IMPERIAL COLLEGE LONDON

EE4-12
EE9AO2
EE9SC1

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

MSc and EEE/ISE PART IV: MEng and ACGI

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Tuesday, 1 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer Question 1 and any TWO other questions

Question 1 is worth 40% of the marks and other questions are worth 30%

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : D.M. Brookes

Second Marker(s) : M.K. Gurcan

[Corrected]

Information for Candidates:

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms. the signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.

Abbreviations

BIBO	Bounded Input, Bounded Output
CTFT	Continuous-time Fourier Transform
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
DTFT	Discrete-Time Fourier Transform
LTI	Linear Time-Invariant
MDCT	Modified Discrete Cosine Transform
SNR	Signal-to-Noise Ratio

Notation

- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ \dots \ , x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{condition}[n] = 1$ whenever "$condition$" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Convolution

- $v[n] = x[n] * y[n] \ \Rightarrow \ v[n] = \sum_{r=-\infty}^{\infty} x[r]y[n-r]$
- $v[n] = x[n] \circledast_N y[n] \ \Rightarrow \ v[n] = \sum_{r=0}^{N-1} x[r]y[(n-r)_{\text{mod } N}]$

Forward and Inverse Transforms

CTFT	$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$
DTFT	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$	$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$
DFT	$X[k] = \sum_{0}^{N-1} x[n]e^{-j2\pi\frac{kn}{N}}$	$x[n] = \frac{1}{N}\sum_{0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$
DCT	$X[k] = \sum_{n=0}^{N-1} x[n]\cos\frac{2\pi(2n+1)k}{4N}$	$x[n] = \frac{1}{N}X[0] + \frac{2}{N}\sum_{k=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$
MDCT	$X[k] = \sum_{n=0}^{2N-1} x[n]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$
z	$X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$	$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

(1) $M \approx \frac{a}{3.5\Delta\omega}$

(2) $M \approx \frac{a-8}{2.2\Delta\omega}$

(3) $M \approx \frac{a-20\log_{10} b-1.2}{4.6\Delta\omega}$

where $a =$ stop band attenuation in dB, $b =$ peak-to-peak passband ripple in dB and $\Delta\omega =$ width of smallest transition band in normalized rad/s.

1. (a) The sequences $x[n]$ and $y[n]$ are of length N and M respectively with $M \leq N$. $x[n]$ is zero outside the range $0 \leq n < N$ and $y[n]$ is zero outside the range $0 \leq n < M$. Consider the convolutions (defined in the formula sheet)

$$v[n] = x[n] * y[n]$$

$$w[n] = x[n] \circledast_N y[n]$$

(i) State the values of n for which $v[n] = w[n]$ is necessarily true. [3]

(ii) If $x[n] = [5, 2, 3, 4, 7]$ and $y[n] = [1, 2, 3]$ determine the values of $v[1]$ and $w[1]$. [2]

(b) (i) Explain what is meant by saying that a linear time invariant system is "BIBO stable". [2]

(ii) Prove that if a linear time invariant system is BIBO stable, then its impulse response, $h[n]$, satisfies $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$. [3]

(c) The DTFT of a normalized rectangular window of length $M + 1$ samples is given by $W\left(e^{j\omega}\right) = \frac{\sin 0.5(M+1)\omega}{(M+1)\sin 0.5\omega}$.

(i) Determine the smallest positive value of ω for which $W\left(e^{j\omega}\right) = 0$. [1]

(ii) Determine the value of $20\log_{10}\left|W(e^{j\omega})\right|$ at $\omega = 0$. [2]

(iii) The value of $20\log_{10}\left|W(e^{j\omega})\right|$ at $\omega = \frac{3\pi}{M+1}$ under the assumption that $\sin 0.5\omega \approx 0.5\omega$ for sufficiently large M. [2]

(d) A filter impulse response $h[n]$ is of length $M + 1$ and satisfies the symmetry condition $h[M - n] = h[n]$ for $0 \leq n \leq M$.

(i) Show that if M is odd, the frequency response $H(e^{j\omega})$ may be written

$$H\left(e^{j\omega}\right) = Ae^{j\theta(\omega)} \sum_{n=0}^{\frac{M-1}{2}} h[n] \cos\left(n - \frac{M}{2}\right)\omega$$

and determine expressions for A and $\theta(\omega)$. [4]

(ii) Determine an expression for the group delay of the filter:

$$\tau_H\left(e^{j\omega}\right) = -\frac{d\angle H\left(e^{j\omega}\right)}{d\omega}.$$ [1]

(e) *Figure 1.1* shows the channel selection portion of a digital receiver in which complex signals are shown as bold lines.

The real-valued input signal, $x[r]$, at a sample frequency of 80 MHz, is multiplied by a complex exponential, $s[r]$, to shift the centre frequency of the desired channel to 0 Hz. The resultant complex signal is lowpass filtered by $H(z)$ and downsampled to 400 kHz.

(i) If the desired channel has a centre frequency of 19.1 MHz, give an expression for $s[r]$ as a function of r. [2]

(ii) Each channel in $x[r]$ has a bandwidth of 150 kHz. Calculate the widths of the passband and transition band of the filter $H(z)$ as normalized angular frequencies (i.e. $0 \le \omega \le \pi$). [3]

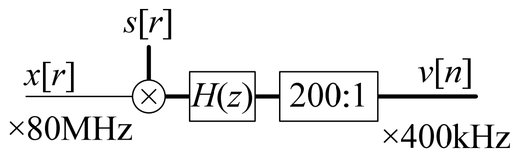


Figure 1.1

(f) *Figure 1.2* shows a decimator followed by a filter which has the transfer function $H(z) = 1 + 2z^{-1}$.

