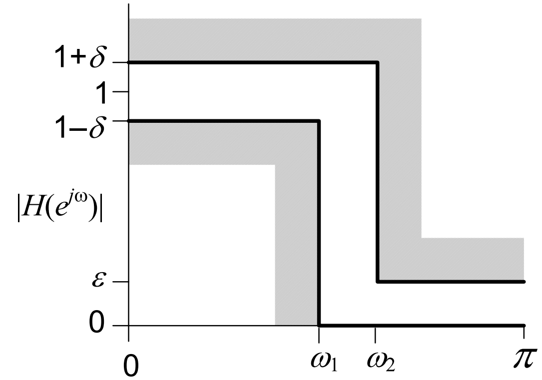
12: Polyphase Filters

Heavy Lowpass filtering

Filter Specification:

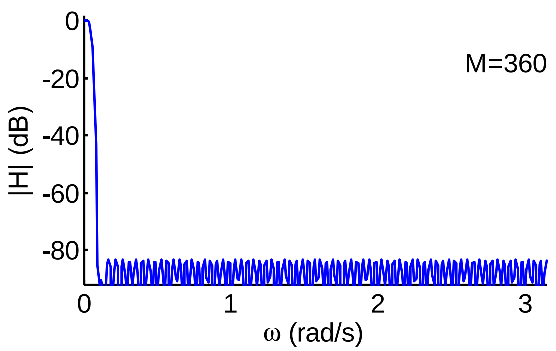
Sample Rate: 20 kHz
Passband edge: 100 Hz $(\omega_1 = 0.03)$
Stopband edge: 300 Hz $(\omega_2 = 0.09)$
Passband ripple: ± 0.05 dB $(\delta = 0.006)$
Stopband Gain: -80 dB $(\epsilon = 0.0001)$

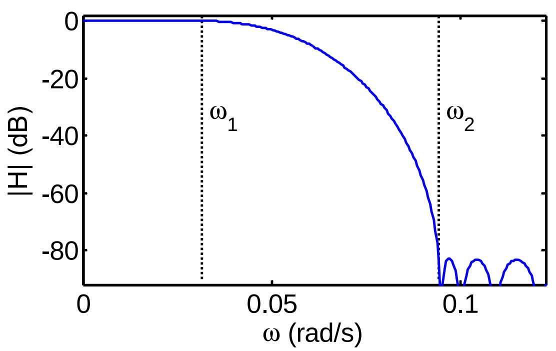


This is an extreme filter because the cutoff frequency is only 1% of the Nyquist frequency.

Symmetric FIR Filter:

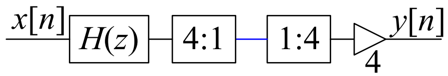
Design with Remez-exchange algorithm
Order = 360

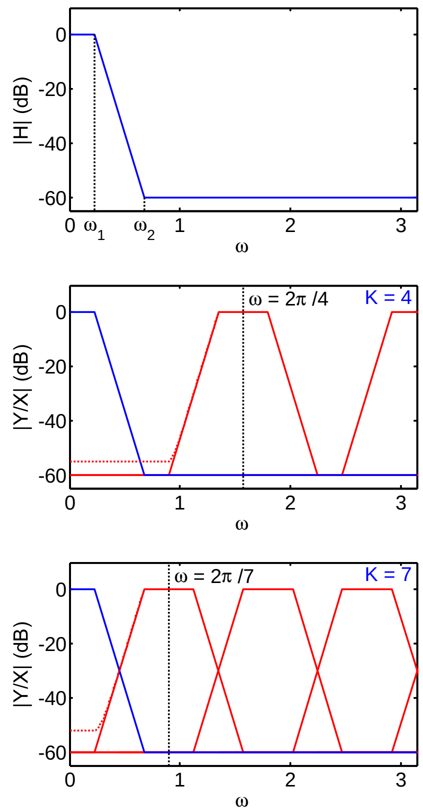




Maximum Decimation Frequency

If a filter passband occupies only a small fraction of $[0, \pi]$, we can downsample then upsample without losing information.





