Paper Number(s): **E4.12**
AS1
SC1
ISE4.7

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

MSc and EEE/ISE PART IV: M.Eng. and ACGI

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Friday, 4 May 10:00 am

There are FIVE questions on this paper.

Answer THREE questions.

Time allowed: 3:00 hours

Corrected Copy

Examiners: Constantinides,A.G. and Stathaki,T.

Special instructions for invigilators: None

Information for candidates: None

1. A real lowpass transfer function $G(z)$ of impulse response $\{h(n)\}$ is given as the sum of two allpass transfer functions in the form

$$G(z) = A_1(z^2) + z^{-1}A_2(z^2)$$

where on the unit circle $z = e^{j\theta}$

$$A_1(e^{j2\theta}) = e^{j\phi_1(\theta)} \text{ and } A_2(e^{j2\theta}) = e^{j(\theta+\phi_2(\theta))}.$$

The allpass transfer functions are to be designed such that $\left|G(e^{j\theta})\right|$ is equiripple both in the passband and stopband, where

$$2-\varepsilon_1 \le \left|G(e^{j\theta})\right| \le 2 \text{ within } -\theta_c \le \theta \le \theta_c \text{ (Passband)}$$

$$0 \le \left|G(e^{j\theta})\right| \le \varepsilon_2 \text{ within } \theta_c \le |\theta| \le \pi \text{ (Stopband).}$$

The ripple widths ε_1 and ε_2 are small. θ_c is the cutoff frequency.

i) Determine an expression for $\left|G(e^{j\theta})\right|$ in terms of $\Delta\phi = \left|\phi_1(\theta) - \phi_2(\theta)\right|$. [5]

ii) Find the maximum passband and the minimum stopband values of $\Delta\phi$ in [7]
terms of ε_1 and ε_2 respectively. [7]

iii) Show that $G_1(z) = A_1(z^2) - z^{-1}A_2(z^2)$ is a highpass transfer function. [6]

iv) Express the cutoff frequency and impulse response of the highpass filter in [4]
terms of the corresponding lowpass parameters. [4]

2. Define the normalised group delay $\tau(\theta)$ of a discrete time system of transfer function $H(z)$. Derive the relationship [8]

$$\tau(\theta) = -\operatorname{Im}\left[\frac{d}{d\theta}\left(\ln H(e^{j\theta})\right)\right].$$

Let the transfer function of a real allpass system of order m be given by [5]

$$H(z) = \prod_{i=1}^{m} A_i(z)$$

where $A_i(z) = \left(\frac{1-\alpha_i^* z}{z-\alpha_i}\right)$, $\alpha_i = \rho_i e^{j\psi_i}$ and $|\rho_i| < 1$.

i) Show that the phase response of $A_i(z)$ is given by [10]

$$\arg(A_i(e^{j\theta})) = -\theta - 2\arctan\frac{\rho_i \sin(\theta-\psi_i)}{1-\rho_i\cos(\theta-\psi_i)},$$

ii) Show that [6]

$$\arg(A_i(e^{j0})) - \arg(A_i(e^{j\pi})) = \pi.$$

iii) Hence, or otherwise, determine the value of the integral for the allpass system [4]

$$\int_0^{\pi} \tau(\theta)d\theta.$$

3 Explain what is meant by terms *computational complexity* and *twiddle factors* in the context of evaluating the Discrete Fourier Transform (DFT). Derive the computational complexity of a N-point DFT. [6]

It is given that $N = N_1 N_2$ with N_1 and N_2 co-prime. On the data array $\{x(n)\}$, $0 \le n \le N-1$, it is proposed to carry out the following 1-D to 2-D mapping

$$n = \langle An_1 + Bn_2 \rangle_N \quad \begin{Bmatrix} 0 \le n_1 \le N_1 - 1 \\ 0 \le n_2 \le N_2 - 1 \end{Bmatrix}$$

$$k = \langle Ck_1 + Dk_2 \rangle_N \quad \begin{Bmatrix} 0 \le k_1 \le N_1 - 1 \\ 0 \le k_2 \le N_2 - 1 \end{Bmatrix}$$

where $\langle M \rangle_N$ means a reduction of the number M modulo N.

