13: Resampling Filters

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Suppose we want to change the sample rate while preserving information: e.g. Audio 44.1 kHz↔48 kHz↔96 kHz
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**Downsample:**
LPF to new Nyquist bandwidth: \( \omega_0 = \frac{\pi}{K} \)

\[
\begin{array}{c}
x[n] \xrightarrow{\text{LPF}} \frac{\pi}{K} \xrightarrow{\text{K:1}} y[i]
\end{array}
\]
Resampling

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e.g. Audio 44.1 kHz ↔ 48 kHz ↔ 96 kHz

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**Rational ratio:** $f_s \times \frac{P}{Q}$
LPF to lower of old and new Nyquist bandwidths: $\omega_0 = \frac{\pi}{\max(P,Q)}$
Suppose we want to change the sample rate while preserving information: e.g. Audio $44.1\ \text{kHz} \leftrightarrow 48\ \text{kHz} \leftrightarrow 96\ \text{kHz}$

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| $x[n]$ | LPF | K:1 | $y[i]$ |

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- Filter order \( M \approx \frac{d}{3.5\Delta\omega} \) where \( d \) is stopband attenuation in dB and \( \Delta\omega \) is the transition bandwidth (Remez-exchange estimate).
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  e.g. $\alpha = 0.05 \Rightarrow M \approx \frac{dK}{7\pi\alpha} = 0.9dK \quad \text{(where } \omega_0 = \frac{\pi}{K})$
If $K = 2$ then the new Nyquist frequency is $\omega_0 = \frac{\pi}{2}$. 
Halfband Filters

If $K = 2$ then the new Nyquist frequency is
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We multiply ideal response $\frac{\sin \omega_0 n}{\pi n}$ by a Kaiser window.
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If \( 4 \mid M \) and we make the filter causal (\( \times z^{-\frac{M}{2}} \)),
\[
H(z) = 0.5z^{-\frac{M}{2}} + z^{-1} \sum_{r=0}^{\frac{M}{2}-1} h_1[r]z^{-2r}
\]
where \( h_1[r] = h[2r + 1 - \frac{M}{2}] \)
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$H_1(z)$ is symmetrical with $\frac{M}{2}$ coefficients so we need $\frac{M}{4}$ multipliers in total (input gain of 0.5 can usually be absorbed elsewhere).
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\( H_1(z) \) is symmetrical with \( \frac{M}{2} \) coefficients so we need \( \frac{M}{4} \) multipliers in total (input gain of 0.5 can usually be absorbed elsewhere).

Computation: \( \frac{M}{4} \) multiplies per input sample
Dyadic 1:8 Upsampler

Suppose $X(z)$: BW = $0.8\pi \Leftrightarrow \alpha = 0.2$
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Upsample 1:2 $\rightarrow U(z)$:
Suppose $X(z): \text{BW} = 0.8\pi \iff \alpha = 0.2$

**Upsample 1:2 $\rightarrow U(z)$:**

Filter $H_P(z)$ must remove image: $\Delta \omega = 0.2\pi$
Dyadic 1:8 Upsampler

Suppose $X(z) \colon \text{BW} = 0.8\pi \iff \alpha = 0.2$

Upsample $1:2 \rightarrow U(z)$:

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For attenuation = 60 dB, $P \approx \frac{60}{3.5\Delta \omega} = 27.3$
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Round up to a multiple of 4: $P = 28$
Dyadic 1:8 Upsampler

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Upsample 1:2 → $U(z)$:
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Upsample 1:2 → $V(z)$: $\Delta\omega = 0.6\pi \Rightarrow Q = 12$
Suppose $X(z)$: $\text{BW} = 0.8\pi \iff \alpha = 0.2$

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**Upsample 1:2 $\rightarrow V(z)$:** $\Delta\omega = 0.6\pi \Rightarrow Q = 12$

**Upsample 1:2 $\rightarrow Y(z)$:** $\Delta\omega = 0.8\pi \Rightarrow R = 8$
Dyadic 1:8 Upsampler

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[diminishing returns + higher sample rate]
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Multiplication Count:
$\left(1 + \frac{P}{4}\right) \times f_x + \frac{Q}{4} \times 2f_x + \frac{R}{4} \times 4f_x = 22f_x$
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**Alternative approach using direct 1:8 upsampling:**
$\Delta\omega = 0.05\pi \Rightarrow M = 110 \Rightarrow 111f_x$ multiplications (using polyphase)
Rational Resampling

