▷ 2: Fourier Series
Periodic Functions
Fourier Series
Why Sin and Cos Waves?
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2: Fourier Series

Periodic Functions

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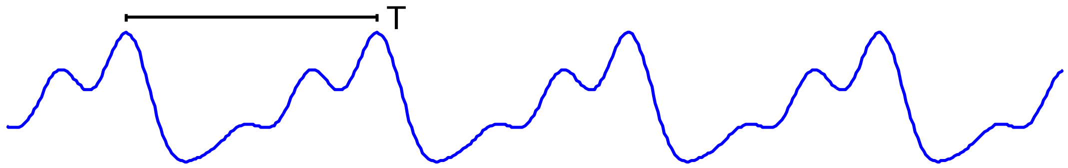
A function, $u(t)$, is periodic with period T if $u(t+T) = u(t)\ \forall t$

- Periodic with period $T \quad \Rightarrow \quad$ Periodic with period $kT\ \forall k \in \mathbb{Z}^+$

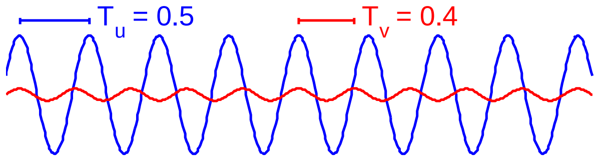
The fundamental period is the smallest $T > 0$ for which $u(t+T) = u(t)$

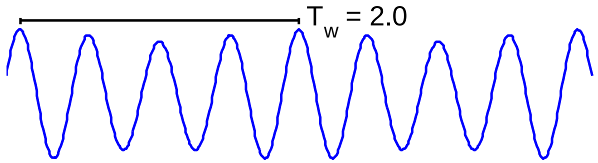


If you add together functions with different periods the fundamental period is the lowest common multiple (LCM) of the individual fundamental periods.

Example:

- $u(t) = \cos 4\pi t \quad \Rightarrow \quad T_u = \frac{2\pi}{4\pi} = 0.5$
- $v(t) = \cos 5\pi t \quad \Rightarrow \quad T_v = \frac{2\pi}{5\pi} = 0.4$
- $w(t) = u(t) + 0.1v(t) \quad \Rightarrow \quad T_w = \mathrm{lcm}(0.5,\ 0.4) = 2.0$





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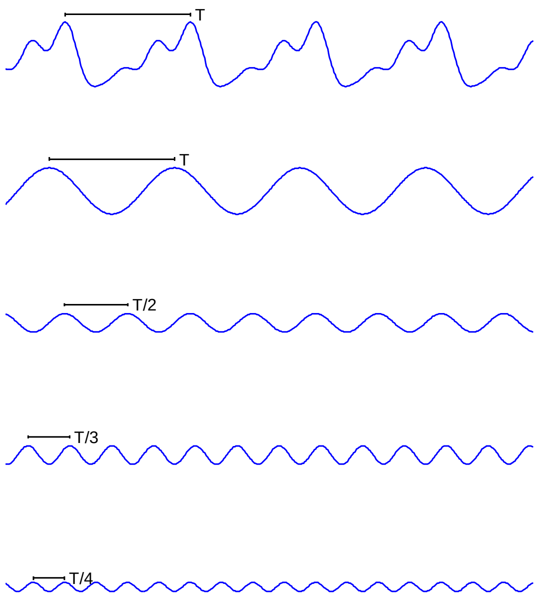
If $u(t)$ has fundamental period T and fundamental frequency $F = \frac{1}{T}$ then, in most cases, we can express it as a (possibly infinite) sum of sine and cosine waves with frequencies 0, F, $2F$, $3F$, $\cdots$.



$u(t) =$

$\sin 2\pi Ft$ $\quad [b_1 = 1]$

$-0.4 \ \sin 2\pi 2Ft$ $\quad [b_2 = -0.4]$

$+0.4 \ \sin 2\pi 3Ft$ $\quad [b_3 = 0.4]$

$-0.2 \ \cos 2\pi 4Ft$ $\quad [a_4 = -0.2]$

The Fourier series for $u(t)$ is

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

The Fourier coefficients of $u(t)$ are a_0, a_1, $\cdots$ and b_1, b_2, $\cdots$.

The n^{th} harmonic of the fundamental is the component at a frequency nF.

Why Sin and Cos Waves?

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Why are engineers obsessed with sine waves?
Answer: Because ...