Derive the conditions that must prevail on the products AC, BD, AD, and BC in order that all possible twiddle factors in the 2-D DFT computation are eliminated. [12]

Show that the following set of parameters satisfies these conditions

$$A = N_2,\ B = N_1,\ C = N_2 \langle N_2^{-1} \rangle_{N_1},\ D = N_1 \langle N_1^{-1} \rangle_{N_2}$$

where $\langle L^{-1} \rangle_P$ denotes the multiplicative inverse of L evaluated modulo P. [12]

Hence outline the algorithm for the computation of the N-point DFT. [3]

4. Consider an ideal linear-phase lowpass digital filter transfer function $H(z)$. On the unit circle $z = e^{j\theta}$, $H(z)$ takes the values

$$H(e^{j\theta}) = \begin{cases} e^{jM\theta} & -\frac{\pi}{M} \le \theta \le \frac{\pi}{M} \\ 0 & elsewhere \end{cases}$$ [4]

where M is a positive integer.

Sketch the amplitude response of $H(e^{j(\theta - \frac{2\pi}{M}r)})$ for $r = 0$, $r = 1$ and $r = 2$.

Hence show that the frequency response shown below is allpass. [7]

$$G(e^{j\theta}) = \sum_{r=0}^{M-1} H(e^{j(\theta - \frac{2\pi}{M}r)}).$$

Let $H(z)$ be expressed as

$$H(z) = \sum_{r=0}^{M-1} z^{-r} H_r(z^M)$$

where $H_r(z)$ are some appropriate subfilter transfer functions. [9]

By replacing z by $ze^{-j\frac{2\pi}{M}k}$ in the expression above for $H(z)$ and summing over k, or otherwise, show that the subfilter transfer function $H_0(z^M)$ is given by the expression [6]

$$H_0(e^{jM\theta}) = \frac{1}{M} \sum_{r=0}^{M-1} H(e^{j(\theta - \frac{2\pi}{M}r)}).$$ [4]

What is the amplitude response of $H_0(z^M)$? [3]

5 The signal flow graph of an oversampling A/D converter is shown in Figure 2. The connecting block S is a two-input, single-output linear system described by $V = \alpha X + \beta U$ where α and β are appropriate transfer functions. The block labelled $Q[\bullet]$ is a bipolar one-bit quantiser, which introduces quantisation noise Q as indicated.

By making appropriate assumptions derive an expression for the output Y in terms of X, α and β and the quantisation noise Q. Comment on the validity of your assumptions in practice. [10]

In a specific realisation it is required that a) the output has real unity gain with respect to the input and b) the noise shaping transfer function is $F(z)$.

Show that under these conditions $\alpha = \dfrac{1}{F(z)}$ and $\beta = \dfrac{F(z)-1}{F(z)}$. [7]

Give an account of the factors that influence the choice for $F(z)$. [8]

Draw the signal flow graph of the interconnecting block **S** when $F(z) = (1 - z^{-1})$ and reduce it to a form that contains only one accumulator. [8]

