3: Complex Fourier Series

Euler's Equation

Euler's Equation: $e^{i\theta} = \cos\theta + i\sin\theta$ [see RHB 3.3]

Hence: $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = -\frac{1}{2}ie^{i\theta} + \frac{1}{2}ie^{-i\theta}$$

Most maths becomes simpler if you use $e^{i\theta}$ instead of $\cos\theta$ and $\sin\theta$

The Complex Fourier Series is the Fourier Series but written using $e^{i\theta}$

Examples where using $e^{i\theta}$ makes things simpler:

Using $e^{i\theta}$	Using $\cos\theta$ and $\sin\theta$
$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$	$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$
$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$	$\cos\theta\cos\phi = \frac{1}{2}\cos(\theta+\phi) + \frac{1}{2}\cos(\theta-\phi)$
$\frac{d}{d\theta}e^{i\theta} = ie^{i\theta}$	$\frac{d}{d\theta}\cos\theta = -\sin\theta$

Fourier Series: $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$

Substitute: $\cos\theta = \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}$ and $\sin\theta = -\frac{1}{2}ie^{i\theta} + \frac{1}{2}ie^{-i\theta}$

$$
\begin{aligned}
u(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \left(\tfrac{1}{2}e^{i\theta} + \tfrac{1}{2}e^{-i\theta}\right) + b_n \left(-\tfrac{1}{2}ie^{i\theta} + \tfrac{1}{2}ie^{-i\theta}\right)\right) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\left(\tfrac{1}{2}a_n - \tfrac{1}{2}ib_n\right) e^{i2\pi nFt}\right) \qquad [\theta = 2\pi nFt] \\
&\qquad + \sum_{n=1}^{\infty} \left(\left(\tfrac{1}{2}a_n + \tfrac{1}{2}ib_n\right) e^{-i2\pi nFt}\right) \\
&= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}
\end{aligned}
$$

where $[b_0 \triangleq 0]$

$$
U_n = \begin{cases} \frac{1}{2}a_n - \frac{1}{2}ib_n & n \geq 1 \\ \frac{1}{2}a_0 & n = 0 \\ \frac{1}{2}a_{|n|} + \frac{1}{2}ib_{|n|} & n \leq -1 \end{cases} \quad \Leftrightarrow \quad U_{\pm n} = \frac{1}{2}\left(a_{|n|} \mp ib_{|n|}\right)
$$

The U_n are normally complex except for U_0 and satisfy $U_n = U_{-n}^*$

Complex Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$ [simpler ☺]

Averaging Complex Exponentials

If $x(t)$ has period $\frac{T}{n}$ for some integer n (i.e. frequency $\frac{n}{T} = nF$):

$$\langle x(t)\rangle \triangleq \frac{1}{T}\int_{t=0}^{T} x(t)dt$$

This is the average over an integer number of cycles.

For a complex exponential:

$$\begin{aligned}\left\langle e^{i2\pi nFt}\right\rangle &= \langle \cos(2\pi nFt) + i\sin(2\pi nFt)\rangle \\ &= \langle \cos(2\pi nFt)\rangle + i\,\langle \sin(2\pi nFt)\rangle \\ &= \begin{cases} 1+0i & n=0 \\ 0+0i & n\neq 0\end{cases}\end{aligned}$$

Hence:

$$\left\langle e^{i2\pi nFt}\right\rangle = \begin{cases} 1 & n=0 \\ 0 & n\neq 0\end{cases}$$

[☺]

Complex Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

To find the coefficient, U_n, we multiply by something that makes all the terms involving the other coefficients average to zero.

$$\begin{aligned}\left\langle u(t)e^{-i2\pi nFt}\right\rangle &= \left\langle \sum_{r=-\infty}^{\infty} U_r e^{i2\pi rFt} e^{-i2\pi nFt}\right\rangle \\ &= \left\langle \sum_{r=-\infty}^{\infty} U_r e^{i2\pi (r-n)Ft}\right\rangle \\ &= \sum_{r=-\infty}^{\infty} U_r \left\langle e^{i2\pi (r-n)Ft}\right\rangle\end{aligned}$$

All terms in the sum are zero, except for the one when $n = r$ which equals U_n:

$$U_n = \left\langle u(t)e^{-i2\pi nFt}\right\rangle \qquad [☺]$$

This shows that the Fourier series coefficients are unique: you cannot have two different sets of coefficients that result in the same function $u(t)$.