1. A sine wave remains a sine wave of the same frequency when you
   (a) multiply by a constant,
   (b) add onto to another sine wave of the same frequency,
   (c) differentiate or integrate or shift in time

2. Almost any function can be expressed as a sum of sine waves
   - Periodic functions $\rightarrow$ Fourier Series
   - Aperiodic functions $\rightarrow$ Fourier Transform

3. Many physical and electronic systems are
   (a) composed entirely of constant-multiply/add/differentiate
   (b) linear: $u(t) \rightarrow x(t)$ and $v(t) \rightarrow y(t)$
   means that $u(t) + v(t) \rightarrow x(t) + y(t)$
   $\Rightarrow$ sum of sine waves $\rightarrow$ sum of sine waves

In these lectures we will use T for the fundamental period and $F = \frac{1}{T}$ for the fundamental frequency.

Dirichlet Conditions

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Not all $u(t)$ can be expressed as a Fourier Series.

Peter Dirichlet derived a set of sufficient conditions.

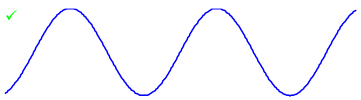
The function $u(t)$ must satisfy:

- periodic and single-valued
- $\int_0^T |u(t)|\, dt < \infty$
- finite number of maxima/minima per period
- finite number of finite discontinuities per period

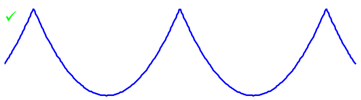
Peter Dirichlet

1805-1859

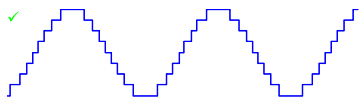
Good:



$\sin(t)$

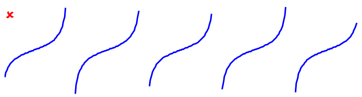


t^2

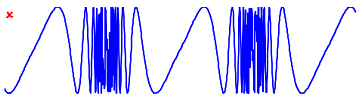


quantized

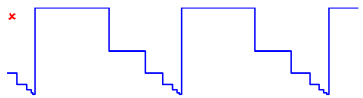
Bad:



$\tan(t)$



$\sin\left(\frac{1}{t}\right)$



∞ halving steps

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Suppose that $u(t)$ satisfies the Dirichlet conditions so that

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Question: How do we find a_n and b_n?

Answer: We use a clever trick that relies on taking averages.

$\langle x(t) \rangle$ equals the average of $x(t)$ over any integer number of periods:

$$\langle x(t) \rangle = \frac{1}{T} \int_{t=0}^{T} x(t)dt$$

Remember, for any integer n, $\langle \sin 2\pi nFt \rangle = 0$

$$\langle \cos 2\pi nFt \rangle = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

Finding a_n and b_n is called Fourier analysis.

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$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\Rightarrow \sin x \cos y = \tfrac{1}{2}\sin(x+y) + \tfrac{1}{2}\sin(x-y)$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\Rightarrow \cos x \cos y = \tfrac{1}{2}\cos(x+y) + \tfrac{1}{2}\cos(x-y)$$
$$\sin x \sin y = \tfrac{1}{2}\cos(x-y) - \tfrac{1}{2}\cos(x+y)$$

Set $x = 2\pi mFt$, $y = 2\pi nFt$ (with $m+n \neq 0$) and take time-averages:

- $\langle \sin(2\pi mFt)\cos(2\pi nFt)\rangle$
  $= \left\langle \tfrac{1}{2}\sin(2\pi(m+n)Ft)\right\rangle + \left\langle \tfrac{1}{2}\sin(2\pi(m-n)Ft)\right\rangle = 0$
- $\langle \cos(2\pi mFt)\cos(2\pi nFt)\rangle$
  $= \left\langle \tfrac{1}{2}\cos(2\pi(m+n)Ft)\right\rangle + \left\langle \tfrac{1}{2}\cos(2\pi(m-n)Ft)\right\rangle = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \end{cases}$
- $\langle \sin(2\pi mFt)\sin(2\pi nFt)\rangle$
  $= \left\langle \tfrac{1}{2}\cos(2\pi(m-n)Ft)\right\rangle - \left\langle \tfrac{1}{2}\cos(2\pi(m+n)Ft)\right\rangle = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \end{cases}$

Summary: $\langle \sin\cos\rangle = 0$ [provided that $m+n \neq 0$]
$\langle \sin\sin\rangle = \langle\cos\cos\rangle = \frac{1}{2}$ if $m = n$ or otherwise $= 0$.

