EE4-12
EE9-AO2
EE9-SC1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2015

MSc and EEE/EIE PART IV: MEng and ACGI

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Friday, 1 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer Question 1 and any TWO other questions

Question 1 is worth 40% of the marks and other questions are worth 30%

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	D.M. Brookes
	Second Marker(s) :	P.T. Stathaki

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Information for Candidates:

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms respectively. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.

Abbreviations

BIBO	Bounded Input, Bounded Output
CTFT	Continuous-Time Fourier Transform
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
DTFT	Discrete-Time Fourier Transform
FIR	Finite Impulse Response

IIR	Infinite Impulse Response
LTI	Linear Time-Invariant
MDCT	Modified Discrete Cosine Transform
PSD	Power Spectral Density
SNR	Signal-to-Noise Ratio

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.
- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$ provided that $\left|\alpha z^{-1}\right| < 1$.

Forward and Inverse Transforms

z:	$X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$	$x[n] = \frac{1}{2\pi j} \oint X(z)z^{n-1}dz$
CTFT:	$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$
DTFT:	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$	$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$
DFT:	$X[k] = \sum_{0}^{N-1} x[n]e^{-j2\pi\frac{kn}{N}}$	$x[n] = \frac{1}{N} \sum_{0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$
DCT:	$X[k] = \sum_{n=0}^{N-1} x[n]\cos\frac{2\pi(2n+1)k}{4N}$	$x[n] = \frac{X[0]}{N} + \frac{2}{N}\sum_{n=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k] = \sum_{n=0}^{2N-1} x[n]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n] = \frac{1}{N}\sum_{0}^{N-1} X[k]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r]y[n-r] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)Y\left(e^{j\omega}\right)$

$v[n] = x[n]y[n] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = \frac{1}{2\pi}X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right)Y\left(e^{j(\omega-\theta)}\right)d\theta$

DFT: $v[n] = x[n] \circledast_N y[n] \triangleq \sum_{r=0}^{N-1} x[r]y[(n-r) \ _{\bmod N}] \quad \Leftrightarrow \quad V[k] = X[k]Y[k]$

$v[n] = x[n]y[n] \quad \Leftrightarrow \quad V[k] = \frac{1}{N}X[k] \circledast_N Y[k] \triangleq \frac{1}{N}\sum_{r=0}^{N-1} X[r]Y[(k-r) \ _{\bmod N}]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)}\frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}\,()$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$
2. $M \approx \frac{a-8}{2.2\Delta\omega}$
3. $M \approx \frac{a-1.2-20\log_{10}b}{4.6\Delta\omega}$

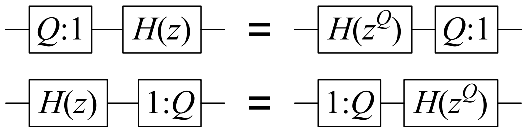
where a =stop band attenuation in dB, b = peak-to-peak passband ripple in dB and $\Delta\omega$ = width of smallest transition band in normalized rad/s.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$, $\rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$, $\rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Noble Identities



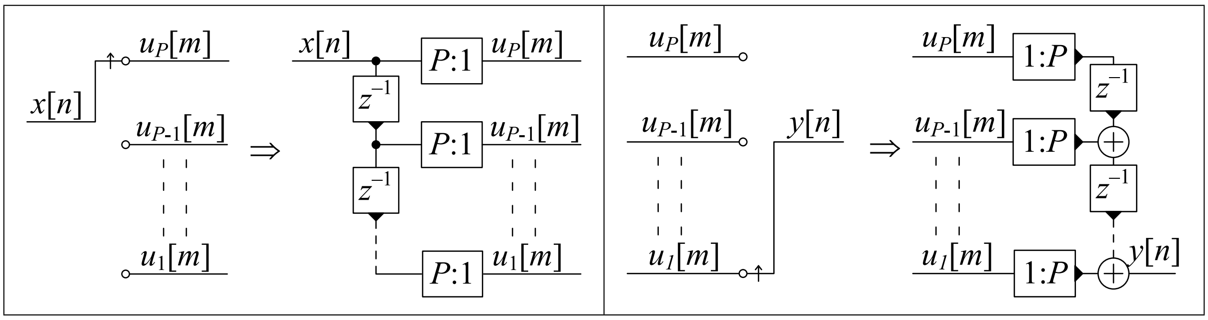
Multirate Spectra

Upsample $v[n]$ by Q: $\quad x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if } Q \mid r \\ 0 & \text{if } Q \nmid r \end{cases} \quad \Rightarrow \quad X(z) = V(z^Q)$

Downsample $v[n]$ by Q: $\quad y[m] = v[Qm] \quad \Rightarrow \quad Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$

Multirate Commutators

Input Commutator	Output Commutator



1. a) The finite length signals $u[0], \cdots, u[M-1]$ and $v[0], \cdots, v[N-1]$ are of length M and N respectively where $M < N$.

The signals $x[n] = u[n] * v[n]$ and $y[n] = u[n] \circledast_N v[n]$ are respectively the convolution and circular convolution of $u[n]$ and $v[n]$ as defined in the data sheet.

i) Prove that $y[n] = x[n]$ for $M-1 \le n \le N-1$. [3]

ii) Determine an expression for $y[n]$ in terms of the $\{x[n]\}$ that is valid for $0 \le n \le M-2$. [2]

iii) If $M = 3$ and $N = 4$ with $u[n] = [1, 2, -1]$ and $v[n] = [1, 1, -1, -1]$ determine both $x[n]$ and $y[n]$ for $0 \le n \le 7$. [3]

b) i) Show that, if $u[n] = \begin{cases} 1 & n \ge 0 \\ 0 & n < 0 \end{cases}$ and a is a complex-valued constant, then $x[n] = a^n u[n]$ and $y[n] = -a^n u[-n-1]$ have the same z-transform but with different regions of convergence. You may use without proof the geometric progression formulae given in the datasheet. [3]

ii) The z-transform $H(z)$ is given by

$$H(z) = \frac{2 + 17z^{-1}}{(2 - z^{-1})(1 + 4z^{-1})}.$$

By expressing $H(z)$ in partial fraction form, determine the sequence, $h[n]$, whose z-transform is $H(z)$ and whose region of convergence includes $|z| = 1$. [4]

c) i) The frequency response of an ideal lowpass filter is given by

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}.$$

By taking the inverse DTFT of $H(e^{j\omega})$, show that the corresponding impluse response is $h[n] = \dfrac{\sin \omega_0 n}{\pi n}$ [3]

ii) By multiplying an ideal filter impulse response by a Hamming window, determine an expression for the real-valued coefficients of an FIR causal bandpass filter of even order, M, whose passband is $1 \le \omega \le 2$.

For even M, a symmetric Hamming window is given by
$w[n] = 0.54 + 0.46 \cos \frac{2\pi n}{M+1}$ for $-0.5M \le n \le 0.5M$. [3]

d) i) Show that if the coefficients $a[r]$ are all real and

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{r=0}^{M} a[M-r]z^{-r}}{\sum_{r=0}^{M} a[r]z^{-r}}$$

then $\left|H(e^{j\omega})\right| \equiv 1$ and $\angle H(e^{j\omega}) = -M\omega - 2\angle A(e^{j\omega})$. [3]

ii) If $H(z) = \dfrac{2-4z^{-1}}{2-z^{-1}}$, sketch graphs of the magnitude and phase of $H(e^{j\omega})$ for $-\pi \le \omega \le \pi$. [3]

e) Figure 1.1 shows the power spectral density (PSD) of a real-valued signal $x[n]$. The horizontal portions of the PSD have values 3, 2 and 1 respectively. The signal $y[n]$ is then obtained by downsampling $x[n]$ by a factor of 3.

Draw a dimensioned sketch showing the PSD of $y[n]$ for $0 \le \omega \le \pi$. You should assume that components of $x[n]$ at different frequencies are uncorrelated and may assume without proof that $Y(z) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{\frac{-j2\pi k}{3}} z^{\frac{1}{3}}\right)$.

Determine the value of each horizontal portion of the PSD and each of the angular frequencies at which its value changes. [5]

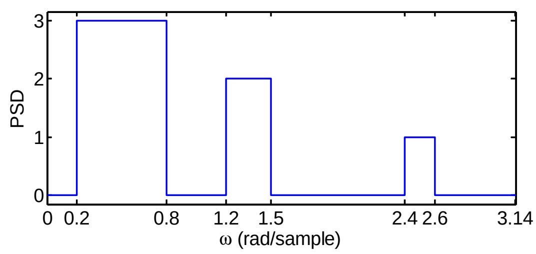


Figure 1.1

f) Figure 1.2 shows the block diagram of a two-band analysis and synthesis processor. You may assume without proof that, for $m = 0$ or 1, $W_m(z) = U_m(z^2)$ and $U_m(z) = \frac{1}{2}\left\{V_m(z^{\frac{1}{2}}) + V_m(-z^{\frac{1}{2}})\right\}$.

i) Derive a simplified expression for $Y(z)$ in terms of $X(z)$. [4]

ii) Explain the relationship between the magnitude responses of the filters $H(z)$ and $H(-z)$. [2]

iii) Explain what is meant by saying that the analysis-synthesis processor shown in Figure 1.2 is “alias-free”. [2]

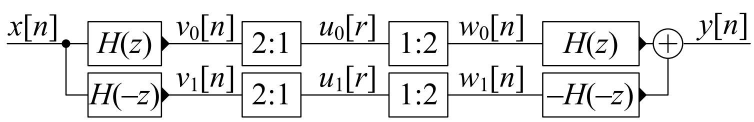


Figure 1.2

2. In this question, filters should be expressed in the standard form $g \times \dfrac{1+b_1 z^{-1}+\cdots}{1+a_1 z^{-1}+\cdots}$ with numerical values given for all coefficients.

a) A bilinear transformation of the z-plane is given by $z = \dfrac{\hat{z}-\lambda}{1-\lambda\hat{z}}$ where the real-valued constant λ satisfies $|\lambda| < 1$.

i) Show that $|z|^2 = 1 + \dfrac{\left(|\hat{z}|^2 - 1\right)\left(1-\lambda^2\right)}{|1-\lambda\hat{z}|^2}$.

Hence show that $|z| < 1$ if and only if $|\hat{z}| < 1$. [4]

ii) Explain why the property shown in part i) is important when using the transformation for filter design. [2]

b) A first-order lowpass filter has the transfer function $G(z) = 1 + z^{-1}$.

i) Determine the gain of the filter at $\omega = 0$ and show that the magnitude of the gain has decreased by a factor of $\sqrt{2}$ at the cutoff frequency, $\omega_G = \frac{\pi}{2}$. [2]

ii) By considering the value of $z^{\frac{1}{2}}G(z)$, determine a trigonometrical expression for $\left|G(e^{j\omega})\right|$ and draw a dimensioned sketch of its value over the range $0 \le \omega \le \pi$. [4]

iii) Using the appropriate z-plane transformation from the datasheet, transform $G(z)$ to a lowpass filter, $H(z)$, with a cutoff frequency of $\omega_H = 0.2$. Calculate the numerical values of the filter coefficients when expressed in the standard form given in the first line of the question. [5]

iv) Draw a dimensioned sketch of $\left|H(e^{j\omega})\right|$ over the range $0 \le \omega \le \pi$. [2]

c) A quadratic transformation of the z-plane is given by $z = -\tilde{z}^2$.

i) Show that $|z| < 1$ if and only if $|\tilde{z}| < 1$. [2]

ii) If $z = e^{j\omega}$ and $\tilde{z} = e^{j\tilde{\omega}}$ sketch a graph of ω versus $\tilde{\omega}$ over the range $-\pi \le \tilde{\omega} \le \pi$. For all $\tilde{\omega}$, the value of ω should be chosen to lie in the range $-\pi < \omega \le \pi$.

[2]

iii) A new filter is defined by $P(\tilde{z}) = H(z)$. Determine the numerical values of the coefficients of $P(\tilde{z})$ when expressed in the standard form given in the first line of the question. [3]

iv) Draw a dimensioned sketch of $\left|P(e^{j\omega})\right|$ over the range $0 \le \omega \le \pi$ and determine the values of ω within this range for which $\left|P(e^{j\omega})\right| = \sqrt{2}$.