(i) Determine the transfer function $G(z)$ so that the system of *Figure 1.3* is equivalent to that of *Figure 1.2*. [2]

(ii) If the input signal is $x[n] = u[n]$, the unit step function, determine $v[n]$ for $0 \le n \le 6$ and $y[r]$ for $0 \le r \le 4$. [3]

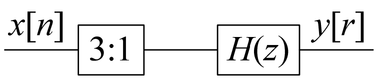


Figure 1.2

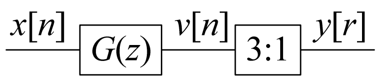


Figure 1.3

(g) *Figure 1.4* shows the power spectrum of a signal $x[n]$ in which signal components at different frequencies are uncorrelated. The signal is both upsampled and downsampled by the system shown in *Figure 1.5*.

On separate graphs draw dimensioned sketches of the two-sided power spectra of (i) $v[m]$ and (ii) $w[r]$. [5]

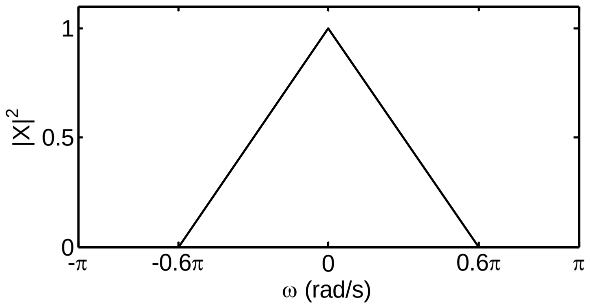


Figure 1.4

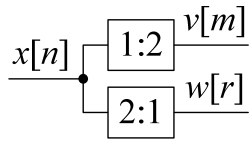


Figure 1.5

(h) In *Figure 1.6* the transfer function $H(z) = 1 + z^{-1}$. Determine the FIR transfer function $G(z)$ so that the overall transfer function is $\frac{Y(z)}{X(z)} = \alpha z^{-1}$ where α is a real-valued constant. [5]

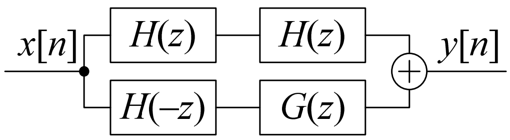


Figure 1.6

2. (a) $w[n]$ is a symmetric window of length 101 where $-50 \le n \le 50$. *Figure 2.1* shows the DTFT, $W(e^{j\omega})$. The axes scales are deliberately incomplete.

A symmetric lowpass FIR filter, $H(z)$, with a cutoff frequency of 1 rad/s is formed by multiplying $w[n]$ by the impulse response of an ideal filter. *Figure 2.2* covers a range of frequencies in the region of $\omega = 1$ and shows $|H(e^{j\omega})|$ in decibels as a solid line and the response of the ideal filter as a dotted line.

The integral $\frac{1}{2\pi}\int_0^\omega W(e^{j\omega})d\omega$ is shown in *Figure 2.3* and has a limiting value of 0.5 for large ω.

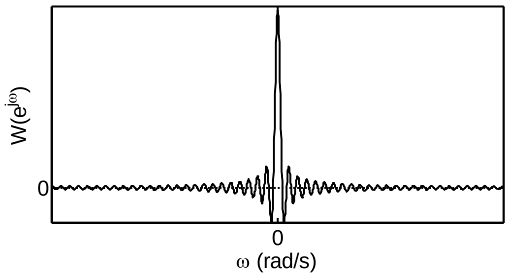


Figure 2.1

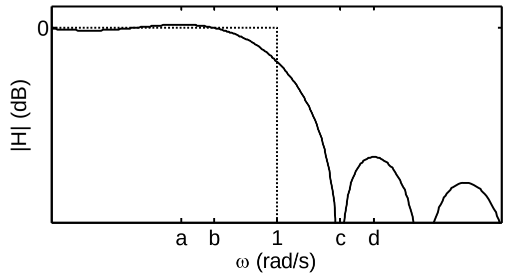


Figure 2.2

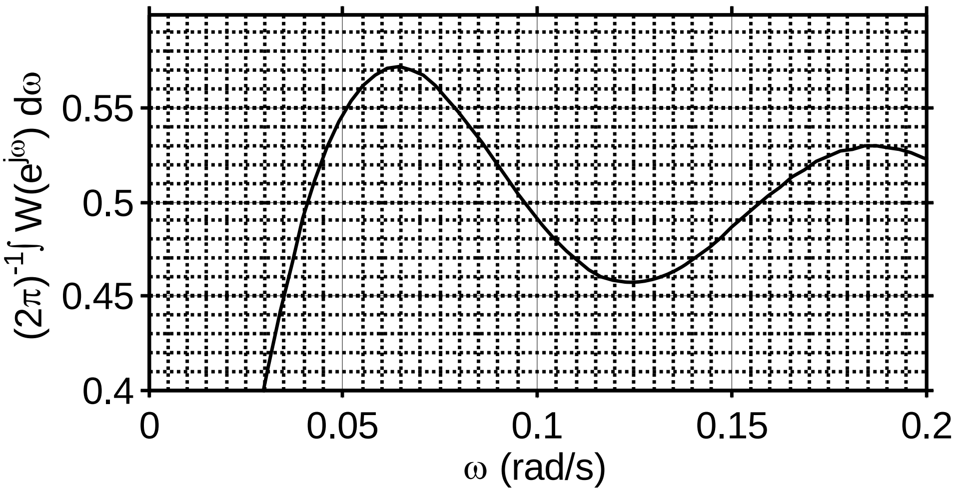


Figure 2.3

(i) Using numerical values taken from *Figure 2.3*, estimate the angular frequencies marked "a", "b", "c" and "d" on *Figure 2.2*. These correspond respectively to (a) the last passband peak, (b) the highest frequency with a gain of 0 dB, (c) the first stopband zero and (d) the first stopband peak. Explain carefully the reasons for your answers. [10]

Estimate also the filter gain in dB at frequencies "*a*" and "*d*". [3]

(ii) Explain how your answers to part (i) would change for a lowpass filter with the same cutoff frequency that is designed using a window, $v[n]$, of length 51 defined by $v[n] = w[2n]$ for $-25 \le n \le 25$. [3]