To resample by \( \frac{P}{Q} \) do 1:\( P \) then LPF, then Q:1.

\[
\begin{align*}
\frac{x[n]}{@f_x} & \quad 1:3 \quad H(z) \quad \frac{y[s]}{@f_y} \\
& \quad \quad 5:1 \quad \frac{y[i]}{@f_y}
\end{align*}
\]
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$$x[n] \quad \times \times \times \times \times \times \times \times$$

$$v[s] \quad \triangle \triangle \triangle \triangle \triangle \triangle \triangle \triangle \triangle$$

Resample by $\frac{P}{Q} \Rightarrow \omega_0 = \frac{\pi}{\max(P, Q)}$
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Resample by \( \frac{P}{Q} \Rightarrow \omega_0 = \frac{\pi}{\max(P, Q)} \)

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\Delta\omega \triangleq 2\alpha\omega_0 = \frac{2\alpha\pi}{\max(P, Q)}
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To resample by $\frac{P}{Q}$ do $1:P$ then LPF, then $Q:1$.

$x[n] \quad \frac{1:3}{@f_x} \quad H(z) \quad \frac{5:1}{\frac{y[i]}{@f_y}}$

$\frac{x[n]}{H_p(z)} \frac{5:1}{\frac{y[i]}{@f_y}}$

$h_0[r] \quad \frac{3}{@} \quad r=0:R$

$h_1[r] \quad \frac{3}{@} \quad r=0:R$

$h_2[r] \quad \frac{3}{@} \quad r=0:R$
### Rational Resampling

To resample by $\frac{P}{Q}$ do 1:$P$ then LPF, then Q:1.

<table>
<thead>
<tr>
<th>$x[n]$</th>
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</tr>
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| $v[s]$ | ◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊◊ sho
Rational Resampling

To resample by $\frac{P}{Q}$ do 1:$P$ then LPF, then Q:1.

$x[n] \rightarrow H(z) \rightarrow v[s] \rightarrow y[i]$

Resample by $\frac{P}{Q} \Rightarrow \omega_0 = \frac{\pi}{\max(P, Q)}$

$\Delta \omega \triangleq 2\alpha \omega_0 = \frac{2\alpha \pi}{\max(P, Q)}$

Polyphase: $H(z) = \sum_{p=0}^{P-1} z^{-p} H_p(z^P)$

Commutate coefficients:

$v[s]$ uses $H_p(z)$ with $p = s \mod P$

Keep only every $Q^{th}$ output:

$x[n] \rightarrow H_p(z) \rightarrow v[s] \rightarrow y[i]$

$h_0[r] \rightarrow$

$h_1[r] \rightarrow @3$

$h_2[r] \rightarrow$

r=0:$R$

$y[i] \rightarrow @^3/5$

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$$\frac{x[n]}{f_x} \xrightarrow[]{1:3} H(z) \xrightarrow[]{\frac{P}{Q}} \frac{v[s]}{f_y} \xrightarrow[]{5:1} \frac{y[i]}{f_y}$$

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Multiplication Count:
$H(z): M + 1 \approx \frac{60 \text{ [dB]}}{3.5\alpha \Delta \omega} = \frac{2.7 \max(P,Q)}{\alpha}$
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$M + 1$ coefficients in all
Rational Resampling

To resample by \( \frac{P}{Q} \) do 1: \( \frac{P}{Q} \) then LPF, then \( Q:1 \).

\[
\begin{align*}
x[n] & \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times \\
v[s] & \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \\
y[i] & \quad \diamond \quad \triangle \quad \circ \quad \circ \quad \diamond \quad \triangle
\end{align*}
\]

Resample by \( \frac{P}{Q} \) \( \Rightarrow \) \( \omega_0 = \frac{\pi}{\max(P, Q)} \)

\( \Delta \omega \triangleq 2\alpha \omega_0 = \frac{2\alpha \pi}{\max(P, Q)} \)