Explain the relationship between the bandwidth of the filter $P(e^{j\omega})$ and the cutoff frequency of the filter $H(e^{j\omega})$. [4]

3. a) The filter $H(z) = \dfrac{1}{1+a_1z^{-1}+a_2z^{-2}}$ where $a_1 = -1.56$ and $a_2 = 0.64$.

i) By multiplying $H(z)$ by its complex conjugate and using the identity $\cos 2\omega = 2\cos^2\omega - 1$, express $\left|H(e^{j\omega})\right|^{-2}$ as a polynomial in $\cos\omega$ giving the coefficients to 5 significant figures. [4]

ii) The filter $H_1(z)$ is the same as $H(z)$ but with coefficient a_1 increased in magnitude by 1% (i.e. multiplied by 1.01). Similarly, the filter $H_2(z)$ is the same as $H(z)$ but with coefficient a_2 increased in magnitude by 1% .

For $\omega_0 = 0.2$, determine the ratios $\left|\dfrac{H_1(e^{j\omega_0})}{H(e^{j\omega_0})}\right|$ and $\left|\dfrac{H_2(e^{j\omega_0})}{H(e^{j\omega_0})}\right|$ in dB. [6]

b) In the block diagram of Figure 3.1 the outputs of all adders are on the right and solid arrows indicate the direction of information flow. Multiplier gains are written adjacent to each multiplier symbol. The parameter p is strictly positive.

i) Show that $G(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1}{1+(p^2-pq-2)z^{-1}+(pq+1)z^{-2}}$.[6]

ii) Determine the conditions on p and q for the filter $G(z)$ to be BIBO stable.

You may assume without proof that the filter $\dfrac{1}{1+b_1z^{-1}+b_2z^{-2}}$ is BIBO stable if and only if $|b_1|-1 < b_2 < 1$. [6]

iii) If

$$G(z) = \frac{1}{1+b_1z^{-1}+b_2z^{-2}},$$

determine expressions for p and q as functions of b_1 and b_2. Calculate the numerical values of p and q if $b_1 = -1.56$ and $b_2 = 0.64$. [3]

iv) The filter $G_p(z)$ is the same as $G(z)$ but with coefficient p increased by 1% (i.e. multiplied by 1.01) from the value determined in part iii). Similarly, the filter $G_q(z)$ is the same as $G(z)$ but with coefficient q increased by 1% from the value determined in part iii).

For $\omega_0 = 0.2$, determine the ratios $\left|\dfrac{G_p(e^{j\omega_0})}{G(e^{j\omega_0})}\right|$ and $\left|\dfrac{G_q(e^{j\omega_0})}{G(e^{j\omega_0})}\right|$ in dB. [5]

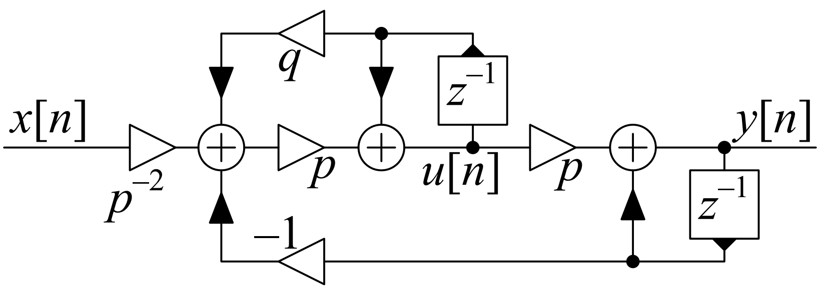


Figure 3.1

4. The FM radio band extends from 87.5 to 108 MHz. Within this band, an FM channel occupies ± 100 kHz around a centre frequency of $c \times 100$ kHz where the channel index, c, is an integer in the range $876 \le c \le 1079$. Figure 4.1 shows the block diagram of an FM radio front-end in which bold lines denote complex-valued signals. The diagram includes a bandpass filter (BPF) whose passband is 87.5 to 108 MHz and an analogue-to-digital converter (ADC) with a sample rate of 78 MHz.

a) Assume the bandpass filter is ideal and the power spectral density of the received signal is constant within the FM band. Sketch the power spectrum of $u[n]$ over the unnormalized frequency range -39 to $+39$ MHz. Determine the maximum width of both the lower transition region and the upper transition region of the BPF block in order to ensure that the FM band image is uncorrupted by aliasing. [3]

b) In Figure 4.1, $u[n]$ is multiplied by the complex-valued $v[n] = \exp(-j\omega_c n)$ where ω_c is the normalized centre frequency of the wanted channel.

i) Give a formula for ω_c in terms of c and state how many multiplications are required per second to multiply $u[n]$ and $v[n]$ (where one multiplication calculates the product of two real numbers). [2]

ii) Assume now that only the FM channels with centre frequencies 99.5, 100 and 100.4 MHz are present. Using an unnormalized frequency axis in kHz, draw a dimensioned sketch of the power spectrum of $w[n]$ when $c = 1000$ covering the range -700 to $+700$ kHz. On your sketch, label the centre frequency of each of the occupied spectral regions. [3]

c) i) Explain the purpose of the lowpass FIR filter, $H(z)$ in Figure 4.1. [2]

ii) Assuming that the centre frequencies of active channels are always at least 400 kHz apart, determine the cutoff frequency and maximum transition width of the filter $H(z)$ in radians/sample. Hence use the formula $M = \frac{a}{3.5\Delta\omega}$ from the datasheet to determine the order of the filter to give a stopband attenuation of 50 dB. [3]

iii) Suppose that $H(z)$ is implemented as a polyphase filter as shown in Figure 4.3. Determine the order of the sub-filters assuming they all have the same order. Give an expression for $h_p[r]$, the impulse response of the sub-filter $H_p(z)$, in terms of $h[n]$, the impulse response of $H(z)$. [2]

iv) Calculate the number of multiplications per second needed to implement Figure 4.3 assuming that all sub-filters have the same order. [3]

d) i) Determine the impulse response of $G_c(z)$ such that Figures 4.1 and 4.2 are functionally identical. [3]

ii) If $G_c(z)$ is implemented as a conventional polyphase filter, give an expression for the impulse response, $g_{c,p}[r]$, of the sub-filter $G_{c,p}(z)$.

Show that if $\alpha_c = \exp\left(\frac{j2\pi c}{780}\right)$, then each coefficient, $\alpha_c^{-p} g_{c,p}[r]$, of $\alpha_c^{-p} G_{c,p}(z)$ is either purely real or purely imaginary. [3]

iii) In Figure 4.4, the subfilter $G_{c,p}(z)$ is implemented as $\alpha_c^{-p}G_{c,p}(z)$ followed by a multiplication by α_c^p. Determine a simplified expression for $s[r]$ so that Figure 4.4 is functionally equivalent to Figure 4.3. [3]

iv) Giving your reasons fully, determine the number of multiplications per second required to implement Figure 4.4. You may exclude negation operations from the multiplication count. [3]

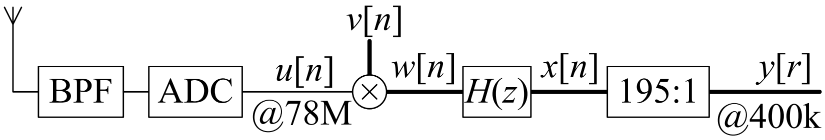


Figure 4.1

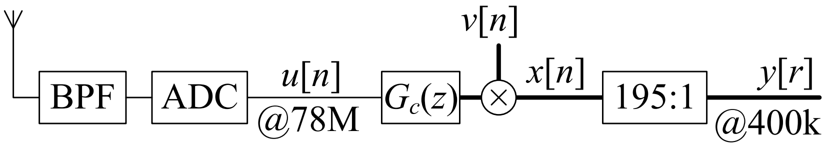


Figure 4.2

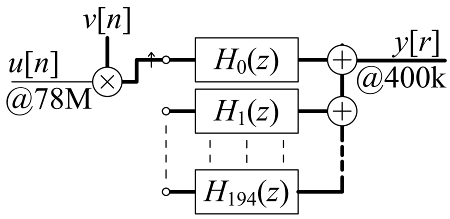


Figure 4.3

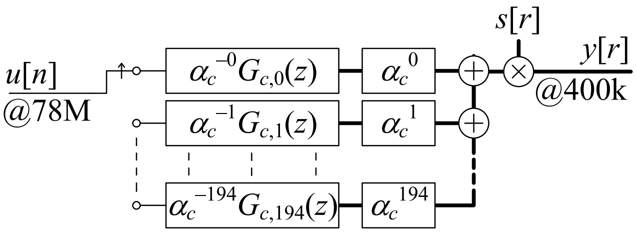


Figure 4.4

Digital Signal Processing and Digital Filters

*********** Solutions ***********

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DFT	Discrete Fourier Transform		PSD	Power Spectral Density
DTFT	Discrete-Time Fourier Transform		SNR	Signal-to-Noise Ratio
FIR	Finite Impulse Response			

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- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

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- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.
- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$ provided that $\left|\alpha z^{-1}\right| < 1$.

Forward and Inverse Transforms

z:	$X(z)=\sum_{-\infty}^{\infty}x[n]z^{-n}$	$x[n]=\frac{1}{2\pi j}\oint X(z)z^{n-1}dz$
CTFT:	$X(j\Omega)=\int_{-\infty}^{\infty}x(t)e^{-j\Omega t}dt$	$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\Omega)e^{j\Omega t}d\Omega$
DTFT:	$X(e^{j\omega})=\sum_{-\infty}^{\infty}x[n]e^{-j\omega n}$	$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$
DFT:	$X[k]=\sum_{0}^{N-1}x[n]e^{-j2\pi\frac{kn}{N}}$	$x[n]=\frac{1}{N}\sum_{0}^{N-1}X[k]e^{j2\pi\frac{kn}{N}}$
DCT:	$X[k]=\sum_{n=0}^{N-1}x[n]\cos\frac{2\pi(2n+1)k}{4N}$	$x[n]=\frac{X[0]}{N}+\frac{2}{N}\sum_{n=1}^{N-1}X[k]\cos\frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k]=\sum_{n=0}^{2N-1}x[n]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n]=\frac{1}{N}\sum_{0}^{N-1}X[k]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n]=x[n]*y[n]\triangleq\sum_{r=-\infty}^{\infty}x[r]y[n-r] \quad\Leftrightarrow\quad V\left(e^{j\omega}\right)=X\left(e^{j\omega}\right)Y\left(e^{j\omega}\right)$

$v[n]=x[n]y[n] \quad\Leftrightarrow\quad V\left(e^{j\omega}\right)=\frac{1}{2\pi}X\left(e^{j\omega}\right)\circledast Y\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{j\theta}\right)Y\left(e^{j(\omega-\theta)}\right)d\theta$

DFT: $v[n]=x[n]\circledast_N y[n]\triangleq\sum_{r=0}^{N-1}x[r]y[(n-r)_{\bmod N}] \quad\Leftrightarrow\quad V[k]=X[k]Y[k]$

$v[n]=x[n]y[n] \quad\Leftrightarrow\quad V[k]=\frac{1}{N}X[k]\circledast_N Y[k]\triangleq\frac{1}{N}\sum_{r=0}^{N-1}X[r]Y[(k-r)_{\bmod N}]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega})=-\frac{d\angle H(e^{j\omega})}{d\omega}=\Re\left(\frac{-z}{H(z)}\frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}}=\Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}\left(\right)$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M\approx\frac{a}{3.5\Delta\omega}$
2. $M\approx\frac{a-8}{2.2\Delta\omega}$
3. $M\approx\frac{a-1.2-20\log_{10}b}{4.6\Delta\omega}$

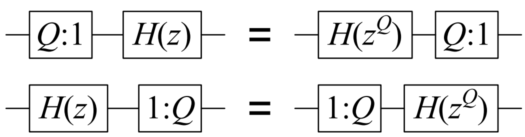
where a =stop band attenuation in dB, b = peak-to-peak passband ripple in dB and $\Delta\omega$ = width of smallest transition band in normalized rad/s.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Noble Identities