(b) Outline the steps taken in the Remez-Exchange algorithm when designing a symmetric FIR filter with a specified target magnitude response. [6]

Explain the sense in which the resultant filter is optimal. [2]

(c) (i) State the “Alternation Theorem” in the context of fitting a polynomial, $y = f(x)$, to a set of empirical data pairs, (x_i, y_i). Explain how this theorem relates to the Remez-Exchange algorithm for filter design. [2]

(ii) Suppose that we wish to approximate a set of empirical data pairs, (x_i, y_i) with a linear function $y = f(x) = px + q$ in order to minimize the worst case absolute approximation error given by $\epsilon = \max_i |y_i - f(x_i)|$. Determine a matrix **A** such that, if the maximal approximation error is attained with alternating signs at x_1, x_2 and x_3, then p, q and ϵ will satisfy [2]

$$\mathbf{A}\begin{pmatrix} p \\ q \\ \epsilon \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

(iii) Show that if both sides of the matrix equation in part (i) are pre-multiplied by the row vector $\mathbf{c}^T = (x_2 - x_3 \quad x_3 - x_1 \quad x_1 - x_2)$, it is possible to determine ϵ without solving the entire equation. Hence give an expression for ϵ. [2]

3. (a) $H(z) = \frac{B(z)}{A(z)}$ is a rational LTI filter with $A(z) = \sum_{r=0}^{M} a[r]z^{-r}$ and $B(z) = \sum_{r=0}^{M} b[r]z^{-r}$ where $a[r]$ and $b[r]$ are real and satisfy $b[r] = a[M-r]$.

(i) Show that $H(z) = \frac{z^{-M}A(z^{-1})}{A(z)}$ [5]

(ii) Show that $\left|H(e^{j\omega})\right| = 1$ for all values of ω. [4]

(b) The block diagram of *Figure 3.1* includes two adders, two multipliers with gains of k and $-k$ respectively, one unit delay and an LTI block with the rational transfer function $G_1(z)$.

(i) Show that $Y_1(z) = G_1(z)X_2(z) - kz^{-1}G_1(z)Y_1(z)$. Derive expressions in terms of k and $G_1(z)$ for $\frac{Y_1(z)}{X_2(z)}$ and hence for $G_2(z) = \frac{Y_2(z)}{X_2(z)}$. [6]

(ii) Suppose that $F_1(z)$ is causal, FIR and of order N with $f_1[0] = 1$ and that

$$G_1(z) = \frac{z^{-N}F_1(z^{-1})}{F_1(z)}$$

Show that $G_2(z) = \frac{Y_2(z)}{X_2(z)} = \frac{z^{-(N+1)}F_2(z^{-1})}{F_2(z)}$ where $f_2[N+1] = k$ and $f_1[r] = \frac{f_2[r] - kf_2[N+1-r]}{1-k^2}$ for $0 \le r \le N$. [10]

(c) The block diagram of *Figure 3.2* consists of two cascaded copies of *Figure 3.1*.

Determine k_2 and k_1 in Figure 3.2 so that $G_2(z) = \frac{Y_2(z)}{X_2(z)} = \frac{0.4 - 0.28z^{-1} + z^{-2}}{1 - 0.28z^{-1} + 0.4z^{-2}}$. [5]

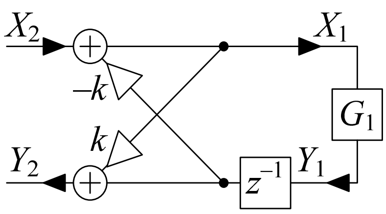


Figure 3.1

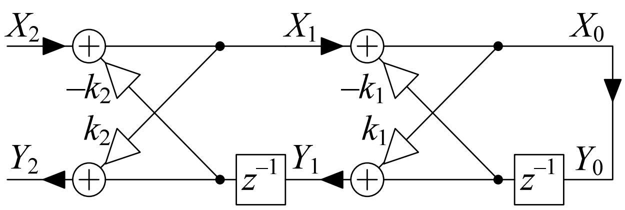


Figure 3.2

4. (a) The sampling rate converter of *Figure 4.1* comprises an upsampler, a symmetric lowpass FIR filter, $H(z)$, and a downsampler. As indicated on the diagram, the signal $v[n]$ is sampled at 16 kHz and it has a bandwidth of ± 7 kHz with a uniform power spectrum in this range.

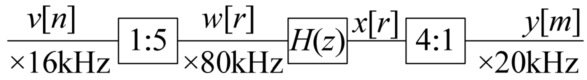


Figure 4.1

(i) Briefly describe the purpose of the lowpass filter $H(z)$. Determine its cutoff frequency and maximum transition bandwidth in rad/s. [4]

(ii) Draw separate dimensioned sketches of the signal power spectrum at V, W, X and Y covering the normalized angular frequency range $0 \leq \omega \leq \pi$. Give the corresponding sample frequency (in Hz) in each case. [8]

(iii) The minimum order of $H(z)$ is given approximately by $\frac{a}{3.5b}$ where a is the stopband attenuation (in dB) and b is the transition bandwidth (in rad/s). Calculate, M, the required order of $H(z)$ if $a = 60$ dB and hence estimate the number of multiplications per second required to implement the system. [2]

(iv) If the SNR of $v[n]$ is 56 dB, estimate the SNR of $y[m]$. [4]

(b) The filter $H(z)$ is now implemented using the polyphase structure shown in *Figure 4.2* using five filters $H_0(z), H_1(z), \ldots, H_4(z)$.