Polyphase: \( H(z) = \sum_{p=0}^{P-1} z^{-p} H_p(z^P) \)

Commutate coefficients:
- \( v[s] \) uses \( H_p(z) \) with \( p = s \mod P \)
- Keep only every \( Q^\text{th} \) output:
- \( y[i] \) uses \( H_p(z) \) with \( p = Qi \mod P \)

Multiplication Count:
- \( H(z): M + 1 \approx \frac{60 \text{ [dB]}}{3.5 \Delta \omega} = \frac{2.7 \max(P, Q)}{\alpha} \)
- \( H_p(z): R + 1 = \frac{M+1}{P} = \frac{2.7}{\alpha} \max \left( 1, \frac{Q}{P} \right) \)

\( M + 1 \) coefficients in all
Rational Resampling

To resample by \( \frac{P}{Q} \) do 1:\( P \) then LPF, then \( Q:1 \).

### Resampling

\[ x[n] \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times \]

\[ v[s] \quad \bigstar \bigstar \bigstar \bigstar \bigstar \bigstar \bigstar \bigstar \bigstar \bigstar \bigstar \bigstar \]

\[ y[i] \quad \bigcirc \quad \bigdiamond \quad \bigtriangleup \quad \bigcirc \quad \bigdiamond \quad \bigtriangleup \]

Resample by \( \frac{P}{Q} \) \( \Rightarrow \) \( \omega_0 = \frac{\pi}{\max(P, Q)} \)

\[ \Delta \omega \triangleq 2\alpha \omega_0 = \frac{2\alpha \pi}{\max(P, Q)} \]

**Polyphase:** \( H(z) = \sum_{p=0}^{P-1} z^{-p} H_p(z^P) \)

Commutate coefficients:

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Keep only every \( Q \)th output:

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\( H_p(z): R + 1 = \frac{M+1}{P} = \frac{2.7}{\alpha} \max \left( 1, \frac{Q}{P} \right) \)

\( M + 1 \) coefficients in all

**Multiplication rate:** \( \frac{2.7}{\alpha} \max \left( 1, \frac{Q}{P} \right) \times f_y \)
Rational Resampling

To resample by $\frac{P}{Q}$ do 1:$P$ then LPF, then $Q$:1.

$$x[n] \xrightarrow{1:3} H(z) \xrightarrow{5:1} y[i] \xrightarrow{3/5}$$

Resample by $\frac{P}{Q} \Rightarrow \omega_0 = \frac{\pi}{\max(P, Q)}$

$$\Delta \omega = 2\alpha \omega_0 = \frac{2\alpha \pi}{\max(P, Q)}$$

Polyphase: $H(z) = \sum_{p=0}^{P-1} z^{-p} H_p(z^P)$

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$v[s]$ uses $H_p(z)$ with $p = s \mod P$

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$M + 1$ coefficients in all

Multiplication rate: $\frac{2.7}{\alpha} \max \left( 1, \frac{Q}{P} \right) \times f_y = \frac{2.7}{\alpha} \max \left( f_y, f_x \right)$
Arbitrary Resampling

Sometimes need very large $P$ and $Q$:

e.g. $\frac{44.1 \text{ kHz}}{48 \text{ kHz}} = \frac{147}{160}$
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**Multiplication rate OK:** $\frac{2.7 \max(f_y, f_x)}{\alpha}$
Arbitrary Resampling

Sometimes need very large $P$ and $Q$:

*e.g.* \( \frac{44.1 \text{ kHz}}{48 \text{ kHz}} = \frac{147}{160} \)

**Multiplication rate OK:**

\[
\frac{2.7 \max(f_y, f_x)}{2.7 \max(P, Q)}
\]

**However # coefficients:**

\[
\frac{2.7 \max(P, Q)}{P, Q}
\]
Arbitrary Resampling

Sometimes need very large $P$ and $Q$:

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Alternatively, use any large integer $P$ and round down to the nearest sample:

- E.g. for $y[i]$ at time $i\frac{Q}{P}$ use $h_p[r]$

  where $p = (\lfloor iQ \rfloor) \mod P$
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Equivalent to converting to analog with zero-order hold and resampling at $f_y = \frac{P}{Q}$. 
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Equivalent to converting to analog with zero-order hold and resampling at $f_y = \frac{P}{Q}$.