[Trigonometric Products Proofs]

Proof that $\cos x \cos y = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)$

We know that

$$\begin{aligned}\cos(x+y) &= \cos x \cos y - \sin x \sin y\\ \cos(x-y) &= \cos x \cos y + \sin x \sin y\end{aligned}$$

Adding these two gives

$$\cos(x+y) + \cos(x-y) = 2\cos x \cos y$$

From which: $\cos x \cos y = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)$

Subtracting instead of adding gives: $\sin x \sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$

Proof that $\left\langle \frac{1}{2}\cos(2\pi\,(m+n)\,Ft)\right\rangle + \left\langle \frac{1}{2}\cos(2\pi\,(m-n)\,Ft)\right\rangle = \begin{cases}0 & m \neq n\\ \frac{1}{2} & m = n\end{cases}$

We are assuming that m and n are integers with $m+n \neq 0$ and we use the result that $\langle\cos 2\pi f t\rangle$ is zero unless $f = 0$ in which case $\langle\cos 2\pi 0 t\rangle = 1$. The frequency of the first term, $\cos(2\pi\,(m+n)\,Ft)$, is $(m+n)\,F$ which is definitely non-zero (because of our assumption that $m+n \neq 0$) and so the average of this cosine wave is zero. The frequency of the second term is $(m-n)\,F$ and this is zero only if $m = n$. So it follows that the entire expression is zero unless $m = n$ in which case the second term gives a value of $\frac{1}{2}$. Since m and n are integers, we can take the averages over a time interval T and be sure that this includes an integer number of periods for both terms.

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Find a_n and b_n in $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nFt + b_n \sin 2\pi nFt)$

Answer: $a_n = 2\langle u(t)\cos(2\pi nFt)\rangle \triangleq \frac{2}{T}\int_0^T u(t)\cos(2\pi nFt)\,dt$

$b_n = 2\langle u(t)\sin(2\pi nFt)\rangle \triangleq \frac{2}{T}\int_0^T u(t)\sin(2\pi nFt)\,dt$

Proof [a_0]: $2\langle u(t)\cos(2\pi 0Ft)\rangle = 2\langle u(t)\rangle = 2 \times \frac{a_0}{2} = a_0$

Proof [$a_n,\ n > 0$]:

$$2\langle u(t)\cos(2\pi nFt)\rangle$$
$$= 2\left\langle \frac{a_0}{2}\cos(2\pi nFt)\right\rangle + \sum_{r=1}^{\infty} 2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle$$
$$+ \sum_{r=1}^{\infty} 2\langle b_r \sin(2\pi rFt)\cos(2\pi nFt)\rangle$$

Term 1: $2\left\langle \frac{a_0}{2}\cos(2\pi nFt)\right\rangle = 0$

Term 2: $2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle = \begin{cases} a_n & r = n \\ 0 & r \neq n \end{cases}$

$$\Rightarrow \sum_{r=1}^{\infty} 2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle = a_n$$

Term 3: $2\langle b_r \sin 2\pi rFt \cos(2\pi nFt)\rangle = 0$

Proof [$b_n,\ n > 0$]: same method as for a_n

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Truncated Series:

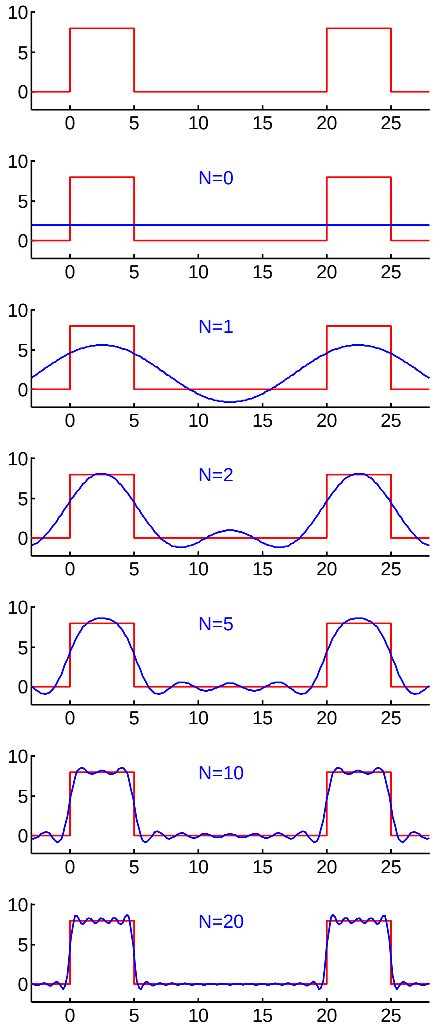
$$u_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Pulse: $T = 20$, width $W = \frac{T}{4}$, height $A = 8$

