4: Parseval's Theorem and Convolution

Suppose we have two signals with the same period, $T = \frac{1}{F}$,

$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$

$$\Rightarrow \quad u^*(t) = \sum_{n=-\infty}^{\infty} U_n^* e^{-i2\pi nFt} \qquad [u(t) = u^*(t) \text{ if real}]$$

$$v(t) = \sum_{n=-\infty}^{\infty} V_n e^{i2\pi nFt}$$

Now multiply $u^*(t)$ and $v(t)$ together and take the average over $[0, T]$.
[Use different "dummy variables", n and m, so they don't get mixed up]

$$\begin{aligned}\langle u^*(t)v(t)\rangle &= \left\langle \sum_{n=-\infty}^{\infty} U_n^* e^{-i2\pi nFt} \sum_{m=-\infty}^{\infty} V_m e^{i2\pi mFt} \right\rangle \\ &= \sum_{n=-\infty}^{\infty} U_n^* \sum_{m=-\infty}^{\infty} V_m \left\langle e^{-i2\pi nFt} e^{i2\pi mFt} \right\rangle \\ &= \sum_{n=-\infty}^{\infty} U_n^* \sum_{m=-\infty}^{\infty} V_m \left\langle e^{i2\pi(m-n)Ft} \right\rangle\end{aligned}$$

The quantity $\langle \cdots \rangle$ equals 1 if $m = n$ and 0 otherwise, so the only non-zero element in the second sum is when $m = n$, so the second sum equals V_n.

Hence Parseval's theorem: $\langle u^*(t)v(t)\rangle = \sum_{n=-\infty}^{\infty} U_n^* V_n$

If $v(t) = u(t)$ we get: $\left\langle |u(t)|^2 \right\rangle = \sum_{n=-\infty}^{\infty} U_n^* U_n = \sum_{n=-\infty}^{\infty} |U_n|^2$

[Manipulating sums]

If you have a multiplicative expression involving two or more sums, then you must use different dummy variables for each of the sums:

$$\sum_n af(n) \sum_m bg(m)$$

(1) You can always move any quantities to the right

$$\begin{aligned}\sum_n af(n) \sum_m bg(m) &= \sum_n a \sum_m bf(n)g(m) \\ &= \sum_n \sum_m abf(n)g(m)\end{aligned}$$

(2) You can move quantities to the left past a summation provided that they do not involve the dummy variable of the summation:

$$\begin{aligned}\sum_n \sum_m abf(n)g(m) &= \sum_n af(n) \sum_m bg(m) \\ &\neq \sum_n af(n)g(m) \sum_m b\end{aligned}$$

The last expression doesn't make sense in any case since m is undefined outside $\sum_m$

(3) You can swap the summation order if the sum converges absolutely

$$\sum_n \sum_m h(n,m) = \sum_m \sum_n h(n,m) \quad \text{provided that } \sum_n \sum_m |h(n,m)| < \infty$$

The equality on the left is not necessarily true if the sum does not converge absolutely. Of course, if the sum has only a finite number of terms, it is bound to converge absolutely.

Power Conservation

The average power of a periodic signal is given by $P_u \triangleq \left\langle |u(t)|^2 \right\rangle$.

This is the average electrical power that would be dissipated if the signal represents the voltage across a $1\,\Omega$ resistor.

Parseval's Theorem: $P_u = \left\langle |u(t)|^2 \right\rangle = \sum_{n=-\infty}^{\infty} |U_n|^2$

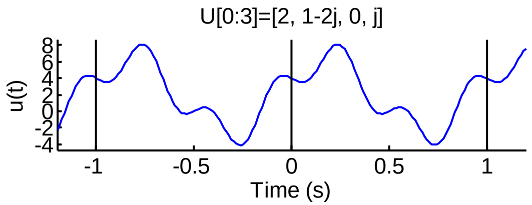
$$= |U_0|^2 + 2\sum_{n=1}^{\infty} |U_n|^2 \qquad \text{[assume } u(t) \text{ real]}$$

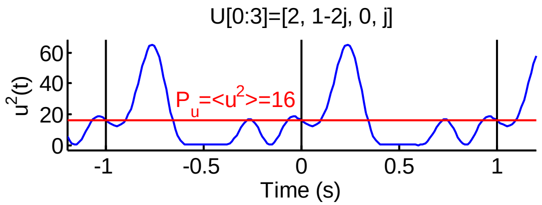
$$= \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \qquad \left[U_{+n} = \frac{a_n - ib_n}{2}\right]$$

Parseval's theorem ⇒ the average power in $u(t)$ is equal to the sum of the average powers in each of its Fourier components.