Zero-order hold convolves with rectangular $\frac{1}{P}$-wide window $\Rightarrow$ multiplies periodic spectrum by $\frac{\sin \frac{\Omega}{2P}}{\Omega/2P}$. 

1) Upsample $\@ P$
2) LPF to $\min(\pi, \pi/P)$
3) Zero-order hold
Arbitrary Resampling

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Equivalent to converting to analog with zero-order hold and resampling at $f_y = \frac{P}{Q}$.

Zero-order hold convolves with rectangular $\frac{1}{P}$-wide window $\Rightarrow$ multiplies periodic spectrum by $\frac{\sin \frac{\Omega P}{2}}{\Omega P}$. Resampling aliases $\Omega$ to $\Omega \mod \frac{2P\pi}{Q}$. 

1) Upsample @ $P$
2) LPF to $\min(\pi, \pi P/Q)$
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Arbitrary Resampling

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Equivalent to converting to analog with zero-order hold and resampling at $f_y = \frac{P}{Q}$.

Zero-order hold convolves with rectangular $\frac{1}{P}$-wide window $\Rightarrow$ multiplies periodic spectrum by $\sin \frac{\Omega}{2P}$. Resampling aliases $\Omega$ to $\Omega \mod \frac{2P\pi}{Q}$.

Unit power component at $\Omega_1$ gives alias components with total power:

$$\sin^2 \frac{\Omega_1}{2P} \sum_{n=1}^{\infty} \left( \frac{2P}{2nP\pi + \Omega_1} \right)^2 + \left( \frac{2P}{2nP\pi - \Omega_1} \right)^2 \approx \frac{\omega_1^2}{4P^2} \frac{2\pi^2}{6\pi^2} = \frac{\Omega_1^2}{12P^2}$$
Arbitrary Resampling

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- e.g. $\frac{44.1 \text{ kHz}}{48 \text{ kHz}} = \frac{147}{160}$

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Equivalent to converting to analog with zero-order hold and resampling at $f_y = \frac{P}{Q}$.

Zero-order hold convolves with rectangular $\frac{1}{P}$-wide window $\Rightarrow$ multiplies periodic spectrum by $\frac{\sin \left( \frac{\Omega Q}{2P} \right)}{\frac{\Omega Q}{2P}}$. Resampling aliases $\Omega$ to $\Omega \mod \frac{2P\pi}{Q}$.

Unit power component at $\Omega_1$ gives alias components with total power:

$$\sin^2 \frac{\Omega_1}{2P} \sum_{n=1}^{\infty} \left( \frac{2P}{2nP\pi + \Omega_1} \right)^2 + \left( \frac{2P}{2nP\pi - \Omega_1} \right)^2 \approx \frac{\omega_1^2}{4P^2} \frac{2\pi^2}{6\pi^2} = \frac{\Omega_1^2}{12P^2}$$

For worst case, $\Omega_1 = \pi$, need $P = 906$ to get $-60$ dB ☹️
Suppose $P = 50$ and $H(z)$ has order $M = 249$. $H(z)$ is lowpass filter with $\omega_0 \approx \frac{\pi}{50}$. 

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\[ \omega_0 \approx \frac{\pi}{50} \]
Suppose $P = 50$ and $H(z)$ has order $M = 249$. $H(z)$ is lowpass filter with $\omega_0 \approx \frac{\pi}{50}$.

Split into 50 filters of length $R + 1 = \frac{M+1}{P} = 5$:

$h_p[0]$ is the first $P$ samples of $h[m]$. 
Suppose \( P = 50 \) and \( H(z) \) has order \( M = 249 \). 

\( H(z) \) is lowpass filter with \( \omega_0 \approx \frac{\pi}{50} \)

Split into 50 filters of length \( R + 1 = \frac{M+1}{P} = 5 \):

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Suppose $P = 50$ and $H(z)$ has order $M = 249$. $H(z)$ is lowpass filter with $\omega_0 \approx \frac{\pi}{50}$.