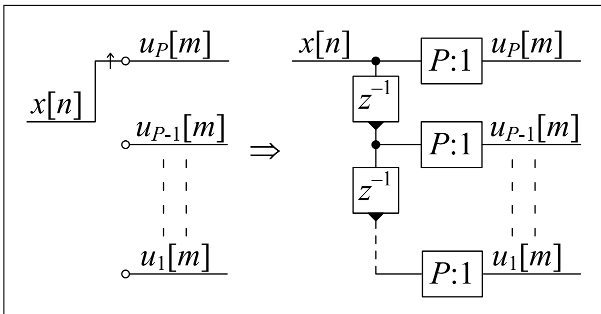
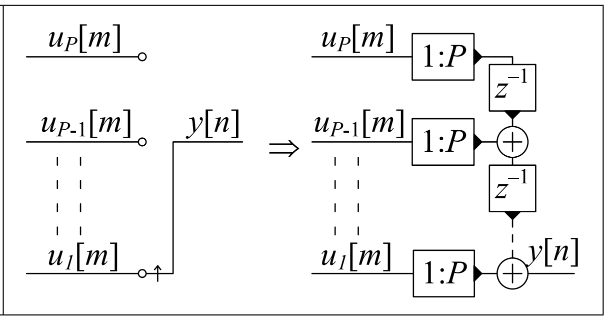


Multirate Spectra

Upsample $v[n]$ by Q: $\quad x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if } Q \mid r \\ 0 & \text{if } Q \nmid r \end{cases} \quad \Rightarrow \quad X(z) = V(z^Q)$

Downsample $v[n]$ by Q: $\quad y[m] = v[Qm] \quad \Rightarrow \quad Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$

Multirate Commutators

Input Commutator	Output Commutator
	

********* Questions and Solutions *********

1. a) The finite length signals $u[0], \cdots, u[M-1]$ and $v[0], \cdots, v[N-1]$ are of length M and N respectively where $M < N$.

The signals $x[n] = u[n] * v[n]$ and $y[n] = u[n] \circledast_N v[n]$ are respectively the convolution and circular convolution of $u[n]$ and $v[n]$ as defined in the data sheet.

i) Prove that $y[n] = x[n]$ for $M-1 \le n \le N-1$. [3]

From the data sheet $y[n] = \sum_{r=0}^{N-1} u[r]v[(n-r)_{\bmod N}]$. Since $u[r] = 0$ outside $0 \le r \le M-1$, we can change the summation limits for both $x[n]$ and $y[n]$ to $0 \le r \le M-1$.

To ensure that $(n-r)_{\bmod N} = n-r$, we need $0 \le n-r \le N-1$ which is equivalent to $r \le n \le N-1+r$. We need this to be true for the entire summing range $0 \le r \le M-1$. We therefore take the maximum of the lower limit and the minimum of the upper limit to obtain $\max(r) \le n \le N-1+\min(r)$ which gives $M-1 \le n \le N-1$ as required. For n within this range, the "$\bmod N$" is redundant and we can write $y[n] = \sum_{r=0}^{M-1} u[r]v[n-r] = x[n]$.

Several people kept the upper summation limit for the crcular convolution at $N-1$ which is equivalent to assuming that $M = N$ and makes the answer impossible to obtain. Quite a large number of people merely stated the answer without proving it as the question asked. Writing $y[n] = \sum_{r=0}^{N-1} v[r]u[(n-r)_{\bmod N}]$ is also a valid expression but it is much harder to deal with because (a) the upper summation limit cannot now be reduced to $M-1$ and (b) the index of $u[\cdots]$ ranges over $[0, N-1]$ and it is necessary to take into account that the last $N-M$ of these values are zero.

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ii) Determine an expression for $y[n]$ in terms of the $\{x[n]\}$ that is valid for $0 \le n \le M-2$. [2]

From the answer to part i), $y[n] = \sum_{r=0}^{M-1} u[r]v[(n-r)_{\bmod N}]$. We can split the summation up into two parts $y[n] = \sum_{r=0}^{n} u[r]v[(n-r)_{\bmod N}] + \sum_{r=n+1}^{M-1} u[r]v[(n-r)_{\bmod N}]$ and for n in the range $0 \le n \le M-2$ both summations include at least one term. For the first summation, $n-r$ is always ≥ 0 since $r \le n$ and so it follows that $(n-r)_{\bmod N} = n-r$. For the second summation, $n-r$ ranges from a minimum of $\min(n) - \max(r) = -(M-1)$ to a maximum of -1 since $r > n$ always. For this range, $(n-r)_{\bmod N} = n-r+N$ since $M < N$ implies that $\min(n-r) = -(M-1) > -N$. Thus we can write $y[n] = \sum_{r=0}^{n} u[r]v[n-r] + \sum_{r=n+1}^{M-1} u[r]v[n-r+N]$. The first term equals $x[n]$ since for r outside the summing range, either $u[r]$ or $v[n-r]$ is zero. The second term equals $x[n+N] = \sum_{r=n+1}^{M-1} u[r]v[n+N-r]$ for the same reason. Thus

$$y[n] = x[n] + x[n+N] \quad \text{for} \quad 0 \le n \le M-2.$$

This result may also be determined graphically by considering the overlap between $v[n]$ and a time-reversed, time-shifted version of

$x[n]$.

Surprisingly few people got this right although it is fairly obvious if you consider the graphical method of performing convolution. Many people did not give an expression in terms of the $\{x[n]\}$ as the question asked. Instead of answering the question, some people gave a condition for ensuring that $y[n] = x[n]$ in this range (e.g. the last $M-1$ values of $v[n]$ shoud be zero).

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iii) If $M = 3$ and $N = 4$ with $u[n] = [1, 2, -1]$ and $v[n] = [1, 1, -1, -1]$ determine both $x[n]$ and $y[n]$ for $0 \le n \le 7$. [3]

$x[n] = [1, 3, 0, -4, -1, 1, 0, 0]$ *and* $y[n] = [0, 4, 0, -4, 0, 4, 0, -4] = [1, 3, 0, -4, -1, 1, 0, 0] + [-1, 1, 0, 0, 1, 3, 0, -4]$.

Note that the convolution is a finite signal but that circular convolution is periodic. The second expression given for $y[n]$ illustrates the answer to part ii).

One of several ways to perform the convolution is to make a table of products:

	$v[0] = 1$	1	-1	-1
$u[0] = 1$	1_0	1_1	-1_2	-1_3
2	2_1	2_2	-2_3	$-2_{4,0}$
-1	-1_2	-1_3	$1_{4,0}$	$1_{5,1}$

Each entry in the table contributes to the $x[i]$ and $y[j]$ indicated by the first and second subscript (with the second omitted if equal to the first). Thus summing along the anti-diagonals gives $x[i]$ and doing the same with wrap-around gives $y[j]$.

Mostly done OK but sometimes with a lot of calculation. Many people only listed $x[n]$ for $0 \le n \le 5$ even though the question asked for $0 \le n \le 7$. Quite a number made $y[n] = 0$ for $n \ge 4$ instead of making it periodic. Several people calculated $y[n] = u[n] \circledast_7 v[n]$ or $y[n] = u[n] \circledast_8 v[n]$ instead of what the question asked.

---

b) i) Show that, if $u[n] = \begin{cases} 1 & n \ge 0 \\ 0 & n < 0 \end{cases}$ and a is a complex-valued constant, then $x[n] = a^n u[n]$ and $y[n] = -a^n u[-n-1]$ have the same z-transform but with different regions of convergence. You may use without proof the geometric progression formulae given in the datasheet. [3]

Using the formula in the datasheet, $X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n} = \sum_0^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}}$ *provided that* $|az^{-1}| < 1 \Leftrightarrow |z| > |a|$.

Similarly, $Y(z) = \sum_{-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{-1} -a^n z^{-n} = \sum_{r=1}^{\infty} -a^{-r} z^r = 1 - \sum_0^{\infty} a^{-r} z^r = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}}$ *provided that* $|a^{-1}z| < 1 \Leftrightarrow |z| <$

$|a|$. In the third step we substituted $r=-n$ and also reversed the summation order (which makes no difference to the sum within the region of absolute convergence).

In the step $\sum_{n=-\infty}^{-1} -a^n z^{-n} = \sum_{r=1}^{\infty} -a^{-r} z^r$, we are substituting $r=-n$; some people made the substitution in the limits but not in the exponents of the summand. Some people just stated $|az^{-1}|<1$ without explicitly turning it into a ROC, i.e. a condition on $|z|$. Several people did not even mention the ROCs for the two cases even though the question asked about them and the convergence condition is explicitly given in the datasheet. Note that writing $z>a$ makes no sense if z and/or a are complex; an inequality requires real-valued operands and you must write $|z|>|a|$.

---

ii) The z-transform $H(z)$ is given by

$$H(z) = \frac{2+17z^{-1}}{(2-z^{-1})(1+4z^{-1})}.$$

By expressing $H(z)$ in partial fraction form, determine the sequence, $h[n]$, whose z-transform is $H(z)$ and whose region of convergence includes $|z|=1$. [4]

We wish to write $H(z) = \frac{b}{2-z^{-1}} + \frac{c}{1+4z^{-1}} = \frac{(b+2c)+(4b-c)z^{-1}}{(2-z^{-1})(1+4z^{-1})}$. By matching coefficients, we obtain $b+2c=2$ and $4b-c=17$ from which $b=4$ and $c=-1$. These coefficients can also be derived using the residue theorem: $b = \left.\frac{2+17z^{-1}}{1+4z^{-1}}\right|_{z=0.5} = 4$ and $c = \left.\frac{2+17z^{-1}}{2-z^{-1}}\right|_{z=-4} = -1$. Hence $H(z) = \frac{2}{1-0.5z^{-1}} - \frac{1}{1+4z^{-1}}$.

The corresponding poles are at $z=0.5$ and $z=-4$, so the sequence we need is $2\times 0.5^n u[n] - (-(-4)^n u[-n-1]) = 2^{1-n}u[n] + (-4)^n u[-n-1]$.

Several people multiplied numerator and denominator by z^2 (sometimes incorrectly) before splitting up as partial fractions to give $H(z) = 1 + \frac{10z+4}{(2z-1)(z+4)} = 1 + \frac{2}{2z-1} + \frac{4}{z+4}$. This makes life harder because it introduces an additional constant term and also leaves the partial fractions in the wrong form to apply part (i) directly. Many people gave the second term as $(-4)^n u[n]$ (which grows exponentially with n) instead of $(-4)^n u[-n-1]$ even though they got part (i) correct. Quite a few put $2^{1-n}u[n] - (-4)^n u[-n-1]$ with an incorrect minus sign for the second term.

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c) i) The frequency response of an ideal lowpass filter is given by

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}.$$

By taking the inverse DTFT of $H(e^{j\omega})$, show that the corresponding impluse response is $h[n] = \dfrac{\sin \omega_0 n}{\pi n}$ [3]

From the datasheet, $h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega n}d\omega = \frac{1}{j2n\pi}\left[e^{j\omega}\right]_{-\omega_0}^{\omega_0} = \frac{2j\sin\omega_0}{j2n\pi} = \frac{\sin\omega_0 n}{\pi n}$.

Most got this right. A few said that $\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = 2\int_{0}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$ *which is only valid if the integrand is an even function (not true in this case). A few people substituted* $e^{j\omega n} = \cos\omega n + j\sin\omega n$ *in the original integral which gives the correct answer but with additional effort.*

ii) By multiplying an ideal filter impulse response by a Hamming window, determine an expression for the real-valued coefficients of an FIR causal bandpass filter of even order, M, whose passband is $1 \le \omega \le 2$.