Downsample: aliased components at offsets of $\frac{2\pi}{K}$ are almost zero because of $H(z)$

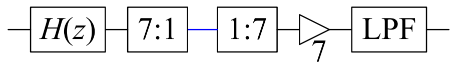
Upsample: Images spaced at $\frac{2\pi}{K}$ can be removed using another low pass filter

To avoid aliasing in the passband, we need

$$\frac{2\pi}{K} - \omega_2 \geq \omega_1 \quad \Rightarrow \quad K \leq \frac{2\pi}{\omega_1+\omega_2}$$

Centre of transition band must be $\leq$ intermediate Nyquist freq, $\frac{\pi}{K}$

We must add a lowpass filter to remove the images:



Passband noise = noise floor at output of $H(z)$ plus $10\log_{10}(K-1)$ dB.

Polyphase decomposition

For our filter: original Nyquist frequency $= 10$ kHz and transition band centre is at 200 Hz so we can use $K = 50$.

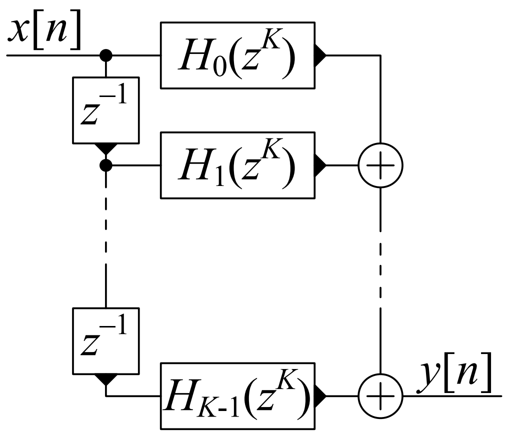
We will split $H(z)$ into K filters each of order $R-1$. For convenience, assume $M+1$ is a multiple of K (else zero-pad $h[n]$).

Example: $M = 399,\ K = 50 \Rightarrow R = \frac{M+1}{K} = 8$

$$
\begin{aligned}
H(z) &= \sum_{m=0}^{M} h[m] z^{-m} \\
&= \sum_{m=0}^{K-1} h[m] z^{-m} + \sum_{m=0}^{K-1} h[m+K] z^{-(m+K)} + \cdots \qquad [R \text{ terms}] \\
&= \sum_{r=0}^{R-1} \sum_{m=0}^{K-1} h[m+Kr] z^{-m-Kr} \\
&= \sum_{m=0}^{K-1} z^{-m} \sum_{r=0}^{R-1} h_m[r] z^{-Kr}
\end{aligned}
$$