Note the sign of the exponent: "+" in the Fourier Series but "−" for Fourier Analysis (in order to cancel out the "+").

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$
$$= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$

We can easily convert between the two forms.

Fourier Coefficients $\rightarrow$ Complex Fourier Coefficients:

$$U_{\pm n} = \tfrac{1}{2}\left(a_{|n|} \mp i b_{|n|}\right)$$ $[U_n = U_{-n}^*]$

Complex Fourier Coefficients $\rightarrow$ Fourier Coefficients:

$$a_n = U_n + U_{-n} = 2\Re\left(U_n\right)$$ [$\Re$ = "real part"]

$$b_n = i\left(U_n - U_{-n}\right) = -2\Im\left(U_n\right)$$ [$\Im$ = "imaginary part"]

The formula for a_n works even for $n = 0$.

[Complex functions of time]

In these lectures, we are assuming that $u(t)$ is a periodic real-valued function of time. In this case we can represent $u(t)$ using either the Fourier Series or the Complex Fourier Series:

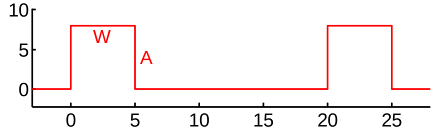
$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nFt + b_n \sin 2\pi nFt) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$

We have seen that the U_n coefficients are complex-valued and that U_n and U_{-n} are complex conjugates so that we can write $U_{-n} = U_n^*$.

In fact, the complex Fourier series can also be used when $u(t)$ is a complex-valued function of time (this is sometimes useful in the fields of communications and signal processing). In this case, it is still true that $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$, but now U_n and U_{-n} are completely independent and normally $U_{-n} \neq U_n^*$.

Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



Method 1:

$$U_{\pm n} = \tfrac{1}{2}a_n \mp i\tfrac{1}{2}b_n$$

Method 2:

$$U_n = \left\langle u(t)e^{-i2\pi nFt}\right\rangle$$

$$= \tfrac{1}{T}\int_0^T u(t)e^{-i2\pi nFt}dt$$

$$= \tfrac{1}{T}\int_0^W Ae^{-i2\pi nFt}dt$$

$$= \frac{A}{-i2\pi nFT}\left[e^{-i2\pi nFt}\right]_0^W$$

$$= \frac{A}{i2\pi n}\left(1 - e^{-i2\pi nFW}\right)$$

$$= \frac{Ae^{-i\pi nFW}}{i2\pi n}\left(e^{i\pi nFW} - e^{-i\pi nFW}\right)$$

$$= \frac{Ae^{-i\pi nFW}}{n\pi}\sin\left(n\pi FW\right)$$

$$= \frac{8}{n\pi}\sin\left(\frac{n\pi}{4}\right)e^{-i\frac{n\pi}{4}}$$

n	a_n	b_n	U_n
-6			$i\frac{8}{6\pi}$
-5			$\frac{4}{5\pi} + i\frac{4}{5\pi}$
-4			0
-3			$\frac{-4}{3\pi} + i\frac{4}{3\pi}$
-2			$i\frac{8}{2\pi}$
-1			$\frac{4}{\pi} + i\frac{4}{\pi}$
0	4		2
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	$\frac{4}{\pi} + i\frac{-4}{\pi}$
2	0	$\frac{16}{2\pi}$	$i\frac{-8}{2\pi}$
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	$\frac{-4}{3\pi} + i\frac{-4}{3\pi}$
4	0	0	0
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	$\frac{4}{5\pi} + i\frac{-4}{5\pi}$
6	0	$\frac{16}{6\pi}$	$i\frac{-8}{6\pi}$

Complex Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

If $v(t)$ is the same as $u(t)$ but delayed by a time τ: $v(t) = u(t-\tau)$

$$v(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nF(t-\tau)} = \sum_{n=-\infty}^{\infty} \left(U_n e^{-i2\pi nF\tau}\right) e^{i2\pi nFt}$$
$$= \sum_{n=-\infty}^{\infty} V_n e^{i2\pi nFt}$$