For even M, a symmetric Hamming window is given by
$w[n] = 0.54 + 0.46\cos\frac{2\pi n}{M+1}$ for $-0.5M \le n \le 0.5M$. [3]

The windowed response is $h[n]w[n]$ *for* $-0.5M \le n \le 0.5M$ *where the ideal impulse response is given by the difference of two lowpass filters:* $h[n] = \frac{\sin 2n - \sin n}{\pi n}$. *In order to make the filter causal, we need to delay the impulse response by* $0.5M$ *samples, and so we need* $w[n-0.5M]h[n-0.5M]$ *for* $0 \le n \le M$. *Thus the coefficients are*

$$g[n] = \left(0.54 + 0.46\cos\frac{2\pi n - \pi M}{M+1}\right)\frac{\sin(2n-M) - \sin(n-0.5M)}{\pi n - 0.5\pi M}.$$

This is the standard windowing method of designing an FIR filter. Instead of subtracting two lowpass filters to get a bandpass filter, some peole shifted the lowpass response in the frequency domain. This resuts in an asymmetric frequency response and hence requires complex coefficients (unless you add together two complementary shifts). Also, the protoype filter needs a 2-sided bandwidth of unity. Others tried to apply the lowpass-to-bandpass transformation from the datasheet. The problem with this approach is that you are applying the tansformation to a filter that is not described by a rational polynomial; no-one did this successfully. Several people tried to calculated the convolution $h[n] * w[n]$ *instead of the product* $h[n]w[n]$*; this is much harder as well as being incorrect.*

d) i) Show that if the coefficients $a[r]$ are all real and

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{r=0}^{M} a[M-r]z^{-r}}{\sum_{r=0}^{M} a[r]z^{-r}}$$

then $\left|H(e^{j\omega})\right| \equiv 1$ and $\angle H(e^{j\omega}) = -M\omega - 2\angle A(e^{j\omega})$. [3]

We can express $B(z) = z^{-M}A(z^{-1})$. *Hence* $H(e^{j\omega}) = e^{-jM\omega}\frac{A(e^{-j\omega})}{A(e^{j\omega})} = e^{-jM\omega}\frac{A^*(e^{j\omega})}{A(e^{j\omega})}$ *where the final step requires all the coefficients of* $A(z)$ *to be real-valued.*

Hence

$$\left|H(e^{j\omega})\right| = \left|e^{-jM\omega}\right| \frac{\left|A^*(e^{j\omega})\right|}{\left|A(e^{j\omega})\right|} = 1 \times \frac{\left|A(e^{j\omega})\right|}{\left|A(e^{j\omega})\right|} = 1$$

and

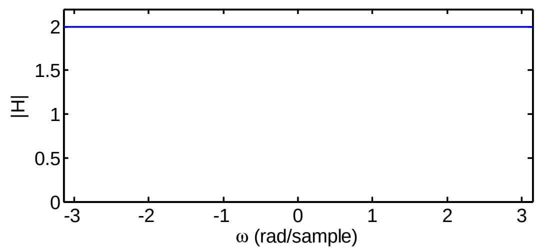
$$\angle H(e^{j\omega}) = \angle e^{-jM\omega} + \angle A^*(e^{j\omega}) - \angle A(e^{j\omega}) = -M\omega - 2\angle A(e^{j\omega}).$$

The equivalence $A(z^{-1}) = A^*(z)$ *is only true if* $|z| = 1$ *and also depends on the coefficients,* $a[n]$, *being real-valued; very few people mentioned either of thee conditions. Quite a few people thought that* $\left|\sum_{r=0}^{M} a[r]z^{-r}\right| = \sum_{r=0}^{M} |a[r]z^{-r}|$; *this is entirely false since the magnitude of a sum does not equal the sum of the magnitudes unless all the summands are real and positive (e.g.* $|-1+2| \neq 1+2$*). A similar error, made by some, is to say incorrectly that* $\left(\sum_{r=0}^{M} a[r]z^{-r}\right)\left(\sum_{r=0}^{M} a[r]z^{-r}\right)^* = \sum_{r=0}^{M} a[r]z^{-r}a[r]^*z^{+r}$; *If you multiply two sums, you must change the dummy variable in one of them to avoid a conflict. We therefore have* $\left(\sum_{r=0}^{M} a[r]z^{-r}\right)\left(\sum_{s=0}^{M} a[s]z^{-s}\right)^* = \sum_{r,s=0}^{M} a[r]z^{-r}a[s]^*z^{+s}$ *and the sum is now over* $(M+1)^2$ *combinations or r and s.*

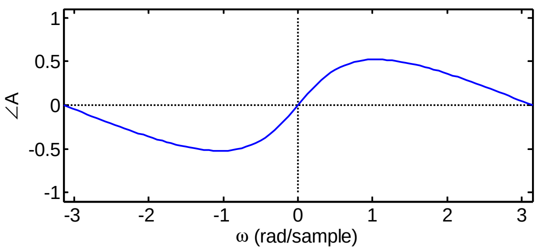
---

ii) If $H(z) = \dfrac{2-4z^{-1}}{2-z^{-1}}$, sketch graphs of the magnitude and phase of $H(e^{j\omega})$ for $-\pi \le \omega \le \pi$. [3]

We can write $H(z) = -2z^{-1}\frac{2-z}{2-z^{-1}}$. *Hence* $|H(z)| = 2\forall\omega$.

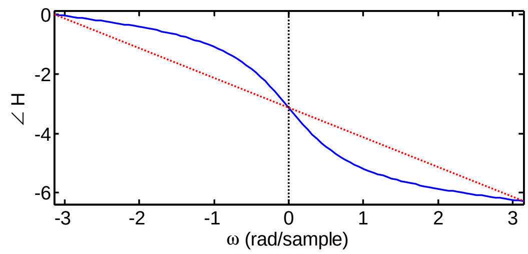


We can write $\angle A(e^{j\omega}) = \angle\left(2 - e^{-j\omega}\right) = \angle(2 - \cos\omega + j\sin\omega) = \tan^{-1}\frac{\sin\omega}{2-\cos\omega}$. *The denominator of the fraction varies between* 1 *(at* $\omega = 0$*) and* 3 *(at* $\omega = \pm\pi$*) and, for* $x < 1$, $\tan^{-1} x \approx x$, *so the graph looks like a distorted sine wave:*



Note that since $A(z)$ has one pole and one zero and both are within the unit circle, the total phase change over $-\pi \le \omega \le \pi$ is equal to zero.

Since $M = 1$, we have $\angle H(z) = -\pi - \omega - 2\angle\left(2 - e^{-j\omega}\right)$. The first two terms are plotted as the dashed line in the lower graph below, and onto this we add $-2\angle A(e^{j\omega})$ to get the final answer.



Not everyone noticed that this was a mltiple of an allpass filter; some people worked out its magnitude and phase response from scratch rather than using the results from part (i). Allpass filters can be recognised by either of two properties: (a) their numerator coefficients are a multiple of the denominator coefficients in reverse order or (b) the poles are the reiprocals of the zeros. In this case, there is a pole at $z = 0.5$ and a zero at $z = 2$. Several said that $\angle\left(2 - e^{-j\omega}\right) = \angle - e^{-j\omega}$ which is not true (e.g. $\angle(1 + j) \neq \angle j$). Several drew the magnitude response as going to zero outsidethe range $\pm\pi$; this is not correct since the response of any discrete time filter is periodic in ω with period 2π. Many people omitted the factor of 2 and said the gain was $|H(z)| = 1\,\forall\omega$.

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e) Figure 1.1 shows the power spectral density (PSD) of a real-valued signal $x[n]$. The horizontal portions of the PSD have values 3, 2 and 1 respectively. The signal $y[n]$ is then obtained by downsampling $x[n]$ by a factor of 3.

Draw a dimensioned sketch showing the PSD of $y[n]$ for $0 \le \omega \le \pi$. You should assume that components of $x[n]$ at different frequencies are uncorrelated and may assume without proof that $Y(z) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{\frac{-j2\pi k}{3}} z^{\frac{1}{3}}\right)$.

Determine the value of each horizontal portion of the PSD and each of the angular frequencies at which its value changes. [5]

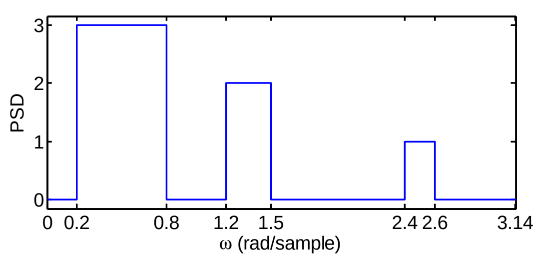
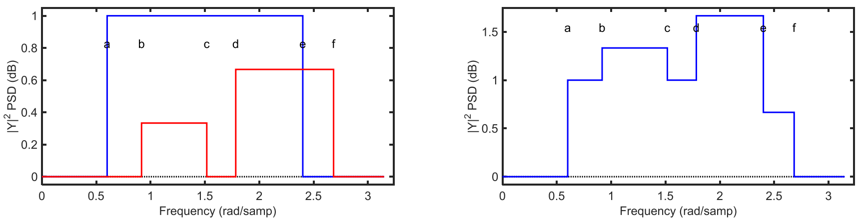


Figure 1.1

Each portion of the original PSD will be expanded horizontally by a factor of 3 and its amplitude reduced by a factor of 3 (i.e. the energy per second is reduced by 3^2 *but since there are now fewer samples, the energy per sample is reduced only by a factor of 3). Thus the portion between* $(0.2, 0.8)$ *will be mapped to* $(\mathrm{a}, \mathrm{e}) = (0.6, 2.4)$ *with amplitude* $\frac{3}{3} = 1$. *The portion between* $(1.2, 1.5)$ *will be mapped to* $(3.6, 4.5) = (3.6 - 2\pi, 4.5 - 2\pi) = (-2.683, -1.783)$. *The symmetric part of this image will therefore be at* $(\mathrm{d}, \mathrm{f}) = (1.783, 2.683)$ *with an amplitude of* $\frac{2}{3} = 0.67$. *Finally, the portion between* $(2.4, 2.6)$ *will be mapped to* $(7.2, 7.8) = (7.2 - 2\pi, 7.8 - 2\pi) = (0.917, 1.517) = (\mathrm{b}, \mathrm{c})$ *with an amplitude of* $\frac{1}{3} = 0.33$.



The figures above show, on the left, the mapped spectral blocks and, on the right, their sum. The frequencies at which the value changes are $\{a, b, c, d, e, f\} = \{0.6, 0.917, 1.517, 1.783, 2.4, 2.683\}$ *and the amplitudes of the flat portions are* $\{0, 3, 4, 3, 5, 2, 0\} \times \frac{1}{3} = \{0, 1, 1.33, 1, 1.67, 0.67, 0\}$. *The total power (integral of the graph) is the same as that of the original signal.*

A few people divided the frequencies by 3 *instead of multiplying them by* 3 *and, in some cases, included the images that would be introduced by upsampling. Note that aliasing moves an image by an integer multiple of* 2π; *several people mapped the spectral portion* $(1.2, 1.5)$ *to* $(3.6 - \pi, 4.5 - \pi) = (0.46, 1.36)$ *which involves a shift of* π *and is incorrect. Several people correctly expanded the width of each block by 3 but kept the centre frequency of each block at the same frequency as before; this is wrong and makes no logical sense. On person took* $\pi = 3$ *which is quite a severe approximation.*

f) Figure 1.2 shows the block diagram of a two-band analysis and synthesis processor. You may assume without proof that, for $m = 0$ or 1, $W_m(z) = U_m(z^2)$ and $U_m(z) = \frac{1}{2}\left\{V_m(z^{\frac{1}{2}}) + V_m(-z^{\frac{1}{2}})\right\}$.

i) Derive a simplified expression for $Y(z)$ in terms of $X(z)$. [4]

We can write

$$\begin{aligned} W_0(z) &= U_0(z^2) = \frac{1}{2}\{V_0(z) + V_0(-z)\} \\ &= \frac{1}{2}\{H(z)X(z) + H(-z)X(-z)\} \end{aligned}$$

Similarly

$$
\begin{aligned}
W_1(z) &= U_1(z^2) = \frac{1}{2}\{V_1(z) + V_1(-z)\} \\
&= \frac{1}{2}\{H(-z)X(z) + H(z)X(-z)\}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
Y(z) &= H(z)W_0(z) - H(-z)W_1(z) \\
&= \frac{1}{2}\left\{H^2(z)X(z) + H(z)H(-z)X(-z) - H^2(-z)X(z) - H(-z)H(z)X(-z)\right\} \\
&= \frac{1}{2}\left\{H^2(z) - H^2(-z)\right\}X(z)
\end{aligned}
$$

Most people got this correct. A surprising number reached the penultimat line above but did not notice that $H(z)H(-z)X(-z)$ and $-H(-z)H(z)X(-z)$ cancelled out.