where $h_m[r] = h[m+Kr]$

$$
= \sum_{m=0}^{K-1} z^{-m} H_m\left(z^K\right)
$$



Example: $M = 399,\ K = 50,\ R = 8$

$h_3[r] = [h[3],\ h[53],\ \cdots,\ h[303],\ h[353]]$

This is a polyphase implementation of the filter $H(z)$

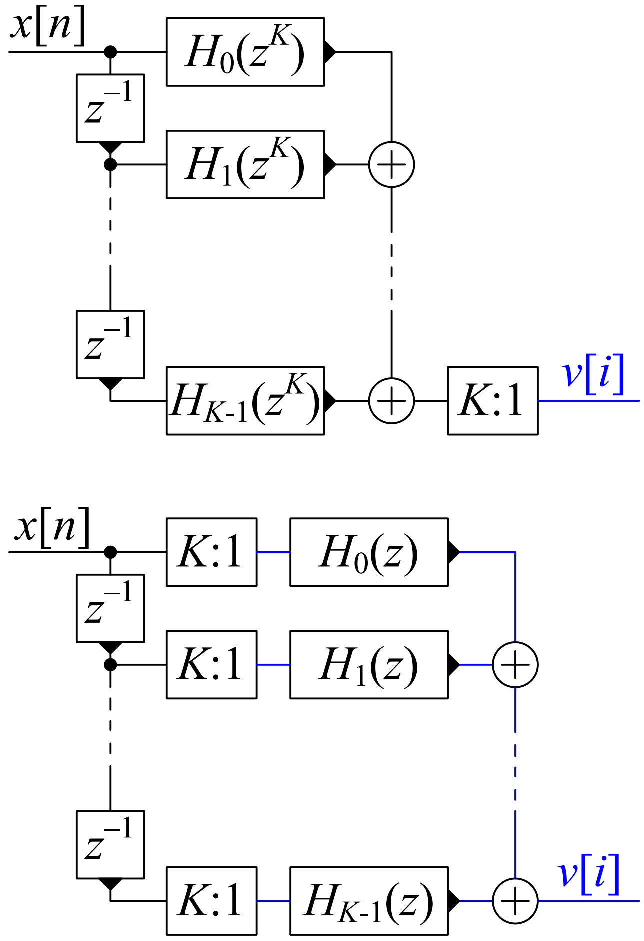
Downsampled Polyphase Filter

$H(z)$ is low pass so we downsample its output by K without aliasing.

The number of multiplications per input sample is $M + 1 = 400$.

Using the Noble identities, we can move the resampling back through the adders and filters. $H_m(z^K)$ turns into $H_m(z)$ at a lower sample rate.

We still perform 400 multiplications but now only once for every K input samples.



Multiplications per input sample $= 8$ (down by a factor of 50 ☺) but $v[n]$ has the wrong sample rate (☹).

Polyphase Upsampler

To restore sample rate: upsample and then lowpass filter to remove images

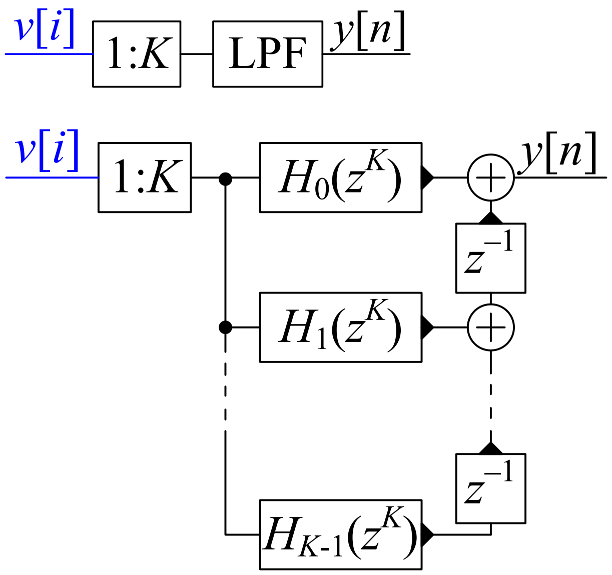
We can use the same lowpass filter, $H(z)$, in polyphase form:

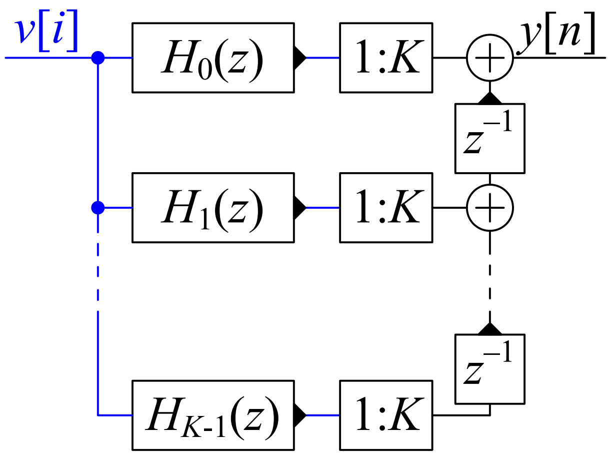
$$\sum_{m=0}^{K-1} z^{-m} \sum_{r=0}^{R-1} h_m[r] z^{-Kr}$$

This time we put the delay z^{-m} after the filters.
Multiplications per output sample $=$ 400

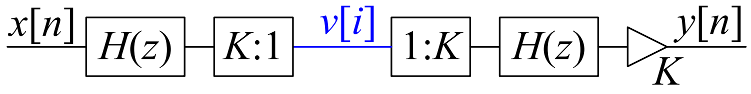
Using the Noble identities, we can move the resampling forwards through the filters. $H_m(z^K)$ turns into $H_m(z)$ at a lower sample rate.