(i) State the required order of the filter $H_2(z)$ and give a formula for its coefficients, $h_2[n]$, in terms of $h[r]$, the coefficients of $H(z)$. [3]

(ii) Estimate the number of multiplications per second needed to implement the block diagram of *Figure 4.2*. [2]

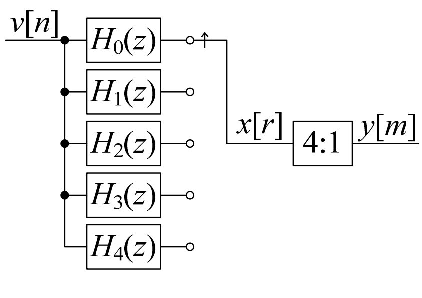


Figure 4.2

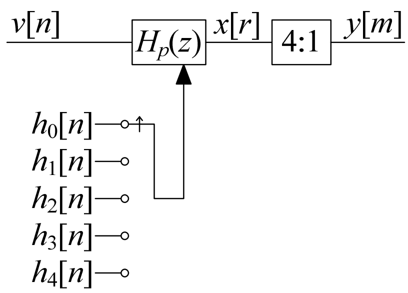


Figure 4.3

(c) The system is now implemented as shown in *Figure 4.3* using a single filter, $H_p(z)$, but with commutating coefficients.

(i) Draw a block diagram showing how $H_p(z)$ is constructed from delays, multipliers and adders. [3]

(ii) Explain how the output downsampler can now be merged with the filter and estimate the number of multiplications per second needed for the resultant implementation. [2]

(iii) Explaining your reasons carefully, determine the input sample numbers and the coefficient set that are used to generate output sample $y[99]$. [2]

2012 E4.13/EE9SC1: DSP and Digital Filters - Solutions

Key to letters on mark scheme: B=Bookwork, C=New computed example, A=Analysis of new example, D=design of new example

1. (a) (i) The geometric way of seeing this is to derive the convolution $v[n]$ by flipping $y[n]$ and then sliding it along $x[n]$. We can divide the convolution into three portions:

(a) part of $y[n]$ extends before $x[0]$,

(b) $y[n]$ fits entirely within $x[n]$,

(c) part of $y[n]$ extends past $x[N-1]$.

Only when $y[n]$ fits entirely within $x[n]$ will $v[n] = w[n]$. This corresponds to $M - 1 \leq n \leq N - 1$. [3B]

To derive this algebraically, we note that for $v[n] = w[n]$, we need $y[n-r] = y[(n-r)_{\text{mod } N}]$ for the summing range $r = 0, \dots, N-1$.

We know that $y[i] = y[i_{\text{mod } N}]$ for $0 \leq i \leq N-1$ since the "mod N" then has no effect. Less obviously, it will also be true when $M - N \leq i \leq -1$ since for these values $y[i] = y[i+N] = 0$. So overall $y[i] = y[i_{\text{mod } N}]$ whenever $M - N \leq i \leq N - 1$.

So when $r = 0$, $y[n-r] = y[(n-r)_{\text{mod } N}]$ requires $n - r = n \leq N - 1$. When $r = N - 1$, $y[n-r] = y[(n-r)_{\text{mod } N}]$ requires $n - r = n - N + 1 \geq M - N \Rightarrow n \geq M - 1$. Combining these gives $M - 1 \leq n \leq N - 1$.

Very few got this right.

(ii) $v[1] = 1 \times 2 + 2 \times 5 = 12$ and $w[1] = 1 \times 2 + 2 \times 5 + 3 \times 7 = 33$. [2C]

For completeness, $v[n] = [5, 12, 22, 16, 24, 26, 21]$ and $w[n] = [31, 33, 22, 16, 24]$.

Note that $N = 5$ in this example (the length of $x[n]$) whereas quite a few people took it to be 3. A surprisingly large number of people couldn't do these convolutions correctly. Some forgot that the first element of $x[n]$ is $x[0]$ rather than $x[1]$.

(b) (i) An LTI system is BIBO stable if a bounded input sequence, $x[n]$ always gives a bounded output sequence, $y[n]$. That is, [2B]

$$|x[n]| < B\ \forall n\ \Rightarrow |y[n]| < f(B) \forall n$$

Many people quoted equivalent conditions from the notes but did not actually define what BIBO means. Quite a lot of people did not seem to know what "bounded" meant, i.e. $\exists B$ such that $|x[n]| < B \forall n$. Using the wrong definition of bounded made part (ii) very hard.

(ii) Define $x[n] = +1$ if $h[-n] \geq 0$ and $x[n] = -1$ otherwise.

Then $y[0] = \sum_{r=-\infty}^{\infty} h[r]x[0-r] = \sum_{r=-\infty}^{\infty} |h[r]|$ which is finite from the BIBO assumption since $x[n] \leq 1\ \forall n$ and thus is bounded. [3B]

Only a minority knew what a "proof" was. You can only use the BIBO property by defining an input sequence $x[n]$ that is bounded and then using the fact that each output sample, $y[n]$, must be bounded.

(c) (i) $W = 0$ when the numerator is zero but the denominator is non-zero which is when $0.5(M+1)\omega = k\pi$ for $k \neq 0$. This happens first when $k = 1$ giving $\omega = \frac{2\pi}{M+1}$. [1C]

Some thought $\sin\theta = 0$ implies θ is a multiple of 2π rather than π.

(ii) Using the small angle formula $\sin\theta \approx \theta$ gives $W = \frac{0.5(M+1)\omega}{(M+1)\times 0.5\omega} = 1 = 0$ dB for small ω (including $\omega = 0$). Equivalently, you can use L'Hôpital's rule. [2C]

Many people were unable to do this. Quite a few assumed that when $\omega = 0$, $\frac{\sin 0.5(M+1)\omega}{\sin 0.5\omega} = \frac{0}{0} = 1$ but this is not valid. You need to either use L'Hôpital's rule or else, more easily, the small angle approximation (i.e. 1-term Taylor series) for $\sin\theta$.

(iii) At $\omega = \frac{3\pi}{M+1}$, $|W| = \frac{1}{(M+1)\sin\frac{1.5\pi}{M+1}} \approx \frac{1}{1.5\pi} = 0.212 = -13.5$ dB [2C]

Some used the small angle approximation for the numerator as well as the denominator. This is wrong since $0.5(M+1)\omega = \frac{3\pi}{2}$ which is clearly not "small". The approximation given in the question is valid for sufficiently large M. Several had their calculators set to "degrees" when calculating $\sin 1.5\pi$. Some calculated the correct value but did not express it in dB as the question asked.