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- $h_p[0]$ is the first $P$ samples of $h[m]$
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- $h_p[r] = h[p + rP]$
Suppose $P = 50$ and $H(z)$ has order $M = 249$.

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Use a polynomial of order $L$ to approximate each segment:

\[
h_p[r] \approx f_r\left(\frac{P}{P}\right) \quad \text{with} \quad 0 \leq \frac{P}{P} < 1
\]
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$h[m]$ is smooth, so errors are low. E.g. error $< 10^{-3}$ for $L = 4$.
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- Resultant filter almost as good
Polynomial Approximation

Suppose $P = 50$ and $H(z)$ has order $M = 249$. $H(z)$ is a lowpass filter with $\omega_0 \approx \frac{\pi}{50}$.

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- Resultant filter almost as good
- Instead of $M + 1 = 250$ coefficients we only need $(R + 1)(L + 1) = 25$

\[\frac{\beta}{\omega^2 + \beta^2} = \frac{H_p(z)}{z^{\text{order}} + \text{other terms}}\]
Polynomial Approximation

Suppose $P = 50$ and $H(z)$ has order $M = 249$. $H(z)$ is a lowpass filter with $\omega_0 \approx \frac{\pi}{50}$. Split into 50 filters of length $R + 1 = \frac{M+1}{P} = 5$:

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Use a polynomial of order $L$ to approximate each segment:

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- Resultant filter almost as good
- Instead of $M + 1 = 250$ coefficients we only need

$$(R + 1)(L + 1) = 25$$

where

$$R + 1 = \frac{2.7}{\alpha} \max \left(1, \frac{Q}{P}\right)$$
Filter coefficients depend on fractional part of $i \frac{Q}{P}$:

$$\Delta[i] = i \frac{Q}{P} - n$$

where $n = \left\lfloor i \frac{Q}{P} \right\rfloor$.

$$R + 1 = \frac{M+1}{P} = 5$$

$$x[n] \xrightarrow[@P/Q]{H_{\Delta}(z)} y[i]$$

$$\Delta[i] [f_0(\Delta), \ldots, f_R(\Delta)]$$
Filter coefficients depend on fractional part of $i \frac{Q}{P}$:

$$\Delta[i] = i \frac{Q}{P} - n$$

where $n = \lfloor i \frac{Q}{P} \rfloor$

$$y[i] = \sum_{r=0}^{R} f_r(\Delta[i]) x[n - r]$$

$$R + 1 = \frac{M+1}{P} = 5$$
Farrow Filter

Filter coefficients depend on fractional part of $i \frac{Q}{P}$:

$$\Delta [i] = i \frac{Q}{P} - n \text{ where } n = \left\lfloor i \frac{Q}{P} \right\rfloor$$

$$y[i] = \sum_{r=0}^{R} f_r(\Delta[i]) x[n - r]$$

where $f_r(\Delta) = \sum_{l=0}^{L} b_l[r] \Delta^l$

$R + 1 = \frac{M+1}{P} = 5$

$H_{\Delta}(z) \uparrow_{P/Q}^P [f_0(\Delta), \ldots, f_R(\Delta)]$
Filter coefficients depend on fractional part of $i \frac{Q}{P}$:

\[ \Delta[i] = i \frac{Q}{P} - n \quad \text{where} \quad n = \left\lfloor i \frac{Q}{P} \right\rfloor \]

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$$R + 1 = \frac{M+1}{P} = 5$$

$$x[n] \rightarrow 1 \frac{H_{\Delta(z)}}{P/Q} y[i]$$

$$\Delta[i] \rightarrow \frac{f_0(\Delta), \ldots, f_R(\Delta)}{P/Q}$$
Filter coefficients depend on fractional part of $i\frac{Q}{P}$:

\[ \Delta[i] = i\frac{Q}{P} - n \text{ where } n = \left\lfloor i\frac{Q}{P} \right\rfloor \]

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where \( f_r(\Delta) = \sum_{l=0}^{L} b_l[r]\Delta^l \)