Multiplications per output sample $= 8$ (down by a factor of 50 ☺).





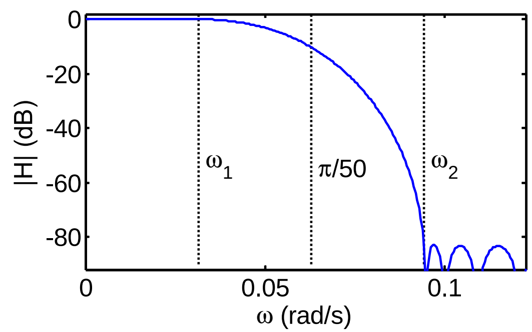
Complete Filter

The overall system implements: 

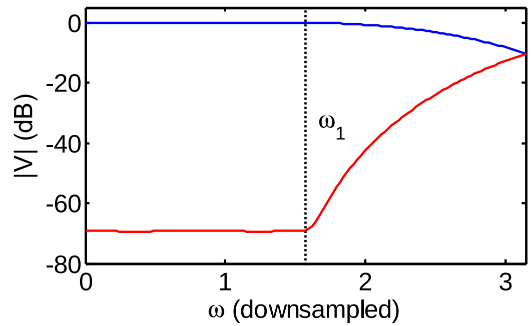
Need an extra gain of K to compensate for the downsampling energy loss.

Filtering at downsampled rate requires 16 multiplications per input sample (8 for each filter). Reduced by $\frac{K}{2}$ from the original 400.

$H(e^{j\omega})$ reaches -10 dB at the downsampler Nyquist frequency of $\frac{\pi}{K}$.
Spectral components $> \frac{\pi}{K}$ will be aliased down in frequency in $V(e^{j\omega})$.



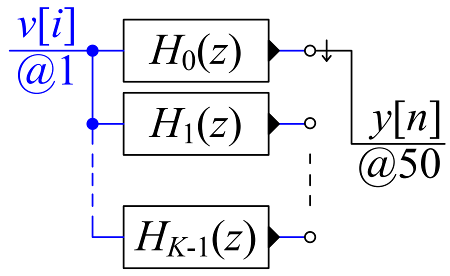
For $V(e^{j\omega})$, passband gain (blue curve) follows the same curve as $X(e^{j\omega})$.
Noise arises from K aliased spectral intervals.
Unit white noise in $X(e^{j\omega})$ gives passband noise floor at -69 dB (red curve) even though stop band ripple is below -83 dB (due to $K-1$ aliased stopband copies).



Upsampler Implementation

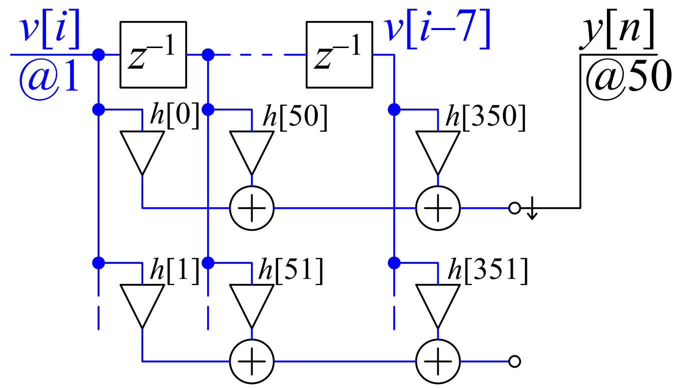
We can represent the upsampler compactly using a commutator. Sample $y[n]$ comes from $H_k(z)$ where $k = n \mod K$.

