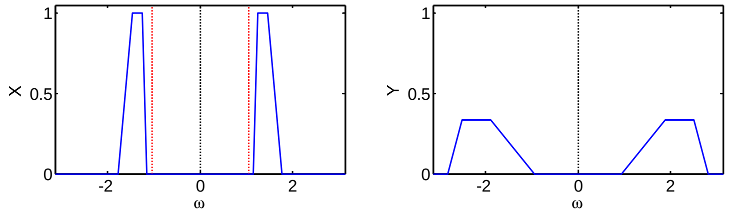
$K = 3$
Not quite limited to $\pm\frac{\pi}{K}$
Shaded region shows aliasing



Energy decreases: $\frac{1}{2\pi}\int\left|Y(e^{j\omega})\right|^2 d\omega \approx \frac{1}{K}\times\frac{1}{2\pi}\int\left|X(e^{j\omega})\right|^2 d\omega$

Example 2:

$K = 3$
Energy all in $\frac{\pi}{K} \le |\omega| < 2\frac{\pi}{K}$
No aliasing: 



No aliasing: If all energy is in $r\frac{\pi}{K} \le |\omega| < (r+1)\frac{\pi}{K}$ for some integer r
Normal case ($r = 0$): If all energy in $0 \le |\omega| \le \frac{\pi}{K}$

Downsampling: Total energy multiplied by $\approx \frac{1}{K}$ ($= \frac{1}{K}$ if no aliasing)
Average power $\approx$ unchanged ($=$ energy/sample)

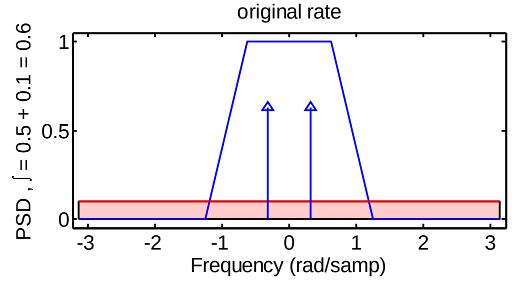
Example: Signal in $\omega \in \pm 0.4\pi$ + Tone @ $\omega = \pm 0.1\pi$ + White noise

Power = Energy/sample = Average PSD

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{PSD}(\omega)d\omega \ = 0.6$$

Component powers:

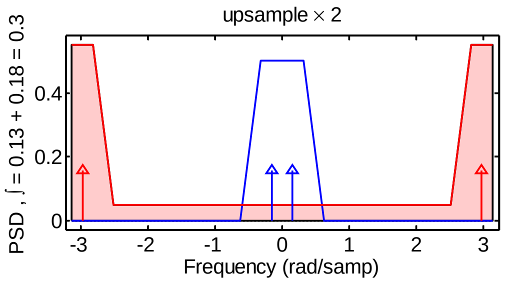
Signal $= 0.3$, Tone $= 0.2$, Noise $= 0.1$

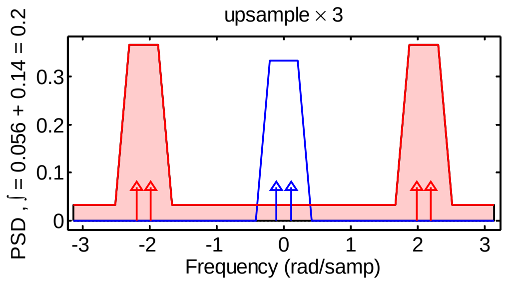


Upsampling:

Same energy per second

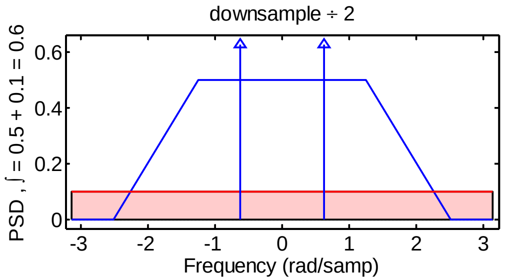
$\Rightarrow$ Power is $\div K$

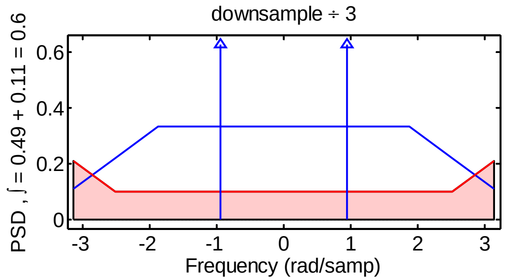




Downsampling:

Average power is unchanged.
$\exists$ aliasing in the $\div 3$ case.