(d) (i) $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \sum_{n=0}^{\frac{M-1}{2}} h[n]e^{-j\omega n} + \sum_{n=\frac{M+1}{2}}^{M} h[n]e^{-j\omega n}$.

Substituting $r = M - n$ in the second summation and reversing the order of summation gives

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} h[n]e^{-j\omega n} + \sum_{r=0}^{\frac{M-1}{2}} h[M-r]e^{j\omega(r-M)}$$

Since $h[M-r] = h[r]$ we can combine the sums to give

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} h[n]\left(e^{-j\omega r} + e^{j\omega(r-M)}\right)$$

$$= e^{-\frac{-j\omega M}{2}} \sum_{n=0}^{\frac{M-1}{2}} h[n]\left(e^{-j\omega(r-\frac{M}{2})} + e^{j\omega(r-\frac{M}{2})}\right)$$

$$= 2e^{-\frac{-j\omega M}{2}} \sum_{n=0}^{\frac{M-1}{2}} h[n]\cos\left(n - \frac{M}{2}\right)\omega$$

Hence $A = 2$ and $\theta(\omega) = -\frac{M}{2}\omega$. [4B]

Whenever you have two exponentials added together, $e^{a} \pm e^{b}$, it is a very common trick to take out a common factor of $e^{\frac{a+b}{2}}$ to turn it into a cos or sine. We do this in the derivation above:

(ii) $\angle H(e^{j\omega}) = \theta(\omega) = -\frac{M}{2}\omega$ so $\tau_H(e^{j\omega}) = \frac{M}{2}$. [1B]

Some didn't notice that $\angle H(e^{j\omega})$ is just $\theta(\omega)$. Indeed the main purpose the the decomposition in part (i) is to separate the magnitude and phase of $H(e^{j\omega})$. Some tried to take the derivative of the imaginary part of $\ln H$ which is a valid but complicated way of getting the answer.

(e) (i) The continuous time expression for $s(t)$ is $s(t) = \exp(-2\pi j f_c t)$. Substituting $t = \frac{r}{f_s}$ results in $s[r] = \exp\left(-\frac{2\pi j f_c r}{f_s}\right) = \exp\left(-\frac{191\pi j r}{400}\right) = \exp(-1.5001 jr)$. [2C]

Many people omitted the r resulting in an expression for $s[r]$ that did not depend on r at all. Quite a few omitted the f_s in the denominator.

(ii) The wanted channel covers ± 75 kHz and the first aliasing image covers 325 to 475 kHz.

Therefore the 75 Hz single-sided bandwidth is $2\pi \times \frac{75}{80000} = \frac{3\pi}{1600} = 0.00589\ \text{rad/s}$.

The 250 Hz transition bandwidth is $2\pi \times \frac{325-75}{80000} = \frac{\pi}{160} = 0.0196\ \text{rad/s}$. [3C]

Most people took the bandwidth to be ± 150 kHz either side of the centre frequency rather than the correct value of ± 75 kHz. Many also took the new Nyquist frequency (200k $= \frac{\pi}{200}$) to be the upper edge of the transition band whereas in fact it should be the centre of the transition band.

(f) (i) From the Noble identities: $G(z) = 1 + 2z^{-3}$. [2A]

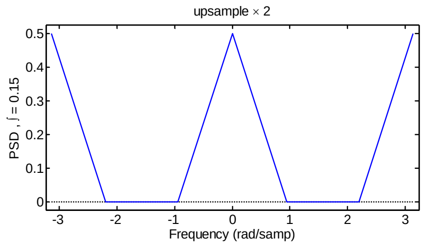
Some said $G(z) = H(z^3)$ rather than working it out explicitly. They got full marks but should consider themselves lucky.

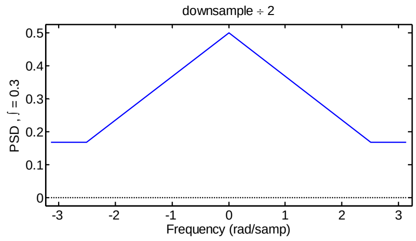
(ii) $v[n] = g[n] * x[n] = [1,1,1,3,3,3,3,\cdots]$. $y[n]$ is a downsampled version of $v[n]$ and therefore $y[n] = [1,3,3,3,3,\cdots]$. [3C]

Some tried to do this via the z-transforms $V(z)$ and $Y(z)$ but invariable got it wrong. A 2-tap filter is much easier to apply directly in the time domain.

(g) Mathematically, $V(z) = X(z^2)$ and $W(z) = \frac{1}{2}\left(X\left(z^{\frac{1}{2}}\right) + X(-z^{\frac{1}{2}}\right)$.

The spectrum of $V(z)$ shrinks horizontally by 2, halves in amplitude and replicates; note that an upsampled energy spectrum stays the same amplitude but an upsampled power spectrum halves in amplitude because there are twice as many samples for the same energy. The power spectrum of $W(z)$ halves in amplitude (unlike an energy spectrum which would quarter in amplitude) and expands by 2 in frequency and we get some aliasing because the original spectrum extended past $\pm 0.5\pi$. The alias components add onto the original spectrum to give flat portions at either side. [5A]





Some people omitted the images from V or got them in the wrong place. Some omitted the aliasing in $|W|^2$.

(h) The overall transfer function is:

$$H^2(z) + H(-z)G(z) = 1 + 2z^{-1} + z^{-2} + (1 - z^{-1})G(z) = \alpha z^{-1}$$

Rearranging this gives $G(z) = -\frac{1+(2-\alpha)z^{-1}+z^{-2}}{1-z^{-1}}$.

Because $G(z)$ is FIR, the denominator must be a factor of the numerator so we have $G(z) = a + bz^{-1}$ with

$$1 + (2 - \alpha)z^{-1} + z^{-2} = -(1 - z^{-1})G(z) = -a + (a - b)z^{-1} + bz^{-2}$$

Now, identifying the z^0 *and* z^{-2} coefficients gives $a = -1$ and $b = +1$.