---

ii) Explain the relationship between the magnitude responses of the filters $H(z)$ and $H(-z)$. [2]

The magnitude response of $H(-e^{j\omega})$ is the same as that of $H(e^{j\omega})$ but reflected around the frequency $\omega = \frac{\pi}{2}$ since $H(-e^{j\omega}) = H\left(e^{j(\omega-\pi)}\right) = H^*\left(e^{j(\pi-\omega)}\right)$ wher the last step assumes that the coefficients of $H(z)$ are all real. Another way to express this is that the complex response of $H(-e^{j\omega}) = H\left(e^{j(\omega-\pi)}\right)$ is the same as that of $H(e^{j\omega})$ but shifted in frequency by π.

Some misunderstood the questions and instead gave conditions on $H(z)$ for perfect reconstruction. Several people said that $H(-e^{j\omega})$ and $H(e^{j\omega})$ have the same magnitude reponse. Some of these people thought that $H(-z)$ was phase-shifted by π; this would have been $-H(z)$.

---

iii) Explain what is meant by saying that the analysis-synthesis processor shown in Figure 1.2 is "alias-free". [2]

The analuysis-synthesis process is alias free because the term $X(-z)$ does not appear in the expression for $Y(z)$. The power spectrum of $X(-z)$ is the same as that of $X(z)$but reflected around $\omega = \frac{\pi}{2}$.

No comment yet

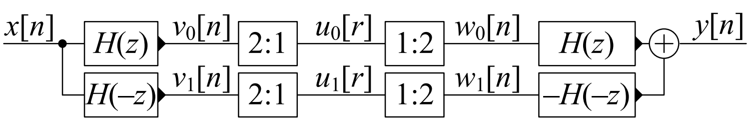


Figure 1.2

2. In this question, filters should be expressed in the standard form $g \times \frac{1+b_1 z^{-1}+\cdots}{1+a_1 z^{-1}+\cdots}$ with numerical values given for all coefficients.

a) A bilinear transformation of the z-plane is given by $z = \frac{\hat{z}-\lambda}{1-\lambda\hat{z}}$ where the real-valued constant λ satisfies $|\lambda| < 1$.

i) Show that $|z|^2 = 1 + \frac{\left(|\hat{z}|^2-1\right)\left(1-\lambda^2\right)}{|1-\lambda\hat{z}|^2}$.

Hence show that $|z| < 1$ if and only if $|\hat{z}| < 1$. [4]

We can write

$$
\begin{aligned}
1+\frac{\left(|\hat{z}|^2-1\right)\left(1-\lambda^2\right)}{|1-\lambda\hat{z}|^2} &= \frac{\left(1-\lambda\left(\hat{z}+\hat{z}^*\right)+\lambda^2|\hat{z}|^2\right)+|\hat{z}|^2-\lambda^2|\hat{z}|^2-1+\lambda^2}{|1-\lambda\hat{z}|^2} \\
&= \frac{|\hat{z}|^2-\lambda\left(\hat{z}+\hat{z}^*\right)+\lambda^2}{|1-\lambda\hat{z}|^2} \\
&= \frac{\left(\hat{z}-\lambda\right)\left(\hat{z}^*-\lambda\right)}{|1-\lambda\hat{z}|^2} \\
&= \frac{|\hat{z}-\lambda|^2}{|1-\lambda\hat{z}|^2} = |z|^2
\end{aligned}
$$

Since $|\lambda| < 1$, the numerator term $\left(1-\lambda^2\right)$ must be strictly positive. In addition, the denominator term satisfies $|1-\lambda\hat{z}|^2 \geq 0$. Hence, assuming for the moment that $1-\lambda\hat{z} \neq 0$, the sign of the fraction is equal to the sign of $\left(|\hat{z}|^2-1\right)$ and is positive or negative according to whether $|\hat{z}| > 1$ or $|\hat{z}| < 1$. Clearly $|\hat{z}| = 1$ makes the fraction zero and hence $|z| = 1$. Putting all this together, we have shown that $|\hat{z}| < 1 \Rightarrow |z| < 1$ and $|\hat{z}| \geq 1 \Rightarrow |z| \geq 1$ which is equivalent to $|z| < 1 \Rightarrow |\hat{z}| < 1$.

The special case, $1-\lambda\hat{z} = 0$, arises when $\hat{z} = \lambda^{-1} > 1$. In this case, the numerator of the fraction is strictly positive and $|z| = +\infty \not< 1$ so the proposition is satisfied.

Some people assumed that $\hat{z}^* = \hat{z}^{-1}$; this is only true if $|\hat{z}| = 1$ which cannot be assumed for this question. The phrase "if and only if" means that you must prove the implication in both directions; quite a few only proved it one way. Very very few people considered the case when the denominator is zero. Note that, by expressing $|z|^2$ as zz^* we eliminate absolute-value operators, $|\cdots|$ from the equations which allows us to use normal algebra rules; a few people tried to manipulate equations that included absolute-value operators and invariably made mistakes such as assuming $|1+z| = 1+|z|$. Surprisingly many people said $|z|^2 = z^2$ or else $|1-\lambda\hat{z}|^2 = (1-\lambda\hat{z})^2$ which, although it neatly avoids any absolute-value issues, is algebraically incorrect when z is complex. Inequalities such as $z < 1$ make no sense whatsoever is z is complex; both sides of an inequality must be real-valued.

ii) Explain why the property shown in part i) is important when using the transformation for filter design. [2]

The property implies that the unit circle maps into itself; this means that if the bilinear transformation is used to transform a filter, the frequency response of a transformed filter is the same as that of the original filter but with a distorted frequency axis. A filter is stable iff all its poles lie strictly inside the unit circle. If this transformation is applied to a stable filter, the property proved in part i) ensures that the transformed filter is also stable. In the same way, it also ensures that a minimum phase filter will transform into another minimum phase filter and that a causal filter will transform into a causal filter.

Surprisingly few people correctly stated any of the above properties. Several just stated that the poles of a stable filter had to be inside the unit circle; this is true but is nothing to do with the transformation.Some made stronger (but untrue) statements such as "the region of convergence is unchanged".

---

b) A first-order lowpass filter has the transfer function $G(z) = 1 + z^{-1}$.

i) Determine the gain of the filter at $\omega = 0$ and show that the magnitude of the gain has decreased by a factor of $\sqrt{2}$ at the cutoff frequency, $\omega_G = \frac{\pi}{2}$. [2]

For $\omega = 0$, the filter gain is $G(e^{j\omega}) = G(1) = 2$.

At $\omega_1 = \frac{\pi}{2}$, the filter gain is $G(e^{j\omega_1}) = G(j) = 1 - j$. Hence $\left|G(e^{j\omega_1})\right| = |1 - j| = \sqrt{2} = \frac{G(1)}{\sqrt{2}}$.

Mostly done OK although a few people calculated complex magnitudes incorrectly. Several people took $|1 + e^{-j0}| = 1$ or even $|1 + e^{-j0}| = 0$.

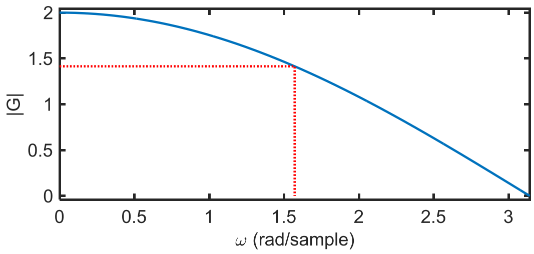
---

ii) By considering the value of $z^{\frac{1}{2}}G(z)$, determine a trigonometrical expression for $\left|G(e^{j\omega})\right|$ and draw a dimensioned sketch of its value over the range $0 \le \omega \le \pi$. [4]

For $z = e^{j\omega}$, we can write

$$
\begin{aligned}
z^{\frac{1}{2}}G(z) &= z^{\frac{1}{2}} + z^{-\frac{1}{2}} \\
e^{j\frac{\omega}{2}}G(e^{j\omega}) &= e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}} \\
&= 2\cos\frac{\omega}{2}
\end{aligned}
$$

Taking the magnitude of each side gives $\left|G(e^{j\omega})\right| = 2\cos\frac{\omega}{2}$ for $|\omega| \le \pi$.



The dashed line shows $|G(\omega_G)| = \sqrt{2}$.

Some had a gradient of 0 at $\omega = \pi$. Some plotted a negative value of $\left|G(e^{j\omega})\right|$ for some values of ω.

---

iii) Using the appropriate z-plane transformation from the datasheet, transform $G(z)$ to a lowpass filter, $H(z)$, with a cutoff frequency of $\omega_H = 0.2$. Calculate the numerical values of the filter coefficients when expressed in the standard form given in the first line of the question. [5]

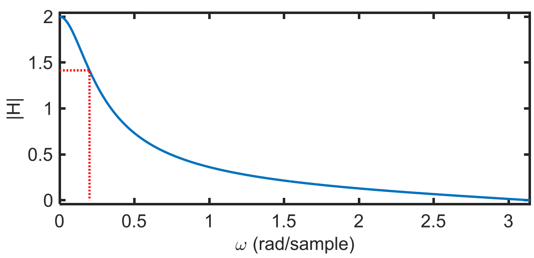
iv) Draw a dimensioned sketch of $\left|H(e^{j\omega})\right|$ over the range $0 \le \omega \le \pi$. [2]

We want a lowpass-to-lowpass transformation with $\omega_0 = \frac{\pi}{2}$ and $\hat{\omega}_1 = 0.2$. So

$$\begin{aligned}\lambda &= \frac{\sin\left(\frac{\omega_0 - \hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0 + \hat{\omega}_1}{2}\right)} \\ &= \frac{\sin 0.6854}{\sin 0.8854} \\ &= \frac{0.633}{0.774} = 0.8176\end{aligned}$$

Substituting for $z^{-1} = \frac{\hat{z}^{-1} - \lambda}{1 - \lambda \hat{z}^{-1}}$ in $G(z)$ gives

$$\begin{aligned}H(\hat{z}) &= 1 + \frac{\hat{z}^{-1} - \lambda}{1 - \lambda \hat{z}^{-1}} \\ &= \frac{1 - \lambda \hat{z}^{-1} + \hat{z}^{-1} - \lambda}{1 - \lambda \hat{z}^{-1}} \\ &= (1 - \lambda)\frac{1 + \hat{z}^{-1}}{1 - \lambda \hat{z}^{-1}} \\ &= 0.1824 \frac{1 + \hat{z}^{-1}}{1 - 0.8176 \hat{z}^{-1}}\end{aligned}$$



The frequency axis has been distorted non-linearly but the gains at $\omega = \{0, \pi\}$ are preserved.

Not everyone copied the formula correctly from the datasheet. The question was very specific about the required form of the answer; even so, some people included the factor g in the numerator coefficients and some people did not calculate the numerical value of λ or the filter coefficients. The transformation maps $\frac{\pi}{2}$ to 0.2; some people also mapped π to 0.4 and said that $\left|H(e^{j\omega})\right| = 0$ for $\omega \geq 0.4$. A small number of people used the substitution formula given in part (a) of the question rather than the one in the datasheet; this gives the wrong answer.

---

c) A quadratic transformation of the z-plane is given by $z = -\tilde{z}^2$.

i) Show that $|z| < 1$ if and only if $|\tilde{z}| < 1$. [2]

We see that $|\tilde{z}| < 1 \Leftrightarrow |\tilde{z}|^2 < 1 \Leftrightarrow |\tilde{z}^2| < 1 \Leftrightarrow |-\tilde{z}^2| < 1 \Leftrightarrow |z| < 1$.