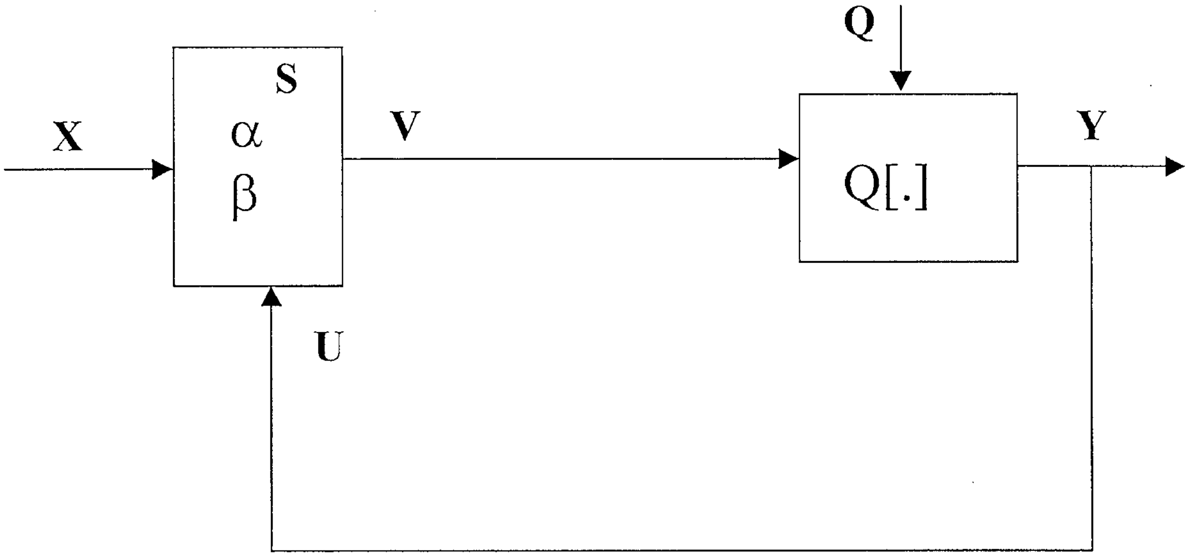


Figure 2

$Y = Q + \frac{1}{F} X + \frac{F-1}{F} Y$

$\left(1 - \frac{F-1}{F}\right) Y = Q + \frac{1}{F} X$

$Y = Q + \frac{1}{F} X$

D.S.P + DIGITAL FILTERS

1. On $z=e^{j\theta}$

(i) $G(e^{j\theta}) = e^{j\phi_1(\theta)} + e^{-j\theta} \cdot e^{j[\phi_2(\theta)+\theta]}$

$$= 2e^{j[\phi_1(\theta)+\phi_2(\theta)]/2} \cdot \cos[\phi_1(\theta)-\phi_2(\theta)]/2$$

or

$$|G(e^{j\theta})| = 2|\cos \Delta\phi/2| > 0$$

5

(ii) In the passband, min $|G(e^{j\theta})|$ corresponds to max $\Delta\phi$ & since all quantities are +ve

$$2-\epsilon_1 = 2\cos\frac{\Delta\phi_{max}}{2}$$

or $\Delta\phi_{max} = 2\cos^{-1}(1-\epsilon_1/2)$ 7

In stopband, max $|G(e^{j\theta})|$ corresponds to min $\Delta\phi$

i.e. $\epsilon_2 = 2\cos\Delta\phi_{min}/2$

or $\Delta\phi_{min} = 2\cos^{-1}\epsilon_2/2$ 7

(iii) Either from frequency transformation $z \leftarrow -z$ (LP to HP)

or $G_1(z)\big|_c = G(-z)\big|_c = e^{j\phi_1(\theta)} - e^{j\phi_2(\theta)}$

hence

$$|G_1(z)| = 2\left|\sin\frac{\Delta\phi}{2}\right| = 2\left|\cos\left(\frac{\pi-\Delta\phi}{2}\right)\right.$$

6

(iv) i.e. passband and stopband regions interchanged hence $G_1(z)$ is highpass with cutoff frequency $\pi-\theta_c$ 4

Impulse response follows from $G(z)=\sum_n h(z) z^{-n}$ and

$$G_1(z) = \sum_n h(z)(-1)^n z^{-n} \Rightarrow \{(-1)^n h(n)\}$$

4

33

Normalised Group delay $\tau(\theta) = -\frac{d\phi(\theta)}{d\theta}$

$\phi(\theta)$ = phase response

Let

$$H(z)|_c = H(e^{j\theta}) = A(\theta)e^{j\phi(\theta)}$$

then $\ln H(e^{j\theta}) = \ln A(\theta) + j\phi(\theta)$

and

$$\frac{d}{d\theta}\ln H(e^{j\theta}) = \frac{d}{d\theta}A(\theta) + j\frac{d\phi(\theta)}{d\theta} = \frac{d}{d\theta}\ln A(\theta) - j\tau(\theta)$$

Hence

$$\tau(\theta) = -\mathrm{Im}\left[\frac{d}{d\theta}\ln H(e^{j\theta})\right]$$

8

$A_i(z) = \frac{1-\alpha_i^* z}{z-\alpha_i}$ and hence $A_1(e^{j\theta}) = e^{-j\theta}\cdot\frac{1-\alpha^* e^{j\theta}}{1-\alpha_i e^{-j\theta}}$

or $A_i(e^{j\theta}) = e^{-j\theta}\cdot[1-\rho_i e^{j(\theta-\psi_i)}]/[1-\rho_i e^{-j(\theta-\psi_i)}]$ 5