Example: $u(t) = 2 + 2\cos 2\pi Ft + 4\sin 2\pi Ft - 2\sin 6\pi Ft$

$$\left\langle |u(t)|^2 \right\rangle = 4 + \frac{1}{2}\left(2^2 + 4^2 + (-2)^2\right) = 16$$





$$U_{0:3} = [2,\ 1-2i,\ 0,\ i] \qquad \Rightarrow \qquad |U_0|^2 + 2\sum_{n=1}^{\infty} |U_n|^2 = 16$$

Magnitude Spectrum and Power Spectrum

The *spectrum* of a periodic signal is the values of $\{U_n\}$ versus nF.

The *magnitude spectrum* is the values of $\{|U_n|\} = \left\{\frac{1}{2}\sqrt{a_{|n|}^2 + b_{|n|}^2}\right\}$.

The *power spectrum* is the values of $\left\{|U_n|^2\right\} = \left\{\frac{1}{4}\left(a_{|n|}^2 + b_{|n|}^2\right)\right\}$.

Example:

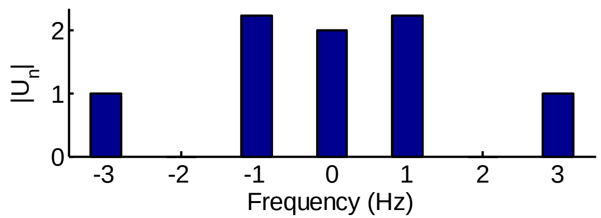
$$u(t) = 2 + 2\cos 2\pi F t + 4\sin 2\pi F t - 2\sin 6\pi F t$$

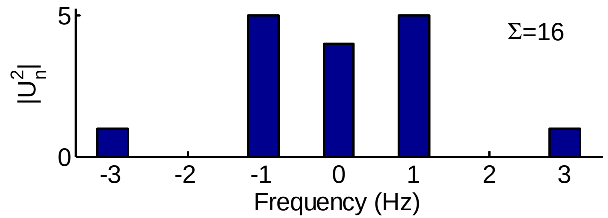
Fourier Coefficients: $a_{0:3} = [4,\ 2,\ 0,\ 0]$ $\qquad b_{1:3} = [4,\ 0,\ -2]$

Spectrum: $U_{-3:3} = [-i,\ 0,\ 1+2i,\ 2,\ 1-2i,\ 0,\ i]$

Magnitude Spectrum: $|U_{-3:3}| = \left[1,\ 0,\ \sqrt{5},\ 2,\ \sqrt{5},\ 0,\ 1\right]$

Power Spectrum: $\left|U_{-3:3}^2\right| = [1,\ 0,\ 5,\ 4,\ 5,\ 0,\ 1]$ $\qquad \left[\sum = \left\langle u^2(t)\right\rangle\right]$





The magnitude and power spectra of a real periodic signal are symmetrical.

A *one-sided power power spectrum* shows U_0 and then $2|U_n|^2$ for $n \geq 1$.

Suppose we have two signals with the same period, $T = \frac{1}{F}$,

$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$
$$v(t) = \sum_{m=-\infty}^{\infty} V_n e^{i2\pi mFt}$$

If $w(t) = u(t)v(t)$ then $W_r = \sum_{m=-\infty}^{\infty} U_{r-m} V_m \triangleq U_r * V_r$

Proof:

$$w(t) = u(t)v(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt} \sum_{m=-\infty}^{\infty} V_m e^{i2\pi mFt}$$
$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U_n V_m e^{i2\pi(m+n)Ft}$$

Now we change the summation variable to use r instead of n:

$$r = m + n \Rightarrow n = r - m$$

This is a one-to-one mapping: every pair (m, n) in the range $\pm\infty$ corresponds to exactly one pair (m, r) in the same range.

$$w(t) = \sum_{r=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U_{r-m} V_m e^{i2\pi rFt} = \sum_{r=-\infty}^{\infty} W_r e^{i2\pi rFt}$$

where $W_r = \sum_{m=-\infty}^{\infty} U_{r-m} V_m \triangleq U_r * V_r$.