Hence the transformation preserves stability.

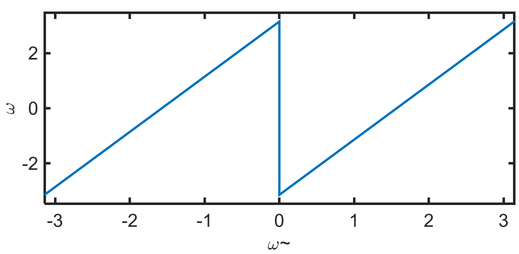
Many people found this difficult and got bogged down in messy complex algebra. Many people proved the implication in only one direction; "if and only if" requires both directions to be proved. Quite a few people wrote expressions like $-1 < z < 1$ which make no sense if z is complex. Many people didn't quite complete their proof; if you are asked to prove that $X \Rightarrow Y$ then your proof must start with X as its first line and end with Y as its last line.

---

ii) If $z = e^{j\omega}$ and $\tilde{z} = e^{j\tilde{\omega}}$ sketch a graph of ω versus $\tilde{\omega}$ over the range $-\pi \leq \tilde{\omega} \leq \pi$. For all $\tilde{\omega}$, the value of ω should be chosen to lie in the range $-\pi < \omega \leq \pi$.

[2]

If $z = -\tilde{z}^2$ then $e^{j\omega} = -e^{j2\tilde{\omega}} = e^{j\pi} \times e^{j2\tilde{\omega}} = e^{j(2\tilde{\omega}+\pi)}$ from which $\omega = (2\tilde{\omega} + \pi)_{\mod 2\pi}$ (or equivalently $\omega = (2\tilde{\omega} - \pi)_{\mod 2\pi}$).



Surprisingly many people found this difficult. Several said $e^{j\omega} = -e^{j2\tilde{\omega}} \Rightarrow j\omega = -j2\tilde{\omega}$ instead of $j\omega = j\pi + j2\tilde{\omega}$. Many people either plotted only the part of the graph for which $2\tilde{\omega} + \pi$ lies in the range $\pm\pi$ or else had the ω axis extending outside the range $\pm\pi$. Angles are only defined modulo 2π and so can always be made to lie in the range $\pm\pi$ as in the graph above (or 0 to 2π if preferred).

iii) A new filter is defined by $P(\tilde{z}) = H(z)$. Determine the numerical values of the coefficients of $P(\tilde{z})$ when expressed in the standard form given in the first line of the question. [3]

We have $H(z) = (1-\lambda)\frac{1+z^{-1}}{1-\lambda z^{-1}}$.

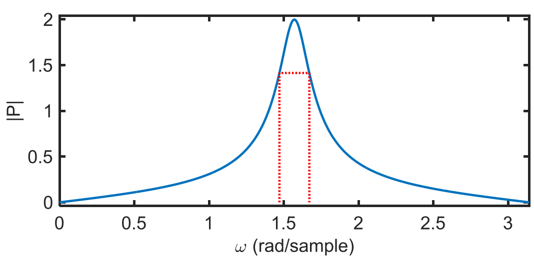
Making the substitution $z^{-1} = -\tilde{z}^{-2}$ gives

$$\begin{aligned} P(\tilde{z}) &= (1-\lambda)\frac{1-\tilde{z}^{-2}}{1+\lambda\tilde{z}^{-2}} \\ &= 0.1824\frac{1-\tilde{z}^{-2}}{1+0.8176\tilde{z}^{-2}} \end{aligned}$$

Mostly done correctly although a few people used the transformation from part (a) instead of $z = -\tilde{z}^2$. Several people did not express the answer in the form requested.

iv) Draw a dimensioned sketch of $\left|P(e^{j\omega})\right|$ over the range $0 \le \omega \le \pi$ and determine the values of ω within this range for which $\left|P(e^{j\omega})\right| = \sqrt{2}$.

Explain the relationship between the bandwidth of the filter $P(e^{j\omega})$ and the cutoff frequency of the filter $H(e^{j\omega})$. [4]



$\left|H(e^{j\omega})\right| = \sqrt{2}$ for $\omega = \pm 0.2$. Hence $\left|P(e^{j\tilde{\omega}})\right| = \sqrt{2}$ when $(2\tilde{\omega}+\pi)_{\bmod 2\pi} = \omega = \pm 0.2$. Solving this equation gives

$$\begin{aligned} 2\tilde{\omega}+\pi &= \pm 0.2 + 2n\pi \\ \tilde{\omega} &= \pm 0.1 + \left(n - \frac{1}{2}\right)\pi \\ &= \cdots, -\frac{3\pi}{2} \pm 0.1, -\frac{\pi}{2} \pm 0.1, \frac{\pi}{2} \pm 0.1, \frac{3\pi}{2} \pm 0.1, , \cdots \end{aligned}$$

The two values of ω in the range $0 \le \omega \le \pi$ for which $\left|P(e^{j\omega})\right| = \sqrt{2}$ are therefore $\omega = \frac{\pi}{2} \pm 0.1 = \{1.4708, 1.6708\}$. This is illustrated by the dotted line on the graph. The bandwidth of the filter is 0.2 and is equal to ω_H, the cutoff frequency of $H(z)$.

Quite a few people tried to determine from first principles the frequencies at which $\left|P(e^{j\omega})\right| = \sqrt{2}$. The entire point of the transformation approach to filter design is that the frequency response stays *the same except for a distorted frequency axis. Many people did not realize that this was a bandpass filter even thugh tey ad the correct expression for $P(z)$; Substituting $z = \pm 1$makes the numerator zero which implies that $P(e^{j\omega}) = 0$for $\omega = 0$ or π.*

3. a) The filter $H(z) = \dfrac{1}{1+a_1 z^{-1} + a_2 z^{-2}}$ where $a_1 = -1.56$ and $a_2 = 0.64$.

i) By multiplying $H(z)$ by its complex conjugate and using the identity $\cos 2\omega = 2\cos^2\omega - 1$, express $\left|H(e^{j\omega})\right|^{-2}$ as a polynomial in $\cos\omega$ giving the coefficients to 5 significant figures. [4]

If $z = e^{j\omega}$, then we know that $z^* = z^{-1}$ and $z^k + z^{-k} = e^{jk\omega} + e^{-jk\omega} = 2\cos k\omega$. So we can write

$$
\begin{aligned}
\left|H(e^{j\omega})\right|^{-2} &= \left(H(e^{j\omega})\right)^{-1}\left(H(e^{-j\omega})\right)^{-1} \\
&= \left(1+a_1 z^{-1}+a_2 z^{-2}\right)\left(1+a_1 z^{1}+a_2 z^{2}\right) \\
&= 1+a_1^2+a_2^2+a_1\left(z^1+z^{-1}\right)+a_2\left(z^2+z^{-2}\right)+a_1 a_2\left(z^1+z^{-1}\right) \\
&= 1+a_1^2+a_2^2+2a_1(1+a_2)\cos\omega+2a_2\cos 2\omega \\
&= 1+a_1^2+a_2^2+2a_1(1+a_2)\cos\omega+2a_2\left(2\cos^2\omega-1\right) \\
&= 1+a_1^2+a_2(a_2-2)+2a_1(1+a_2)\cos\omega+4a_2\cos^2\omega \\
&= 2.560\cos^2\omega-5.1168\cos\omega+2.5632
\end{aligned}
$$

Most people used the correct method but frequently with algebraic errors. The algebra was much messier for those who substituted numerical values for a_1 and a_2 before doing algebraic simplification. In general, it is easient to work with symbolic coefficients until right at the end substituting numerical values as late as possible. Some thought $z^2 + z^{-2} = 2\cos^2\omega$ instead of $2\cos 2\omega$ while others omitted the factor of 2. Some destroyed the symmetry by taking out a factor of z^{-2} from the second line above and others found the poles of $H(z)$ and split it into partial fractions; neither of these approaches was a good idea. Others expanded $z = \cos j\omega + j\sin j\omega$ right at the beginning which creates a lot of messy algebra.

---

ii) The filter $H_1(z)$ is the same as $H(z)$ but with coefficient a_1 increased in magnitude by 1% (i.e. multiplied by 1.01). Similarly, the filter $H_2(z)$ is the same as $H(z)$ but with coefficient a_2 increased in magnitude by 1% .

For $\omega_0 = 0.2$, determine the ratios $\left|\dfrac{H_1(e^{j\omega_0})}{H(e^{j\omega_0})}\right|$ and $\left|\dfrac{H_2(e^{j\omega_0})}{H(e^{j\omega_0})}\right|$ in dB.

[6]

At $\omega_0 = 0.2$, $\cos\omega_0 = 0.9801$ and $\left|H(e^{j\omega_0})\right|^{-2} = 0.007353$ and $\left|H(e^{j\omega_0})\right| = 11.66 = 21.34\,\text{dB}$.

We have $H_1(z)^{-1} = 1 - 1.5756z^{-1} + 0.64z^{-2}$ and so

$$
\begin{aligned}
\left|H_1(e^{j\omega_0})\right|^{-2} &= 1+a_1^2+a_2(a_2-2)+2a_1(1+a_2)\cos\omega+4a_2\cos^2\omega \\
&= 2.56\cos^2\omega-5.1680\cos\omega+2.6121
\end{aligned}
$$

Evaluating this at $\cos\omega = 0.9801$ *gives* $\left|H_1(e^{j\omega_0})\right|^{-2} = 0.006121$ *and* $\left|H_1(e^{j\omega_0})\right| = 12.78 = 22.13\,\text{dB}$. *This is an error of* $\times 1.096 = 0.797\,\text{dB}$.

Similarly $H_2(z)^{-1} = 1 - 1.56z^{-1} + 0.6464z^{-2}$ *and so*

$$\begin{aligned}\left|H_2(e^{j\omega_0})\right|^{-2} &= 1 + a_1^2 + a_2(a_2-2) + 2a_1(1+a_2)\cos\omega + 4a_2\cos^2\omega \\ &= 2.5856\cos^2\omega - 5.1368\cos\omega + 2.5586\end{aligned}$$

Evaluating this at $\cos\omega_0 = 0.9801$ *gives* $\left|H_2(e^{j\omega_0})\right|^{-2} = 0.007806$ *and* $\left|H_2(e^{j\omega_0})\right| = 11.32 = 21.08\,\text{dB}$. *This is an error of* $\times 0.971 = -0.259\,\text{dB}$.

An alternative approach is to evaluate the transfer functions directly at $z_0^{-1} = e^{-0.2j} = 0.9801 - 0.1987j$. *This gives*

$$\begin{aligned}H(z_0)^{-1} &= 0.0606 + 0.0607j \\ &\Rightarrow |H(z_0)| = \frac{1}{0.0858} = 11.662 = \sqrt{136} = 21.33\,\text{dB} \\ H_1(z_0)^{-1} &= 0.0453 + 0.0638j \\ &\Rightarrow |H(z_0)| = \frac{1}{0.0782} = 12.782 = \sqrt{163.38} = 22.13\,\text{dB} \\ H_2(z_0)^{-1} &= 0.0665 + 0.0582j \\ &\Rightarrow |H(z_0)| = \frac{1}{0.0884} = 11.318 = \sqrt{128.1} = 21.08\,\text{dB}.\end{aligned}$$

This part was relatively straightforward if symbolic expressions for the polynomial coefficients has been calculated in part (i). If the coefficients had to be recalculated from scratch using numerical values then it was quite messy and error prone. The numerical values obtained are sensitive to coefficient errors; this is why the previous part asked you to use 5 significant figures for the coefficients. Some gt the conversion to dB wrong by using $10\log_{10}$ *instead of* $20\log_{10}$ *or vice versa.*

---

b) In the block diagram of Figure 3.1 the outputs of all adders are on the right and solid arrows indicate the direction of information flow. Multiplier gains are written adjacent to each multiplier symbol. The parameter p is strictly positive.

i) Show that $G(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1}{1+(p^2-pq-2)z^{-1}+(pq+1)z^{-2}}$. [6]

From the diagram, we can write down that $Y(z) = z^{-1}Y(z) + pU(z)$ *from which* $U(z) = p^{-1}\left(1 - z^{-1}\right)Y(z)$.