$\mathrm{Arg}\,[1-\rho_i e^{j(\theta-\psi_i)}] = -\tan^{-1}\frac{\rho_i\sin(\theta-\psi_i)}{1-\rho_i\cos(\theta-\psi_i)} = -\mathrm{Arg}[1-\rho_i e^{-j(\theta-\psi_i)}]$

i.e. $\mathrm{Arg}\,A_i(e^{j\theta}) = -\theta - 2\tan^{-1}\rho_i\sin(\theta-\psi_i)/((1-\rho_i\cos(\theta-\psi_i))$

and hence

$\mathrm{Arg}\,A_i(e^{j0}) = \pi$

10

$$\tau(\theta) = -\frac{d\phi(\theta)}{d\theta}$$

$$\int_0^{\pi}\tau(\theta)d\theta = -\int_0^{\pi}\frac{d\phi}{d\theta}d\theta = \phi(0) - \phi(\pi)$$ 6

$= m\pi$

or if the bright student remember his complex variable theory

$$\oint \frac{d}{dz}\ln H(z)\,.\,dz = \oint_c\left(\sum_i\frac{-\alpha_i}{1-\alpha_i^* z} - \sum_i\frac{1}{z-\alpha_i}\right)dz$$ 4

$= m.2\pi j$

etc.

33.

⑧

3 Computational complexity in DFT is taken to be the total number of complex multiplications required to compute the DFT. (Sometimes the implicit symmetry is taken into consideration to reduce this by 2)

Twiddle factors are phasing factors in the form $\exp(-j \frac{2\pi}{N_i} k_i n_j)$ between stages that modify partial computations in a multi-stage DFT evaluation.

For N-point DFT there are N complex mults per point producing a total of $O(N^2)$ mults 6

For $n = \langle An_1 + Bn_2 \rangle_N$ $k = \langle Ck_1 + Dk_2 \rangle_N$, $N = N_1 N_2$ we have

$$X(k) = X(\langle Ck_1 + Dk_2 \rangle_N) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(\langle An_1 + Bn_2 \rangle_N) W_N^P$$

where $P = (An_1 + Bn_2)(Ck_1 + Dk_2)$, $W_N = e^{-j2\pi/N}$ $= K$

$$K = W_N^{ACn_1k_1} \cdot W_N^{ADn_1k_2} \cdot W_N^{BDn_2k_2} \cdot W_N^{BCn_2k_1}$$

For the complete removal of twiddle factors 12

$$K = W_{N_1}^{n_1k_1} \cdot W_{N_2}^{n_2k_2}$$

Hence

$\langle AD \rangle_N = 0$ $\langle BC \rangle_N = 0$ $\langle AC \rangle_N = N_2$

$\langle BD \rangle_N = N_1$

Let $\langle N_1^{-1} \rangle_{N_2} = \alpha$ ie $\langle N_1 \alpha \rangle_{N_2} = 1$ $\alpha N_1 = \beta N_2 + 1$

$\langle N_2^{-1} \rangle_{N_1} = \gamma$ $\gamma N_2 = \delta N_1 + 1$

Then $\langle AC \rangle_N = \langle N_2 (\delta N_1 + 1) \rangle_{N_1} = N_2$

$\langle BD \rangle_N = \langle N_1 (\beta N_2 + 1) \rangle_N = N_1$

$\langle AD \rangle_N = \langle N_1 N_2 \langle N_1^{-1} \rangle_{N_2} \rangle_N = 0$ ie. multiple $N_1 N_2$ 12

and similarly with $\langle BC \rangle_N$

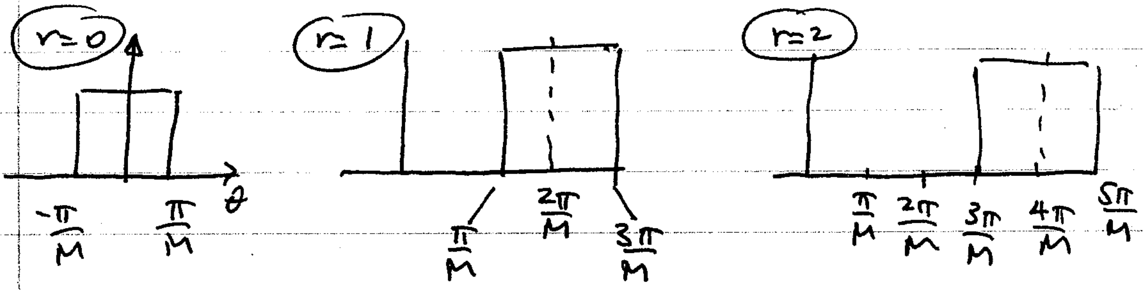
The algorithm maps 1-D array $\{x(n)\}$ to 2-D array $\{x \langle An_1 + Bn_2 \rangle_N\}$ & carries line-by-line followed column-by-column N_1- & N_2-point DFTs respectively (or v.v.) and returns frequency samples to $\{X \langle Ck_1 + Dk_2 \rangle_N\}$ 3