[Power Spectral Density (1)]

The energy of a spectrum is $E_x = \sum_{-\infty}^{+\infty} |x[n]|^2$ and its power is $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{+N} |x[n]|^2$. The energy, E_x, is the total energy in all samples while the power, P_x, is the average energy per sample. If the finite signal $x_N[n]$ is defined as $x_N[n] = \begin{cases} x[n] & |n| \leq N \\ 0 & |n| > N \end{cases}$, then the power spectral density (PSD) is given by $S_{xx}\left(e^{j\omega}\right) = \lim_{N\to\infty} \frac{1}{2N+1} \left|X_N\left(e^{j\omega}\right)\right|^2$. From Parseval's theorem, P_x is the average value of $S_{xx}\left(e^{j\omega}\right)$ or, equivalently, $P_x = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{xx}\left(e^{j\omega}\right) d\omega$.

The signal on the previous slide has three components: (i) a signal component with a power of 0.3 and a trapezoidal PSD with a width of $\pm 0.4\pi$, (ii) a tonal component with a power of 0.2 whose PSD consists of two delta functions and (iii) a white noise component of power 0.1 whose PSD is constant at 0.1. The tonal component might arise from a time-domain waveform $\sqrt{0.4}\cos\left(0.1\pi n + \phi\right)$ where ϕ is arbitrary and does not affect the PSD.

Upsampling by K inserts additional zero-valued samples and so does not affect E_x but, since there are now K times as many samples, P_x is divided by K. The original periodic PSD is shrunk horozontally by a factor of K which means that there are now K images of the original PSD at spacings of $\Delta\omega = \frac{2\pi}{K}$. So, for example, when $K = 2$, the central trapezoidal component has a maximum height of 0.5 and a width of $\pm 0.2\pi$ and there is a second, identical, trapezoidal component shifted by $\Delta\omega = \frac{2\pi}{K} = \pi$. When K is an even number, one of the images will be centred on $\omega = \pi$ and so will wrap around from $+\pi$ to $-\pi$. The power of each image is multiplied by K^{-2} but, since there are K images, the total power is multiplied by K^{-1}. For the white noise, the images all overlap (and add in power), so the white noise PSD amplitude is multiplied by K^{-1}. Finally, the amplitudes of the delta functions are multiplied by K^{-2} so that the total power of all K images is multiplied by K^{-1}.

[Power Spectral Density (2)]

Downsampling by K deletes samples but leaves the average power of the remaining ones unchanged. Thus the total power of the downsampled spectra remains at 0.6. The downsampled PSD is the average of K shifted versions of the original PSD that have been expanded horizontally by a factor of K. The white noise component is the average of K identical expanded but attenuated versions of itself and so its PSD amplitude remains at 0.1. The power of a tonal components is unchanged and so its amplitude is also unchanged.

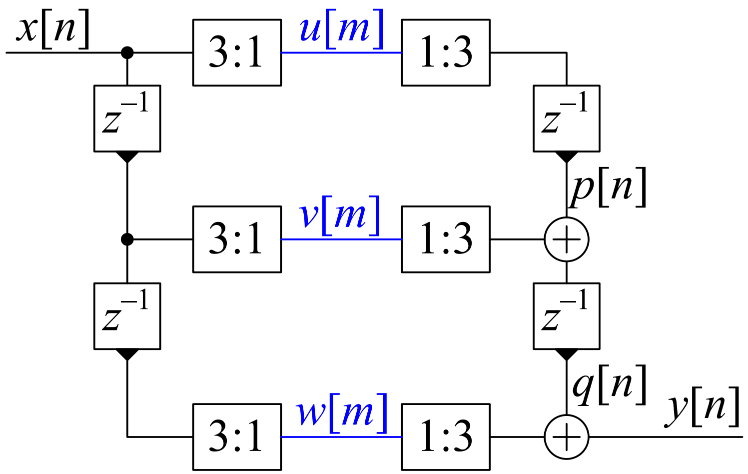
When downsampling by a factor of $K = 3$, the original width of the trapezoidal component expands from $\pm 0.4\pi$ to $\pm 1.2\pi$ which exceeds the $\pm\pi$ range of the graph. Thus, as ω approaches π, the PSD of the signal component is decreasing with ω but has not reached 0 at $\omega = \pi$. This portion of the trapezium wraps around to $\omega = -\pi$ and gives rise to the little triangle of additional noise in the range $-\pi < \omega < -0.8\pi$ where it adds onto the white noise component. In a similar way, the portion of the trapezium that overflows the left edge of the graph gives rise to additional noise at the right of the graph in the range $0.8\pi < \omega < \pi$.