We can also write $U(z) = z^{-1}U(z) + p\left(p^{-2}X(z) + qz^{-1}U(z) - z^{-1}Y(z)\right)$ *from which* $\left(1 - z^{-1} - pqz^{-1}\right)U(z) = p^{-1}X(z) - pz^{-1}Y(z)$.

Substituting the expression we derived in the first line for $U(z)$ *in the* *last equation gives*

$$\begin{aligned}\left(1-z^{-1}-pqz^{-1}\right)p^{-1}\left(1-z^{-1}\right)Y(z) &= p^{-1}X(z)-pz^{-1}Y(z)\\ \left(1-z^{-1}-pqz^{-1}-z^{-1}+z^{-2}+pqz^{-2}+p^2z^{-1}\right)Y(z) &= X(z)\end{aligned}$$

Hence $G(z)=\frac{1}{1+(p^2-pq-2)z^{-1}+(1+pq)z^{-2}}$ *as required.*

Mostly done OK. Some people labelled every node in the block diagram with a different variable name. This is perfectly correct but results in a set of 9 simultaneous equations to solve (albeit very simple ones). Generally the fewer variables the better subject to the constraint that every feedback loop must pass through a named variable. Sometimes the algebra included a step (usually near the end) with a very big jump from the previous line. The extreme version of this was to write down the initial equations and then say "from which we can derive" and then write down the answer. When asked to derive an equation, you will lose marks unless each line clearly follows from the one above. A small number of people used 0.2 degrees rather than 0.2 radians for ω.

---

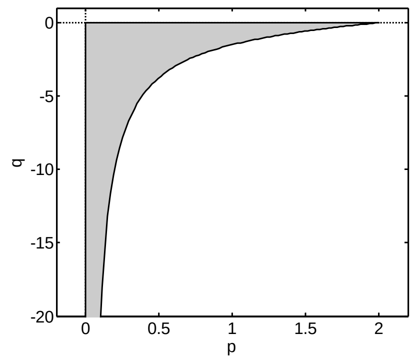
ii) Determine the conditions on p and q for the filter $G(z)$ to be BIBO stable.

You may assume without proof that the filter $\frac{1}{1+b_1z^{-1}+b_2z^{-2}}$ is BIBO stable if and only if $|b_1|-1<b_2<1$. [6]

We can express the condition $|p^2-pq-2|-1<pq+1<1$ *as two separate inequalities. The rightmost inequality is* $pq+1<1 \Leftrightarrow pq<0$ *which means that* p *and* q *are both non-zero and have opposite signs; since* $p>0$ *is stated in the question, we must have* $q<0$.

The leftmost inequality is $|p^2-pq-2|-1<pq+1 \Leftrightarrow |p^2-pq-2|<pq+2$. *An inequality of the form* $|x|<y$ *is the same as* $-y<x<y$, *so we can write* $-pq-2<p^2-pq-2<pq+2$. *This again gives us two inequalities; the left one is* $-pq-2<p^2-pq-2 \Leftrightarrow 0<p^2$ *which just tells us that* p *is non-zero (we knew this already). The right inequality gives us* $p^2-pq-2<pq+2 \Leftrightarrow 2pq>p^2-4 \Leftrightarrow q>\frac{p}{2}-\frac{2}{p}$. *Since we already know* $q<0$, *we can write* $0>q>\frac{p}{2}-\frac{2}{p}$. *Alternatively, by solving the quadratic* $p^2-2pq-4<0$, *we can write* $0<p<q+\sqrt{q^2+4}$.

The condition $\frac{p}{2}-\frac{2}{p}<0$ *implies* $|p|<2$ *so we know that* $0<p<2$ *and* $0>q>\frac{p}{2}-\frac{2}{p}$ *(or equivalently* $q<0$ *and* $0<p<q+\sqrt{q^2+4}$*). This is the shaded region below (plot not requested).*



Note that the constraint $p > 0$ is not actually necessary since changing the sign of both p and q leaves the transfer function unaltered (it just inverts the sign of $u[n]$ and the signals directly connected to it).

Most people got two inequalities, $q < 0$ and $p^2 - 2pq - 4 < 0$ but often had difficulty in transforming the latter into a constraint on p in terms of q.

Note that the inequality $|x| < y$ is equivalent to the two inequalities $+x < y$ and $-x < y$ or, alternatively, $-y < x < y$. This is pretty much the only thing you can do with an inequality that contains absolute value signs; some people tried manipulations like $|x| < y \Rightarrow |x + a| < y + a$ but these are not valid. Another possible approach is to square both sides: $|x| < y \Rightarrow x^2 < y^2$ but, although it does ultimately lead to the correct answer, this doubles the order of the polynomial involved.

Some used the two outer terms of $|p^2 - pq - 2| - 1 < pq + 1 < 1$ to deduce correctly that $|p^2 - pq - 2| - 1 < 1$. However, this is inevitably a weaker inequality than the two involving the central term and so adds nothing useful. Quite a few people said $pq + 1 < 1 \Rightarrow pq < 2$ instead of $pq < 0$. Quite a few people did not notice that the question stated $p > 0$ and included the symmetric solutions for $p < 0$ as well.

---

iii) If

$$G(z) = \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2}},$$

determine expressions for p and q as functions of b_1 and b_2. Calculate the numerical values of p and q if $b_1 = -1.56$ and $b_2 = 0.64$.

[3]

We have $b_1 = p^2 - pq - 2$ and $b_2 = pq + 1$. Adding these equations together gives $b_1 + b_2 = p^2 - 1$ from which $p = \sqrt{b_1 + b_2 + 1}$ (always the positive root since $p > 0$ is given in the question). From the second equation, it is then possible to determine $q = \frac{b_2 - 1}{p} = \frac{b_2 - 1}{\sqrt{b_1 + b_2 + 1}}$.

For the specific values $b_1 = -1.56$ and $b_2 = 0.64$, we get $p = 0.2828$ and $q = -1.2728$.

Most people calculated the numerical values correctly but many omitted the symbolic expressions for p and q which the question asked for.

iv) The filter $G_p(z)$ is the same as $G(z)$ but with coefficient p increased by 1% (i.e. multiplied by 1.01) from the value determined in part iii). Similarly, the filter $G_q(z)$ is the same as $G(z)$ but with coefficient q increased by 1% from the value determined in part iii).

For $\omega_0 = 0.2$, determine the ratios $\left|\frac{G_p(e^{j\omega_0})}{G(e^{j\omega_0})}\right|$ and $\left|\frac{G_q(e^{j\omega_0})}{G(e^{j\omega_0})}\right|$ in dB.

[5]

From part ii), at $\omega_0 = 0.2$, $\cos\omega_0 = 0.9801$ *and* $\left|G(e^{j\omega_0})\right|^{-2} = 0.007353$ *and* $\left|G(e^{j\omega_0})\right| = 11.66 = 21.34\,\text{dB}$.

For G_p, $p = 0.2828 \times 1.01 = 0.2856$ *and* $q = -1.2728$. *This gives* $b_1 = p^2 - pq - 2 = -1.5549$ *and* $b_2 = pq + 1 = 0.6365$ *from which* $G_p(z)^{-1} = 1 - 1.5549z^{-1} + 0.6365z^{-2}$ *and so*

$$\begin{aligned}\left|G_p(e^{j\omega_0})\right|^{-2} &= 1 + b_1^2 + b_2(b_2 - 2) + 2b_1(1 + b_2)\cos\omega + 4b_2\cos^2\omega \\ &= 2.5458\cos^2\omega - 5.0889\cos\omega + 2.5498\end{aligned}$$

Evaluating this at $\cos\omega = 0.9801$ *gives* $\left|G_p(e^{j\omega_0})\right|^{-2} = 0.007618$ *and* $\left|G_p(e^{j\omega_0})\right| = 11.457 = 21.18\,\text{dB}$. *This is an error of* $-0.15\,\text{dB}$.

Similarly, for G_q, $p = 0.2828$ *and* $q = -1.2728 * 1.01 = -1.2855$. *This gives* $b_1 = p^2 - pq - 2 = -1.5565$ *and* $b_2 = pq + 1 = 0.6365$ *from which* $G_q(z)^{-1} = 1 - 1.5565z^{-1} + 0.6365z^{-2}$ *and so*

$$\begin{aligned}\left|G_q(e^{j\omega_0})\right|^{-2} &= 1 + b_1^2 + b_2(b_2 - 2) + 2b_1(1 + b_2)\cos\omega + 4b_2\cos^2\omega \\ &= 2.5458\cos^2\omega - 5.0942\cos\omega + 2.5548\end{aligned}$$

Evaluating this at $\cos\omega_0 = 0.9801$ *gives* $\left|G_q(e^{j\omega_0})\right|^{-2} = 0.007463$ *and* $\left|G_q(e^{j\omega_0})\right| = 11.58 = 21.27\,\text{dB}$. *This is an error of* $-0.064\,\text{dB}$.

Thus $G(z)$ *is significantly less sensitive to coefficient errors (at least at* ω_0*).*

Many people did not attempt this part.

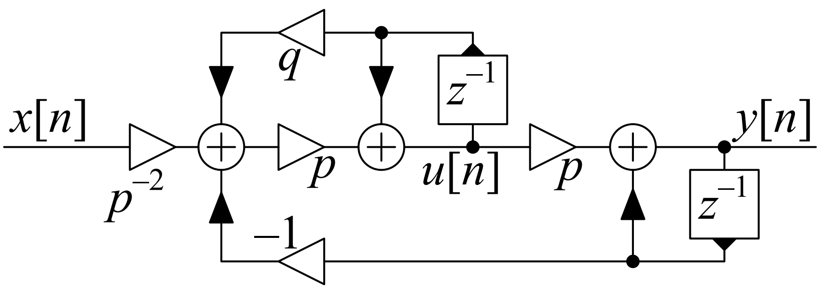
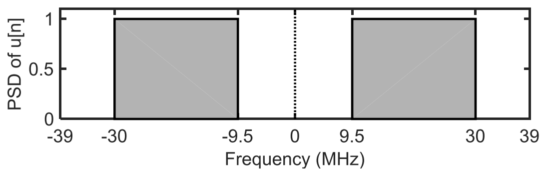


Figure 3.1

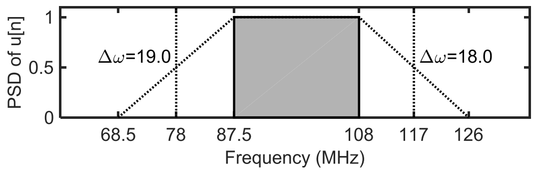
4. The FM radio band extends from 87.5 to 108 MHz. Within this band, an FM channel occupies ± 100 kHz around a centre frequency of $c \times 100$ kHz where the channel index, c, is an integer in the range $876 \leq c \leq 1079$. Figure 4.1 shows the block diagram of an FM radio front-end in which bold lines denote complex-valued signals. The diagram includes a bandpass filter (BPF) whose passband is 87.5 to 108 MHz and an analogue-to-digital converter (ADC) with a sample rate of 78 MHz.

a) Assume the bandpass filter is ideal and the power spectral density of the received signal is constant within the FM band. Sketch the power spectrum of $u[n]$ over the unnormalized frequency range -39 to $+39$ MHz. Determine the maximum width of both the lower transition region and the upper transition region of the BPF block in order to ensure that the FM band image is uncorrupted by aliasing. [3]

The FM band of 87.5 *to* 108 Mhz *will be aliased down by the sample frequency to an image covering* $87.5 - 78 = 9.5$ *to* $108 - 78 = 30$ Mhz.



Frequencies of $78 - 9.5 = 68.5$ Mhz *and* $2 \times 78 - 30 = 126$ Mhz *will be aliased onto the edges of this image and so the widest possible transition bands for the bandpass filter (BPF) are* $68.5 - 87.5 = 19$ Mhz *and* $108 - 129 = 18$ Mhz. *These transition widths can also be deduced from the spectrum plot above as* $2 \times (9.5 - 0) = 19$ Mhz *and* $2 \times (39 - 30) = 18$ Mhz *since* 0 Mhz *and* 39 Mhz *are aliased down from* 78 Mhz *and* 119 Mhz *respectively. Although not asked by the question, these correspond to* $\Delta\omega = \{1.53, 1.45\}$.