33

④

4. $|H(e^{j\theta})| = 1$ in $|\theta| \le \pi/M$

$= 0$ elsewhere hence $|H(e^{j(\theta - \frac{r2\pi}{M})})| = 1$



4

Now

$$G(e^{j\theta}) = \sum_{r=0}^{n-1} H(e^{j(\theta - \frac{2\pi}{M}\cdot r)}) = \sum_{r=0}^{M-1} e^{jM(\theta - \frac{2\pi}{M}\cdot r)} = e^{jM\theta}.M$$ 7

i.e. $|G(e^{j\theta})| = M$ a constant or allpass.

Let $H(e^{j\theta}) = \sum_{k=0}^{M-1} H_k(e^{jM\theta})e^{-jk\theta}$

On replacing $e^{j\theta}$ by $e^{j(\theta - \frac{2\pi}{M}k)}$ we have

$$H(e^{j(\theta - \frac{2\pi}{M}k)}) = \sum_{r=0}^{M-1} H_r(e^{jM\theta}).e^{-jr\theta}.e^{jr\frac{2\pi}{M}.k}$$

Hence $\sum_{k=0}^{M-1} H(e^{j(\theta - \frac{2\pi}{M}k)}) = \sum_{r=0}^{M-1} H_r(e^{jm\theta})e^{-jr\theta}\sum_{k=0}^{n-1} e^{jr\frac{2\pi}{M}k}$ 9

The last summation is zero for $r \ne 0$ & equal to M for $r=0$ 6

ie

$$H_0(e^{jm\theta}) = \frac{1}{M}\sum_{k=0}^{M-1} H(e^{j(\theta - \frac{2\pi}{M}k)})$$ 7

In view of the above $H_0(e^{j\theta})$ must be all pass

33

⑤

5. Assumptions

1) High sampling rate

2) quantisation model is linear

ie $x \rightarrow \boxed{Q(\cdot)} \overset{\downarrow q}{} \rightarrow y \;\equiv\; x \rightarrow \overset{\downarrow q}{\oplus} \rightarrow x+q$.

3) Loop is computable ie. contains at least one delay element

4) Loop is stable, no poles outside $|z|=1$

5) Computational latencies are negigible

Comment:

1) Realisable within reasonable upper limits

2) Oversimplification for linear analysis

3) Easily achievable

4) Can be made to be so

5) Can produce instability if latencies are appreciable

From the figure $\alpha X + \beta Y + Q = Y$

or $Y = \frac{\alpha}{1-\beta} X + \frac{1}{1-\beta} Q$ 10

Requirement (a) imposes $\alpha = (1-\beta) \rightarrow F(z) = \frac{1}{1-\beta} = \frac{1}{\alpha}$

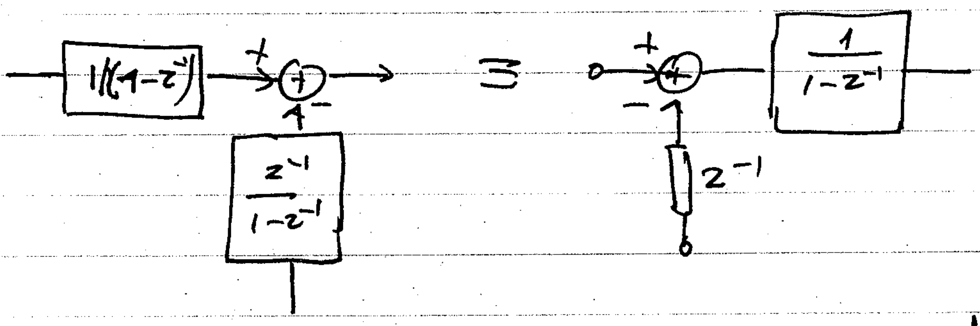
and $\beta = (F(z)-1)/F(z)$ 7

Noise shaping filter F(z) is such that the quantisation noise spectrum at Y is of low amplitude within the signal BW. Then noise can be attenuated by post filtering Y.

If X is lowpass then F(z) is highpass & post filtering Y is lowpass. 8

For $F(z) = 1 - z^{-1}$, $\alpha = \frac{1}{1-z^{-1}}$, $\beta = -\frac{z^{-1}}{1-z^{-1}}$

and hence



8

33