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$$= \sum_{l=0}^{L} \Delta[i]^l v_l[n]$$

where $v_l[n] = b_l[n] \ast x[n]$
Farrow Filter

Filter coefficients depend on fractional part of $i \frac{Q}{P}$:

$$\Delta[i] = i \frac{Q}{P} - n \text{ where } n = \lfloor i \frac{Q}{P} \rfloor$$

$$y[i] = \sum_{r=0}^{R} f_r(\Delta[i]) x[n - r]$$

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where $$v_l[n] = b_l[n] \ast x[n]$$

Horner’s Rule:

$$y[i] = v_0[n] + \Delta (v_1[n] + \Delta (v_2[n] + \Delta (\cdots )))$$

$$R + 1 = \frac{M+1}{P} = 5$$

$$x[n] \star_{\frac{P}{Q}} H_\Delta(z) \frac{y[i]}{\star_{\frac{P}{Q}}}$$

$$\Delta[i] \star_{\frac{P}{Q}} [f_0(\Delta), \ldots, f_R(\Delta)]$$

Horner’s Rule:

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Multiplication Rate:

Each $B_l(z)$ needs $R + 1$ per input sample

Horner needs $L$ per output sample
### Farrow Filter

Filter coefficients depend on fractional part of $i \frac{Q}{P}$:

$$\Delta[i] = i \frac{Q}{P} - n$$

where $n = \left\lfloor i \frac{Q}{P} \right\rfloor$

$$y[i] = \sum_{r=0}^{R} f_r(\Delta[i])x[n-r]$$

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where $v_l[n] = b_l[n] \ast x[n]$

Horner's Rule:

$$y[i] = v_0[n] + \Delta (v_1[n] + \Delta (v_2[n] + \Delta (\cdots )))$$

Multiplication Rate:

Each $B_l(z)$ needs $R + 1$ per input sample

Horner needs $L$ per output sample

Total: $(L + 1)(R + 1)f_x + Lf_y = \frac{2.7(L+1)}{\alpha} \max \left(1, \frac{f_x}{f_y}\right)f_x + Lf_y$
Farrow Filter

Filter coefficients depend on fractional part of \( i \frac{Q}{P} \):
\[
\Delta[i] = i \frac{Q}{P} - n \text{ where } n = \left\lfloor i \frac{Q}{P} \right\rfloor
\]
\[
y[i] = \sum_{r=0}^{R} f_r(\Delta[i]) x[n - r]
\]
where \( f_r(\Delta) = \sum_{l=0}^{L} b_l[r] \Delta^l \)
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\[
= \sum_{l=0}^{L} \Delta[i]^l \sum_{r=0}^{R} b_l[r] x[n - r]
\]
where \( v_l[n] = b_l[n] \ast x[n] \)
[like a Taylor series expansion]

Horner’s Rule:
\[
y[i] = v_0[n] + \Delta (v_1[n] + \Delta (v_2[n] + \Delta (\cdots)))
\]

Multiplication Rate:
Each \( B_l(z) \) needs \( R + 1 \) per input sample
Horner needs \( L \) per output sample
Total: \((L + 1)(R + 1) f_x + L f_y = \frac{2.7(L+1)}{\alpha} \max \left(1, \frac{f_x}{f_y}\right) f_x + L f_y\)}}
Transition band centre at $\omega_0$

- $\omega_0$ = the lower of the old and new Nyquist frequencies
- Transition width = $\Delta \omega = 2\alpha \omega_0$, typically $\alpha \approx 0.1$
Summary

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  - approximate filter impulse response with polynomial segments
  - arbitrary, time-varying, resampling ratios
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For further details see Mitra: 13 and Harris: 7, 8.
# MATLAB routines

<table>
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<tr>
<th>Function</th>
<th>Description</th>
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<tbody>
<tr>
<td>gcd(p,q)</td>
<td>Find $\alpha p + \beta q = 1$ for coprime $p$, $q$</td>
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<tr>
<td>polyfit</td>
<td>Fit a polynomial to data</td>
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<tr>
<td>polyval</td>
<td>Evaluate a polynomial</td>
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<tr>
<td>upfirdn</td>
<td>Perform polyphase filtering</td>
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<tr>
<td>resample</td>
<td>Perform polyphase resampling</td>
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