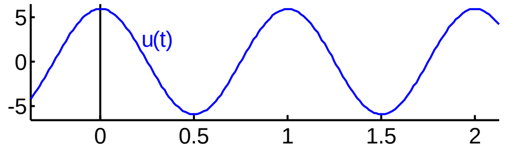
where $V_n = U_n e^{-i2\pi nF\tau}$

Example:

$u(t) = 6\cos(2\pi Ft)$

Fourier: $a_1 = 6,\ b_1 = 0$

Complex: $U_{\pm 1} = \frac{1}{2}a_1 \mp \frac{1}{2}ib_1 = 3$

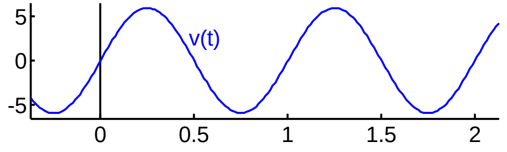


$v(t) = 6\sin(2\pi Ft) = u(t-\tau)$

Time delay: $\tau = \frac{T}{4} \Rightarrow F\tau = \frac{1}{4}$

Complex: $V_1 = U_1 e^{-i\frac{\pi}{2}} = -3i$

$V_{-1} = U_{-1} e^{i\frac{\pi}{2}} = +3i$



Note: If $u(t)$ is a sine wave, U_1 equals half the corresponding phasor.

(1) $u(t)$ real-valued $\Leftrightarrow$ U_n conjugate symmetric $[U_n = U_{-n}^*]$
(2) $u(t)$ even $[u(t) = u(-t)]$ $\Leftrightarrow$ U_n even $[U_n = U_{-n}]$
(3) $u(t)$ odd $[u(t) = -u(-t)]$ $\Leftrightarrow$ U_n odd $[U_n = -U_{-n}]$
(1)+(2) $u(t)$ real & even $\Leftrightarrow$ U_n real & even $[U_n = U_{-n}^* = U_{-n}]$
(1)+(3) $u(t)$ real & odd $\Leftrightarrow$ U_n imaginary & odd $[U_n = U_{-n}^* = -U_{-n}]$

Proof of (2): $u(t)$ even $\Rightarrow$ U_n even

$$U_{-n} = \frac{1}{T}\int_0^T u(t) e^{-i2\pi(-n)Ft} dt$$

$$= \frac{1}{T}\int_{x=0}^{-T} u(-x) e^{-i2\pi nFx} (-dx) \quad \text{[substitute } x = -t\text{]}$$

$$= \frac{1}{T}\int_{x=-T}^{0} u(-x) e^{-i2\pi nFx} dx \quad \text{[reverse the limits]}$$

$$= \frac{1}{T}\int_{x=-T}^{0} u(x) e^{-i2\pi nFx} dx = U_n \quad \text{[even: } u(-x) = u(x)\text{]}$$

Proof of (3): $u(t)$ odd $\Rightarrow$ U_n odd

Same as before, except for the last line:

$$= \frac{1}{T}\int_{x=-T}^{0} -u(x) e^{-i2\pi nFx} dx = -U_n \quad \text{[odd: } u(-x) = -u(x)\text{]}$$

Antiperiodic ⇒ Odd Harmonics Only

A waveform, $u(t)$, is anti-periodic if $u(t+\frac{1}{2}T) = -u(t)$.
If $u(t)$ is anti-periodic then $U_n = 0$ for n even.

Proof:

Define $v(t) = u(t+\frac{T}{2})$, then

(1) $v(t) = -u(t) \Rightarrow V_n = -U_n$

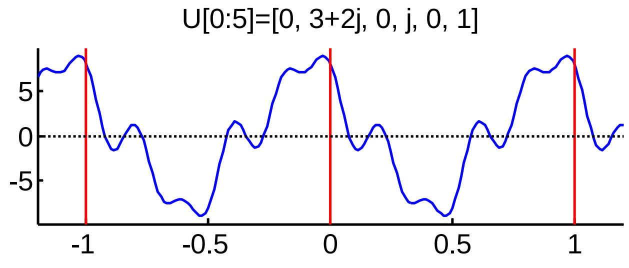
(2) $v(t)$ equals $u(t)$ but delayed by $-\frac{T}{2}$

$$\Rightarrow V_n = U_n e^{i2\pi nF\frac{T}{2}} = U_n e^{in\pi} = \begin{cases} U_n & n\,\text{even} \\ -U_n & n\,\text{odd} \end{cases}$$

Hence for n even: $V_n = -U_n = U_n \Rightarrow U_n = 0$

Example:

$U_{0:5} = [0,\ 3+2i,\ 0,\ i,\ 0,\ 1]$
Odd harmonics only ⇔
Second half of each period is the negative of the first half.