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T u(t)\cos\frac{2\pi nt}{T}dt \\ &= \frac{2}{T}\int_0^W A\cos\frac{2\pi nt}{T}dt \\ &= \frac{2AT}{2\pi nT}\left[\sin\frac{2\pi nt}{T}\right]_0^W \\ &= \frac{A}{n\pi}\sin\frac{2\pi nW}{T} = \frac{A}{n\pi}\sin\frac{n\pi}{2} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{T}\int_0^T u(t)\sin\frac{2\pi nt}{T}dt \\ &= \frac{2AT}{2\pi nT}\left[-\cos\frac{2\pi nt}{T}\right]_0^W \\ &= \frac{A}{n\pi}\left(1-\cos\frac{n\pi}{2}\right) \end{aligned}$$

n	0	1	2	3	4	5	6
a_n	4	$\frac{8}{\pi}$	0	$\frac{-8}{3\pi}$	0	$\frac{8}{5\pi}$	0
b_n		$\frac{8}{\pi}$	$\frac{16}{2\pi}$	$\frac{8}{3\pi}$	0	$\frac{8}{5\pi}$	$\frac{16}{6\pi}$



[Small Angle Approximation]

In the previous example, we can obtain a_0 by noting that $\frac{a_0}{2} = \langle u(t) \rangle$, the average value of the waveform which must be $\frac{AW}{T} = 2$. From this, $a_0 = 4$. We can, however, also derive this value from the general expression.
The expression for a_m is $a_m = \frac{A}{n\pi} \sin \frac{n\pi}{2}$. For the case, $n = 0$, this is difficult to evaluate because both the numerator and denominator are zero. The general way of dealing with this situation is L'Hôpital's rule (see section 4.7 of RHB) but here we can use a simpler and very useful technique that is often referred to as the "small angle approximation". For small values of θ we can approximate the standard trigonometrical functions as: $\sin \theta \approx \theta$, $\cos \theta \approx 1 - 0.5\theta^2$ and $\tan \theta \approx \theta$. These approximations are obtained by taking the first three terms of the Taylor series; they are accurate to $O(\theta^3)$ and are exactly correct when $\theta = 0$. When $m = 0$ we can therefore make an exact approximation to a_n by writing $a_n = \frac{A}{n\pi} \sin \frac{n\pi}{2} \approx \frac{A}{n\pi} \times \frac{n\pi}{2} = \frac{A}{2} = 4$. What we have implicitly done here is to assume that n is a real number (instead of an integer) and then taken the limit of a_n as $n \to 0$.

Linearity

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Fourier analysis maps a function of time onto a set of Fourier coefficients:

$$u(t) \to \{a_n, b_n\}$$

This mapping is linear which means:

(1) For any γ, if $u(t) \to \{a_n, b_n\}$ then $\gamma u(t) \to \{\gamma a_n, \gamma b_n\}$

(2) If $u(t) \to \{a_n, b_n\}$ and $u'(t) \to \{a'_n, b'_n\}$ then

$$(u(t) + u'(t)) \to \{a_n + a'_n, b_n + b'_n\}$$

Proof for a_n: (proof for b_n is similar)

(1) If $\frac{2}{T}\int_0^T u(t)\cos(2\pi nFt)\,dt = a_n$, then

$$\frac{2}{T}\int_0^T (\gamma u(t))\cos(2\pi nFt)\,dt$$
$$= \gamma\frac{2}{T}\int_0^T u(t)\cos(2\pi nFt)\,dt = \gamma a_n$$

(2) If $\frac{2}{T}\int_0^T u(t)\cos(2\pi nFt)\,dt = a_n$ and

$\frac{2}{T}\int_0^T u'(t)\cos(2\pi nFt)\,dt = a'_n$ then

$$\frac{2}{T}\int_0^T (u(t) + u'(t))\cos(2\pi nFt)\,dt$$
$$= \frac{2}{T}\int_0^T u(t)\cos(2\pi nFt)\,dt + \frac{2}{T}\int_0^T u'(t)\cos(2\pi nFt)\,dt$$
$$= a_n + a'_n$$

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- Fourier Series:
  $$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$
- Dirichlet Conditions: sufficient conditions for FS to exist
  - Periodic, Single-valued, Bounded absolute integral
  - Finite number of (a) max/min and (b) finite discontinuities
- Fourier Analysis = "finding a_n and b_n"
  - $a_n = 2\left\langle u(t)\cos\left(2\pi nFt\right)\right\rangle \triangleq \frac{2}{T}\int_0^T u(t)\cos\left(2\pi nFt\right)dt$
  - $b_n = 2\left\langle u(t)\sin\left(2\pi nFt\right)\right\rangle \triangleq \frac{2}{T}\int_0^T u(t)\sin\left(2\pi nFt\right)dt$
- The mapping $u(t) \to \{a_n, b_n\}$ is linear

For further details see RHB 12.1 and 12.2.