So $G(z) = -1 + z^{-1}$ and $\alpha = 4$. [5A]

Some thought $H(-z) = 1 + z$ instead of $1 - z^{-1}$. Many left $G(z)$ as a rational function.

2. (a) (i) The filter response is the convolution of the window response (*Figure 2.1*) with the ideal filter response. Each point in this convolution is therefore an integral of the window response over an interval of length 2 (the double-sided filter bandwidth). Algebraically,

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(e^{j\theta})H_{\text{ideal}}(e^{j(\omega-\theta)})d\theta$$

$$= \frac{1}{2\pi}\int_{\omega-1}^{\omega+1} W(e^{j\theta})d\theta \approx \frac{1}{2\pi}\int_{\omega-1}^{\pi} W(e^{j\theta})d\theta$$

$$= 0.5 - \frac{1}{2\pi}\int_{0}^{\omega-1} W(e^{j\theta})d\theta = 0.5 + \frac{1}{2\pi}\int_{0}^{1-\omega} W(e^{j\theta})d\theta$$

The integration limits in the second line arise because $H_{\text{ideal}}(e^{j(\omega-\theta)}) = 1$ when $-1 < \omega - \theta < 1$ and zero otherwise. The approximation in the second line is good provided that the window length is large enough. The first expression on the last line is most convenient if $\omega > 1$ and the second if $\omega < 1$. [6A]

The final passband peak (a) occurs at a lag corresponding to the first zero crossing of the window response which is the first local maximum of *Figure 2.3*. This is at $a = 1 - 0.064 = 0.936$, with a height $0.5 + 0.57 = 0.59$ dB. [1C]

Many people gave $a = 0.064$ rather than $a = 1 - 0.064$ (and similarly for the other frequencies even though the point 1 was marked on the graph of Fig 2.2.

Frequency b is when the integral crosses 0.5 at $b = 1 - 0.041 = 0.959$ [1C]

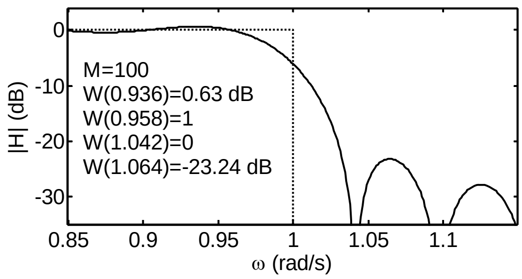
Frequency c is the mirror of b at $c = 1 + 0.041 = 1.041$. [1C]

Several thought that this ought to correspond to a minimum of the plotted graph (e.g. 0.125) rather than where it crosses 0.5.

Frequency d is the mirror of a at $d = 1 + 0.064 = 1.064$ with a height $|0.5 - 0.57| = |-0.07| = -23$ dB. [1C]

Gains at a and d are 0.59 and -23 dB respectively. [3C]

The true values (without the simplifying assumption above, are shown on the plot below:



(ii) A filter of half the order will be twice as wide so the transition will remain the same shape but be expanded by a factor of 2. The critical frequencies will now be: $a = 1 - 0.128 = 0.872$, $b = 1 - 0.082 = 0.918$, $c = 1.082$, $d = 1.128$. The magnitudes will be the same as before. [3A]

(b) The algorithm operates in a transformed frequency domain $\cos\omega$ in which the response of an order M symmetric FIR filter is an $\frac{M}{2}$ order polynomial. The heart of the Remez-exchange algorithm is that for a filter of order M, the weighted frequency-domain error will take on its maximum value $\frac{M}{2}+2$ times with alternating signs. The steps of the algorithm are:

Many omitted to mention that the algorithm operated in a transformed frequency domain in which an FIR filter has a polynomial response.

1. Guess the $\frac{M}{2}+2$ transformed frequencies of the extremal values.
2. Determine the error magnitude and polynomial coefficients of the polynomial that passes through the maximal error locations.

As seen in part c(ii) below, you can determine the maximum error explicitly using a closed-form formula. Some people implied that you had to guess the maximum error.

3. Find the local maxima of the continuous error function by evaluating the polynomial on a dense set of transformed frequencies.
4. Update the the maximal error frequencies to be an alternating set of the local maxima + band edges + $\omega = 0$ and/or $\omega = \pi$. Then go back to step 2 and iterate until convergence.
5. Evaluate the response on $M+1$ evenly spaced frequencies and do an IFFT to obtain the filter coefficients. [6B]

The algorithm was correctly described by almost everyone.

The resultant filter will have the smallest maximal weighted absolute error for any symmetric filter of the specified order. [2B]

The important thing is the minimax criterion.

(c) (i) Alternation Theorem: A polynomial fit of degree n to a bounded set of points is minimax iff it attains its maximal error at $n+2$ points with alternating signs. This theorem confirms that the solution is optimal if the local maxima found in step 3 of the answer to part (b) coincide with those assumed in step 2. [2B]

(ii) The matrix $\mathbf{A}$ will be either

$$\mathbf{A} = \begin{pmatrix} x_1 & 1 & 1 \\ x_2 & 1 & -1 \\ x_3 & 1 & 1 \end{pmatrix}$$

or else the same thing with the final column negated (or equivalently with ϵ negated). [2A]

Correctly done by the few who attempted it except for one person who had the right column all positive.

(iii) If $\mathbf{c}^T = (x_2 - x_3 \quad x_3 - x_1 \quad x_1 - x_2)$, then

$$\mathbf{c}^T\mathbf{A} = (0 \quad 0 \quad 2(x_1 - x_3))$$

The equation therefore becomes

$$2(x_1 - x_3)\epsilon = (x_2 - x_3)y_1 + (x_3 - x_1)y_2 + (x_1 - x_2)y_3$$

from which [2A]

$$\epsilon = \frac{(x_2 - x_3)y_1 + (x_3 - x_1)y_2 + (x_1 - x_2)y_3}{2(x_1 - x_3)}$$

Correctly done by the one person who attempted it.