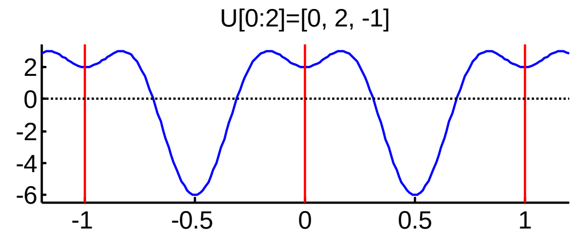


Symmetry Examples

All these examples assume that $u(t)$ is real-valued $\Leftrightarrow U_{-n} = U_{+n}^*$.

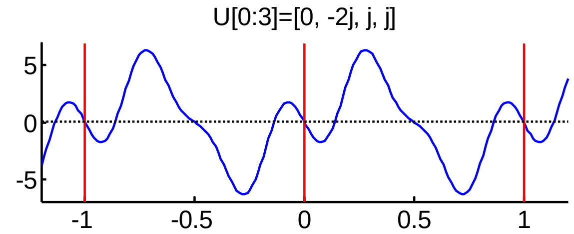
(1) Even $u(t) \Leftrightarrow$ real U_n

$$U_{0:2} = [0,\ 2,\ -1]$$



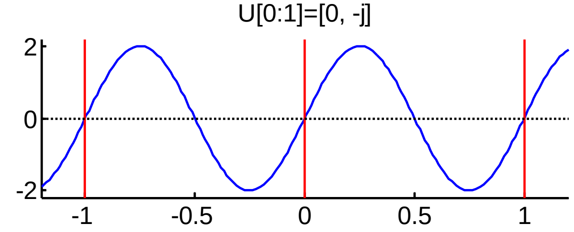
(2) Odd $u(t) \Leftrightarrow$ imaginary U_n

$$U_{0:3} = [0,\ -2i,\ i,\ i]$$



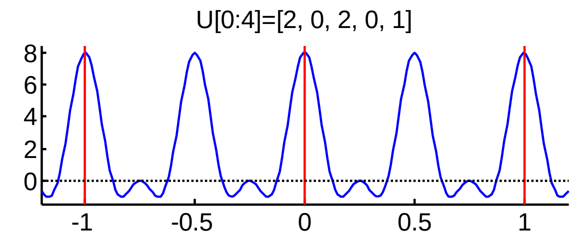
(3) Anti-periodic $u(t)$
$\Leftrightarrow$ odd harmonics only

$$U_{0:1} = [0,\ -i]$$



(4) Even harmonics only
$\Leftrightarrow$ period is really $\frac{1}{2}T$

$$U_{0:4} = [2,\ 0,\ 2,\ 0,\ 1]$$



- Fourier Series:
  $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$
- Complex Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$
  - $U_n = \left\langle u(t) e^{-i2\pi nFt} \right\rangle \triangleq \frac{1}{T}\int_0^T u(t) e^{-i2\pi nFt} dt$
  - Since $u(t)$ is real-valued, $U_n = U_{-n}^*$
  - FS→CFS: $U_{\pm n} = \frac{1}{2} a_{|n|} \mp i \frac{1}{2} b_{|n|}$
  - CFS→FS: $a_n = U_n + U_{-n}$
    $b_n = i\left(U_n - U_{-n}\right)$
- $u(t)$ real and even $\Leftrightarrow u(-t) = u(t)$
  $\Leftrightarrow U_n$ is real-valued and even $\Leftrightarrow b_n = 0\ \forall n$
- $u(t)$ real and odd $\Leftrightarrow u(-t) = -u(t)$
  $\Leftrightarrow U_n$ is purely imaginary and odd $\Leftrightarrow a_n = 0\ \forall n$
- $u(t)$ anti-periodic $\Leftrightarrow u(t + \frac{T}{2}) = -u(t)$
  $\Leftrightarrow$ odd harmonics only $\Leftrightarrow a_{2n} = b_{2n} = U_{2n} = 0\ \forall n$

For further details see RHB 12.3 and 12.7.