Surprisingly, many people did not even attenpt this part and only a minority got the filter transition widths correct. The entire purpose of the bandpass filter is to suppress any frequencies that will alias into the wanted signal band.

---

b) In Figure 4.1, $u[n]$ is multiplied by the complex-valued $v[n] = \exp(-j\omega_c n)$ where ω_c is the normalized centre frequency of the wanted channel.

i) Give a formula for ω_c in terms of c and state how many multiplications are required per second to multiply $u[n]$ and $v[n]$ (where one multiplication calculates the product of two real numbers). [2]

The original unnormalized centre frequency is $\Omega_c = 2\pi c \times 10^5$ *(in the range* 87.5 *to* 107.9 MHz*) but the aliasing has reduced this by* 78 MHz *to* $\Omega'_c = 2\pi(c - 780) \times 10^5$ *(in the range* 19.5 *to* 29.9 MHz*) so the normalized centre frequency is* $\omega_c = \frac{\Omega'_c}{f_s} = \frac{2\pi(c-780)\times 10^5}{78\times 10^6} =$

$\frac{2\pi}{780}c - 2\pi$ meaning that $v[n] = e^{-j\omega_c n} = e^{-j2\pi\frac{(c-780)n}{780}} = e^{-j2\pi\frac{cn}{780}}$. Note that the equivalence $e^{-j2\pi\frac{(c-780)n}{780}} = e^{-j2\pi\frac{cn}{780}}$ means that we can ignore the frequency offset of 2π due to aliasing.

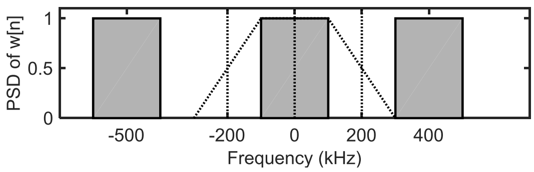
Multiplying a real number, $u[n]$, by a complex number, $v[n]$, requires two multiplications and so the multiplication rate is $2f_s = 156 \times 10^6 = 1.56 \times 10^8$.

Most people got this right except that most had $\omega_c = \frac{2\pi}{780}c$ instead of $\omega_c = \frac{2\pi}{780}c - 2\pi$. As noted in the solution above, this makes no difference to the signal $v[n]$.

---

ii) Assume now that only the FM channels with centre frequencies 99.5, 100 and 100.4 MHz are present. Using an unnormalized frequency axis in kHz, draw a dimensioned sketch of the power spectrum of $w[n]$ when $c = 1000$ covering the range -700 to $+700$ kHz. On your sketch, label the centre frequency of each of the occupied spectral regions. [3]

When $c = 1000$, the spectrum of $u[n]$ is shifted down by 100 MHz to become that of $w[n]$. The shifted centre frequencies of the active FM channels are -0.5, 0 and $+0.4$ MHz. Also marked on the sketch below, but not requested in the question, is the gain of $H(z)$ and $\pm$ the Nyquist frequency, 200 kHz, of the sample rate at $y[n]$.



Almost everyone got this correct. Note that, because the signal is complex, the power spectrum is not necessarily symmetrical (and indeed is asymmetric in this case).

---

c) i) Explain the purpose of the lowpass FIR filter, $H(z)$ in Figure 4.1.[2]

The lowpass filter must remove frequencies outside the range $\pm$200 kHz (which contain unwanted FM channels) in order to prevent aliasing by the downsampler.

Most people got this right.

---

ii) Assuming that the centre frequencies of active channels are always at least 400 kHz apart, determine the cutoff frequency and maximum transition width of the filter $H(z)$ in radians/sample. Hence use the formula $M = \dfrac{a}{3.5\Delta\omega}$ from the datasheet to determine the order of the filter to give a stopband attenuation of 50 dB. [3]

The response of $H(z)$ is shown in the answer to part ii) above. The unnormalized cutoff frequency and transition width are $100\,\text{kHz}$ *and* $200\,\text{kHz}$ *respectively. Multiplying by* $\frac{2\pi}{78\,\text{MHz}}$*, the normalized values are therefore* 8.06×10^{-3} *and* 1.61×10^{-2} *rad/sample. Thus the formula gives* $M = \frac{50}{3.5\times1.61\times10^{-2}} = 887$.

Some took the cutoff frequency as the centre of the transisiton region rather than the correct value which is the edge of the passband.

---

iii) Suppose that $H(z)$ is implemented as a polyphase filter as shown in Figure 4.3. Determine the order of the sub-filters assuming they all have the same order. Give an expression for $h_p[r]$, the impulse response of the sub-filter $H_p(z)$, in terms of $h[n]$, the impulse response of $H(z)$. [2]

The order of an FIR filter is one less than the number of coefficients, so since $H(z)$ has M+1 coefficients, the order of $H_p(z)$ will be $\left\lceil \frac{M+1}{195} - 1 \right\rceil = 4$ *where the brackets denote the ceiling function.*

$h_p[r] = h[p+195r]$ *for* $0 \le p < 195$ *and* $0 \le r \le 4$.

Many people gave the order as the number of coefficients, 5, rather than one less than this number, 4. It is an irritating quirk of nomenclature that the order of an FIR filter is one less than the number of coefficients.

---

iv) Calculate the number of multiplications per second needed to implement Figure 4.3 assuming that all sub-filters have the same order. [3]

The filter coefficients are real but the filer input signal, $w[n]$, is complex. Therefore each of the sub-filters requires $2\times(4+1) = 10$ *multiplications for each of its input samples. Therefore, for each input sample, $u[n]$, we need two multiplications for* $u[n]\times v[n]$ *and* 10 *for the selected sub-filter giving a total of* 12. *The total rate of multiplications per second is therefore* $12\times78\times10^6 = 936\times10^6 = 9.36\times10^8$.

Many people did not include the multiplications for $u[n]\times v[n]$ *and many did not take account of the fact the the filtered signal is complex even though the coefficients are real. Quite a few people wrongly said that only* 5 *complex multiplications were needed per output sample; the number of multiplications is* 5 *per input sample or, almost equivalently,* 888 *per output sample (ignoring the zero coefficients in some of the filters) but definitely not* 5 *per output sample. This can be seen directly from the diagram: each input sample goes to one of the sub-fiters via the commutator (5 multiplications) but each output sample is formed by combining all the subfilter outputs (888multiplications in all).*

---

d) i) Determine the impulse response of $G_c(z)$ such that Figures 4.1 and

4.2 are functionally identical. [3]

From Figure 4.1,

$$
\begin{aligned}
x[n] &= \sum_{m=0}^{M} h[m]w[n-m] \\
&= \sum_{m=0}^{M} h[m]u[n-m]e^{-j\omega_c(n-m)} \\
&= e^{-j\omega_c n}\sum_{m=0}^{M} \left(e^{j\omega_c m}h[m]\right) u[n-m] \\
&= e^{-j\omega_c n}\sum_{m=0}^{M} g_c[m]u[n-m]
\end{aligned}
$$

where $g_c[m] = e^{j\omega_c m}h[m]$. *An alternative way to view this is that* $G_c(z)$ *is a frequency-shifted version of* $H(z)$*where the shift is* $\omega_c - 2\pi$ *(or equivalently* ω_c*). The final expression directly implements Figure 4.2 with* $g_c[m]$ *the impulse response of* $G_c(z)$*.*

Several realized that $G_c(z)$ *was a frequency-shifted version of* $H(z)$ *but often shifted the frequency down rather than up to obtain* $g_c[m] = e^{-j\omega_c m}h[m]$*.*

ii) If $G_c(z)$ is implemented as a conventional polyphase filter, give an expression for the impulse response, $g_{c,p}[r]$, of the sub-filter $G_{c,p}(z)$.

Show that if $\alpha_c = \exp\left(\frac{j2\pi c}{780}\right)$, then each coefficient, $\alpha_c^{-p} g_{c,p}[r]$, of $\alpha_c^{-p} G_{c,p}(z)$ is either purely real or purely imaginary. [3]

From the previous part, we have $g_c[n] = e^{j\omega_c n}h[n] = e^{j2\pi\frac{cn}{780}}h[n]$*. The polyphase implementation therefore has*

$$
\begin{aligned}
g_{c,p}[r] &= g_c[p+195r] \\
&= e^{j2\pi\frac{c(p+195r)}{780}}h[p+195r] \\
&= e^{j2\pi\frac{cp}{780}}e^{j2\pi\frac{cr}{4}}h[p+195r] \\
&= \alpha_c^p j^{cr}h[p+195r]
\end{aligned}
$$

which, as required, is α_c^p *times a quantity that is either purely real or purely imaginary.*

Only a few people got this right.

iii) In Figure 4.4, the subfilter $G_{c,p}(z)$ is implemented as $\alpha_c^{-p}G_{c,p}(z)$ followed by a multiplication by α_c^p. Determine a simplified expression for $s[r]$ so that Figure 4.4 is functionally equivalent to Figure 4.3. [3]

In Figure 4.3 we multiply by $v[n] = e^{-j2\pi\frac{cn}{780}}$ immediately before downsampling. From the noble identities, the is equivalent to multiplying by $s[r] = v[195r] = e^{-j2\pi\frac{195cr}{780}} = e^{-j2\pi\frac{cr}{4}} = -j^{cr}$ after the downsampler as in Figure 4.4.

Only a very few people got this right.

---

iv) Giving your reasons fully, determine the number of multiplications per second required to implement Figure 4.4. You may exclude negation operations from the multiplication count. [3]

Although the subfilter coefficients in Figure 4.4 are complex, only one multiplication per coefficient is required because the input signal is real and the coefficient is either real of purely imaginary. Therefore, for each input sample at $u[n]$, we require 5 multiplies for the filter and 4 for the multiplication by α^p (complex $\times$ complex). Since $s[r]$ is a power of j, it does not involve any actual multiplications. Hence the total number of multiplications is $9 \times 78 \times 10^6 = 702 \times 10^6 = 7.02 \times 10^8$. The reduction relative to part iv) would be larger for larger values of M.

No one got this completely right.

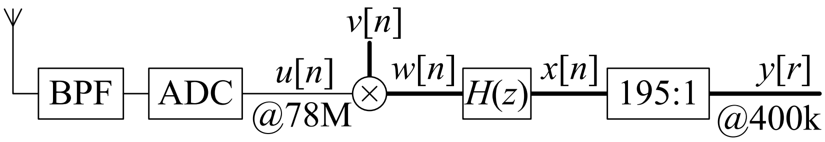


Figure 4.1

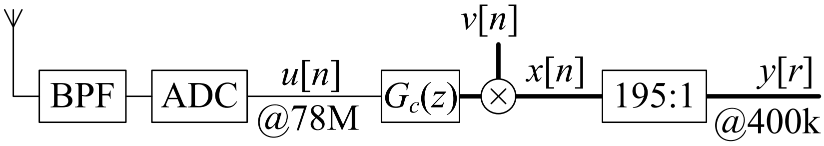


Figure 4.2

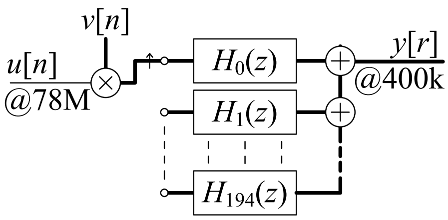


Figure 4.3

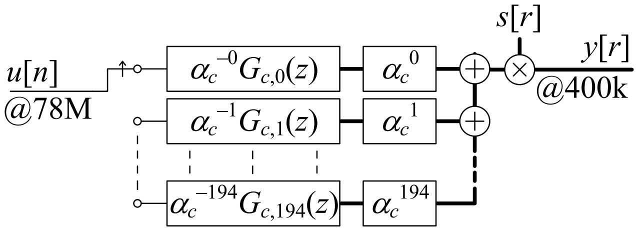


Figure 4.4