3. (a) (i) $B(z) = \sum_{r=0}^{M} b[r]z^{-r} = \sum_{r=0}^{M} a[M-r]z^{-r} = \sum_{s=0}^{M} a[s]z^{-M+s}$

$$= z^{-M}\sum_{s=0}^{M} a[s]z^{+s} = z^{-M}A(z^{-1})$$

where we have made the substitution $s = M - r$ and reversed the summation order. The results follows. [5A]

Quite a few wrote out $B(z)$ in full without the summation sign and showed the transformation that way; this is OK but doing a substitution in the summation is much neater. People found difficulty with knowing the required substitution; the reason we choose $s = M - r$ is so that $a[M-r]$ turns into the simpler $a[s]$. When you make a substitution, you have to transform the limits as well as you do with integrals. Several wrongly transcribed $H(z) = \frac{A(z)}{B(z)}$ and fudged things to get the right answer (it is standard practice for the denominator to be denoted $A(z)$, not just perversity on the part of the examiner).

(ii) If $z = e^{j\omega}$ then $z^{-1} = z^*$ and so, since the coefficients $a[r]$ are real, $A(z^{-1}) = \left(A(z)\right)^*$ and so $|A(z^{-1})| = |A(z)|$.

So $|H(z)| = \frac{|z^{-M}|\times|A(z^{-1})|}{|A(z)|} = 1$. [4B]

One person just stated that since it was an allpass filter its gain had a magnitude of unity. This does not constitute a proof. Many used complicated algebra for this: including splitting up the complex exponentials $e^{j\omega} = \cos\omega + j\sin\omega$.

(b) (i) $Y_1 = G_1X_1 = G_1(X_2 - kz^{-1}Y_1) = G_1X_1 - kz^{-1}G_1Y_1$ from which we get $Y_1 = \frac{G_1}{1+kz^{-1}G_1}X_2$.

$Y_2 = kX_1 + z^{-1}Y_1 = \left(\frac{k}{G_1} + z^{-1}\right)Y_1 = \left(\frac{k}{G_1} + z^{-1}\right)\frac{G_1}{1+kz^{-1}G_1}X_2 = \frac{k+G_1z^{-1}}{1+kz^{-1}G_1}X_2$

from which we get $G_2(z) = \frac{k+G_1(z)z^{-1}}{1+kz^{-1}G_1(z)}$. [6A]

(ii) Substituting for $G_1(z)$ in the result of part (b)(i) and multiplying numerator and denominator by $F_1(z)$ results in

$$G_2(z) = \frac{kF_1(z)+z^{-N}F_1(z^{-1})z^{-1}}{F_1(z)+kz^{-1}z^{-N}F_1(z^{-1})} = \frac{kF_1(z)+z^{-N-1}F_1(z^{-1})}{F_1(z)+kz^{-N-1}F_1(z^{-1})}.$$

If now we define $F_2(z) = F_1(z) + kz^{-N-1}F_1(z^{-1})$, i.e. the denominator of $G_2(z)$, we can write

$$z^{-N-1}F_2(z^{-1}) = z^{-N-1}(F_1(z^{-1}) + kz^{N+1}F_1(z))$$

$= kF_1(z) + z^{-N-1}F_1(z^{-1})$ which is the numerator of $G_2(z)$.

Hence $G_2(z) = \frac{z^{-(N+1)}F_2(z^{-1})}{F_2(z)}$ as required.

From $F_2(z) = F_1(z) + kz^{-N-1}F_1(z^{-1})$, we get $f_2[N+1] = kf_1[0] = k$

We can write $F_2(z^{-1}) = F_1(z^{-1}) + kz^{N+1}F_1(z)$

and, eliminating $F_1(z^{-1})$ gives $F_2(z) - kz^{-N-1}F_2(z^{-1}) = (1-k^2)F_1(z)$

from which $F_1(z) = \frac{F_2(z)-kz^{-N-1}F_2(z^{-1})}{1-k^2}$

Identifying coefficients in z^{-r} gives $f_1[r] = \frac{f_2[r]-kf_2[N+1-r]}{1-k^2}$. [10A]

For $r = N+1$, this gives $f_1[N+1] = \frac{f_2[N+1]-kf_2[0]}{1-k^2} = \frac{k-k\times 1}{1-k^2} = 0$ which is correct since F_1 is of order N.

Generally speaking it is best to avoid introducing summation signs unless you absolutely have to.

(c) Using the result of part (b)(ii), $N = 1$ and $F_2(z) = 1 - 0.28z^{-1} + 0.4z^{-2}$. From this, $k_2 = 0.4$ and $f_1[1] = \frac{f_2[1]-k_2 f_2[1]}{1-k_2^2} = \frac{-0.28-0.4\times -0.28}{1-0.4^2} = \frac{-0.168}{0.84} = -0.2$.

So now, $G_1(z) = \frac{z^{-1}F_1(z^{-1})}{F_1(z)} = \frac{-0.2+z^{-1}}{1-0.2z^{-1}}$. So $k_1 = -0.2$ and $G_0(z) \equiv 1$. [5C]

Most people evaluated an expression or the transfer function of the entire circuit and then matched coefficients: $G_2(z) = \frac{k_2+k_1(1+k_2)z^{-1}+z^{-2}}{1+k_1(1+k_2)z^{-1}+k_2 z^{-2}}$. This gives the correct answer but for large N is much more laborious and error-prone than iterating by setting $k_i = f_i[i]$ at each stage and then calculating the $f_{i-1}[\,]$ coefficients.

4. (a) (i) The function of the lowpass filter is twofold: (a) to remove images introduced by the upsampler and (b) to restrict the signal bandwidth so that no aliasing occurs from the downsampler. Thus the centre of its transition band must be the lower of the input and output Nyquist frequencies: in this case, this is the input Nyquist frequency, $\frac{\pi}{5}$.