["@f" indicates the sample rate]

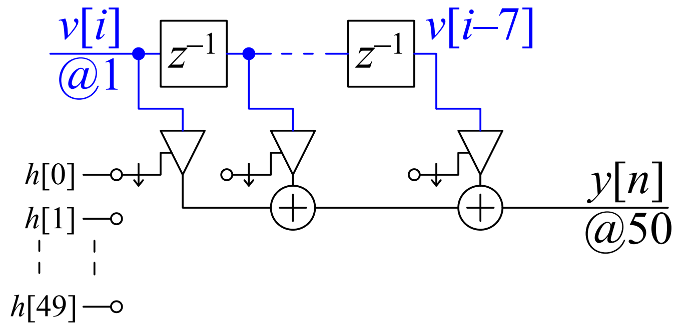


$H_0(z)$ comprises a sequence of 7 delays, 7 adders and 8 gains.

We can share the delays between all 50 filters.



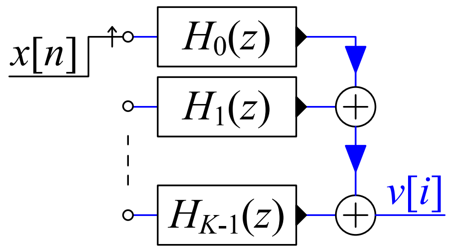
We can also share the gains and adders between all 50 filters and use commutators to switch the coefficients.



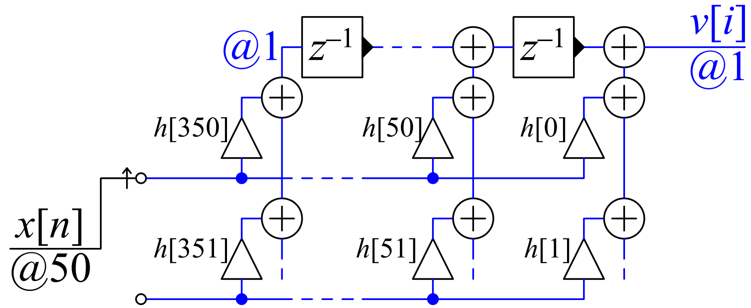
We now need 7 delays, 7 adders and 8 gains for the entire filter.

Downsampler Implementation

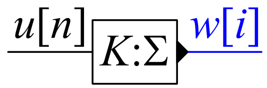
We can again use a commutator. The outputs from all 50 filters are added together to form $v[i]$.



We use the transposed form of $H_m(z)$ because this will allow us to share components.



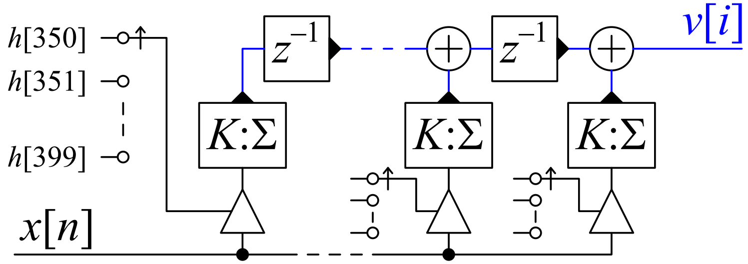
We can sum the outputs of the gain elements using an accumulator which sums blocks of K samples.



$w[i] = \sum_{r=0}^{K-1} u[Ki - r]$

Now we can share all the components and use commutators to switch the gain coefficients.

We need 7 delays, 7 adders, 8 gains and 8 accumulators in total.



Summary

- Filtering should be performed at the lowest possible sample rate
  - reduce filter computation by K
  - actual saving is only $\frac{K}{2}$ because you need a second filter
  - downsampled Nyquist frequency $\geq \max\left(\omega_{\text{passband}}\right) + \frac{\Delta\omega}{2}$
- Polyphase decomposition: split $H(z)$ as $\sum_{m=0}^{K-1} z^{-m} H_m(z^K)$
  - each $H_m(z^K)$ can operate on subsampled data
  - combine the filtering and down/up sampling
- Noise floor is higher because it arises from K spectral intervals that are aliased together by the downsampling.
- Share components between the K filters
  - multiplier gain coefficients switch at the original sampling rate
  - need a new component: accumulator/downsampler $(K : \Sigma)$

For further details see Harris 5.