Summary of Spectral Density Changes: Width $\times$ Height ($\times$Images)

Energy and Power Spectral Densities	Energy Spectral Density		Power Spectral Density	
	Up: $1:K$	Down: $K:1$	Up: $1:K$	Down: $K:1$
Alias-free block	$K^{-1} \times 1\,(\times K)$	$K \times K^{-2}$	$K^{-1} \times K^{-1}\,(\times K)$	$K \times K^{-1}$
Tone: $\delta(\omega - \omega_0)$	$1 \times K^{-1}\ (\times K)$	$1 \times K^{-1}$	$1 \times K^{-2}\ (\times K)$	1×1
White Noise	1×1	$1 \times K^{-1}$	$1 \times K^{-1}$	1×1
Integral $\int d\omega$	$\times 1$	$\approx \times K^{-1}$	$\times K^{-1}$	$\approx \times 1$

Perfect Reconstruction

$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$p[n]$	`-c--f--i--l`
$v[m]$	`b  e  h  k`
$q[n]$	`-bc-ef-hi-kl`
$w[m]$	`a  d  g  j`
$y[n]$	`abcdefghijkl`



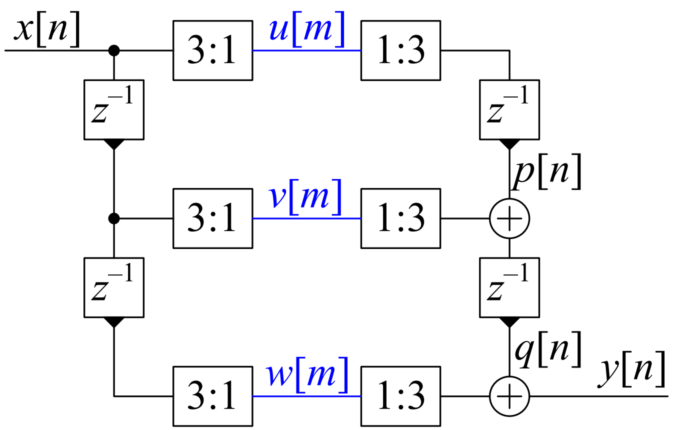
Input sequence $x[n]$ is split into three streams at $\frac{1}{3}$ the sample rate:

$$u[m] = x[3m],\ v[m] = x[3m-1],\ w[m] = x[3m-2]$$

Following upsampling, the streams are aligned by the delays and then added to give:

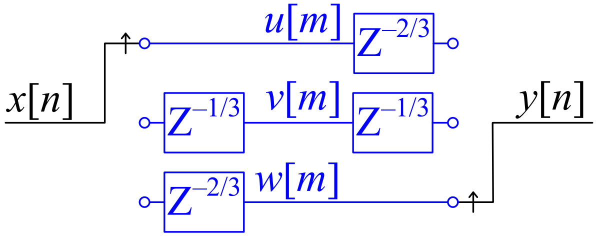
$$y[n] = x[n-2]$$

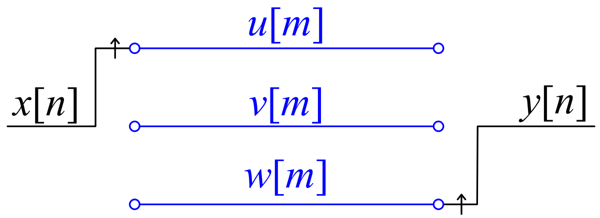
Perfect Reconstruction: output is a delayed scaled replica of the input



$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$v[m]$	`b  e  h  k`
$w[m]$	`a  d  g  j`
$v[m+\frac{1}{3}]$	`e  h  k  l`
$w[m+\frac{2}{3}]$	`d  g  j  m`
$y[n]$	`abcdefghijkl`

The combination of delays and downsamplers can be regarded as a commutator that distributes values in sequence to u, w and v.
Fractional delays, $z^{-\frac{1}{3}}$ and $z^{-\frac{2}{3}}$ are needed to synchronize the streams.
The output commutator takes values from the streams in sequence.
For clarity, we omit the fractional delays and regard each terminal, $\circ$, as holding its value until needed. Initial commutator position has zero delay.





The commutator direction is against the direction of the z^{-1} delays.

Summary

- Multirate Building Blocks
  - Upsample: $X(z) \overset{1:K}{\rightarrow} X(z^K)$
    Invertible, Inserts $K-1$ zeros between samples
    Shrinks and replicates spectrum
    Follow by LP filter to remove images
  - Downsample: $X(z) \overset{K:1}{\rightarrow} \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}} z^{\frac{1}{K}})$
    Destroys information and energy, keeps every K^{th} sample
    Expands and aliasses the spectrum
    Spectrum is the average of K aliased expanded versions
    Precede by LP filter to prevent aliases
- Equivalences
  - Noble Identities: $H(z) \longleftrightarrow H(z^K)$
  - Interchange $P:1$ and $1:Q$ iff $P$and Q coprime
- Commutators
  - Combine delays and down/up sampling

For further details see Mitra: 13.

MATLAB routines

resample	change sampling rate