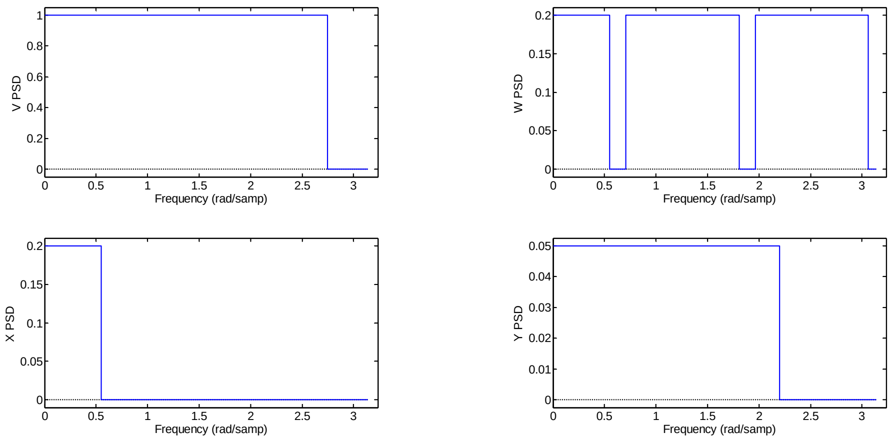
The cutoff frequency is $\frac{2\pi\times 7}{80} = 0.175\pi = 0.550$ rad/s

The transition bandwidth is $\frac{2\pi\times 2}{80} = \frac{\pi}{20} = 0.05\pi = 0.157$ rad/s [4C]

Another expression for the transition bandwidth is $\frac{2\pi}{5} - 2 \times 0.550$.

Several only mentioned only one of the two purposes of $H(z)$, i.e. either removing images introduced by the upsampler or else preventing aliasing by the downsampler but not both. Many gave the wrong cutoff and transition frequencies; the filter transition occupies $0.55 < \omega < 0.707$, which is the gap between the baseband and the first image in the graph of $|W|^2$ below. Several used the alternative expression given for the transition frequency above but used 2π instead of $\frac{2\pi}{5}$. Many got confused about normalized rad/s; a digital filter cutoff frequency is always in the range 0 to π.

(ii) The power spectra are shown below. Note that the upsampler introduces a gain of $\frac{1}{5}$ and the downsampler introduces a gain of $\frac{1}{4}$ into the power spectral density graph. [8A]



Mostly correct. Some gave too many images in W. You get five complete images in $0 < \omega < 2\pi$ so only 2½ images in $0 < \omega < \pi$. Several took the input spectrum to be triangular even though the question said it was uniform.

(iii) The transition bandwidth is $b = 0.157$ so the filter order is $M = \frac{60}{3.5\times 0.157} = 109$. The number of multiplications per second is therefore $(M+1) \times 80\text{ kHz} = 8.8 \times 10^6$. [2C]

Many said M multiplications per sample instead of M+1. Quite a lot of people said the number of multiplications was 109 and did not multiply by the sample frequency. Many people used the wrong sample frequency (16 kHz) even though $H(z)$ is operating at 80 kHz.

(iv) If the signal power in X is 1, the noise power is -56 dB from the original input plus another 4 copies of -60 dB from the filtered images. All this noise gets aliased into the baseband by the downsampler giving a total noise of $10\log(10^{-5.6} + 4 \times 10^{-6}) = 10\log 6.5 \times 10^{-6} = -51.9$ dB. We neglect the factor of $\frac{1}{16}$ introduced by the downsampler since this affects both signal and noise equally. [4C]

Few people realized that the images introduced by the upsampler become the main source of additional noise. The lowpass filter attenuates but does not eliminate this noise which is then aliased into the passband and added to the noise that was originally present in the signal. The noise originally in the signal also generate images but these are attenuated by the filter to negligible levels. The question did not specify how much of the input noise was within the signal band; we assume above that it all is; if the noise is actually uniform in the range 0 to 8 kHz, then the filtering will reduce it by about $10\, log\frac{7}{8} = -0.58$ dB.

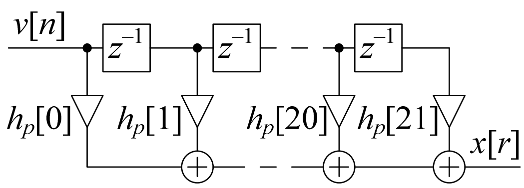
(b) (i) A filter of order M has $M+1$ coefficients, so the required order is now $\frac{M+1}{5} - 1 = 21$.

Coefficients are: $h_2[r] = h[5r+2]$ for $0 \le r \le 21$. [3C]

Almost everyone used $\frac{M}{5}$ *(forgetting that an order* M *FIR filter has* $M+1$ *coefficients). Quite a lot listed the first few coefficients of* $H_2(z)$ *rather than giving a formula for* $h_2[n]$ *as requested. Some of the formulae for* $h_2[n]$ *were actually formulae for* $H(z)$*.*

(ii) The number of coefficients is the same but the sample rate is now 16 kHz so the number of multiplications per second is $(M+1) \times 16\text{ kHz} = 1.76 \times 10^6$. [2C]

(c) (i) The implementation needs 21 delays, 22 multipliers and 21 adders: [3B]



(ii) Since we only keep every fourth output sample, we need only compute the ones we need. So for each output sample we need to increment the coefficient set number, p, by 4. The number of multiplications per second is $(M+1) \div 4 \times 16\text{ kHz} = 0.44 \times 10^6$. [2B]

(iii) The coefficient set is $p = (4m)_{\text{mod } 5} = 1$. So coefficient set $h_1[n]$ is used. The input samples are $[0, -1, \cdots, -21] + \text{floor}\left(\frac{4m}{5}\right) = [79, 78, \cdots, 58]$. [2C]

Another way of looking at it is that $y[99] = x[4 \times 99] = x[396]$. Since $\frac{396}{5} = 79.2$, the latest input sample involved is $v[79]$.