W_r is the sum of all products $U_n V_m$ for which $m + n = r$.

The spectrum $W_r = U_r * V_r$ is called the convolution of U_r and V_r.

Convolution behaves algebraically like multiplication:

1) Commutative: $U_r * V_r = V_r * U_r$
2) Associative: $U_r * V_r * W_r = (U_r * V_r) * W_r = U_r * (V_r * W_r)$
3) Distributive over addition: $W_r * (U_r + V_r) = W_r * U_r + W_r * V_r$
4) Identity Element or "1": If $I_r = \begin{cases} 1 & r = 0 \\ 0 & r \neq 0 \end{cases}$, then $I_r * U_r = U_r$

Proofs: (all sums are over $\pm\infty$)

1) Substitute for m: $n = r - m \Leftrightarrow m = r - n$ [$1 \leftrightarrow 1$ for any r]
$\sum_m U_{r-m} V_m = \sum_n U_n V_{r-n}$

2) Substitute for n: $k = r + m - n \Leftrightarrow n = r + m - k$ [$1 \leftrightarrow 1$]
$\sum_n \left(\left(\sum_m U_{n-m} V_m \right) W_{r-n} \right) = \sum_k \left(\left(\sum_m U_{r-k} V_m \right) W_{k-m} \right)$
$= \sum_k \sum_m U_{r-k} V_m W_{k-m} = \sum_k \left(U_{r-k} \left(\sum_m V_m W_{k-m} \right) \right)$

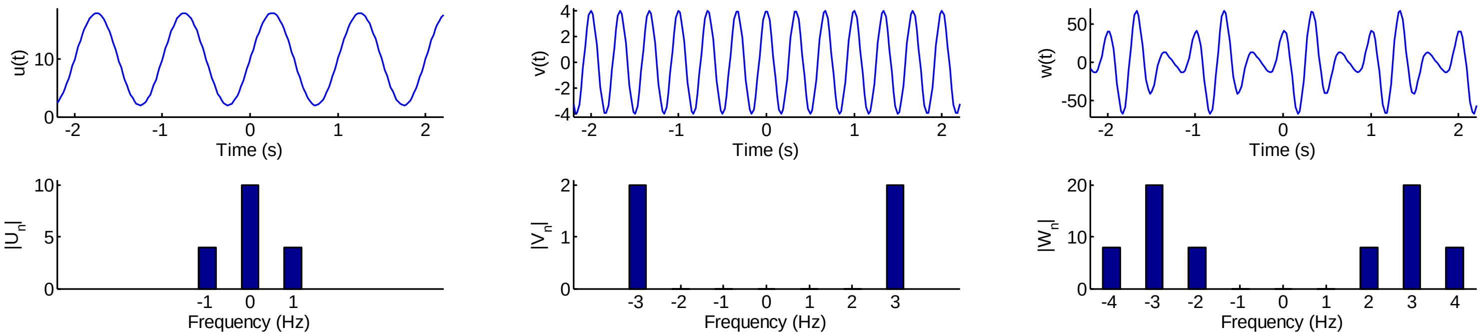
3) $\sum_m W_{r-m} (U_m + V_m) = \sum_m W_{r-m} U_m + \sum_m W_{r-m} V_m$

4) $I_{r-m} U_m = 0$ unless $m = r$. Hence $\sum_m I_{r-m} U_m = U_r$.

Convolution Example

$$u(t) = 10 + 8\sin 2\pi t \qquad v(t) = 4\cos 6\pi t$$

$$U_{-1:1} = [4i,\ \underline{10},\ -4i] \qquad V_{-3:3} = [2,\ 0,\ 0,\ \underline{0},\ 0,\ 0,\ 2] \qquad [\underline{0} = V_0]$$



$$\begin{aligned} w(t) = u(t)v(t) &= (10 + 8\sin 2\pi t)\, 4\cos 6\pi t \\ &= 40\cos 6\pi t + 32\sin 2\pi t \cos 6\pi t \\ &= 40\cos 6\pi t + 16\sin 8\pi t - 16\sin 4\pi t \end{aligned}$$

$$W_{-4:4} = [8i,\ 20,\ -8i,\ 0,\ \underline{0},\ 0,\ 8i,\ 20,\ -8i]$$

To convolve U_n and V_n:

Replace each harmonic in V_n by a scaled copy of the entire $\{U_n\}$ (or vice versa) and sum the complex-valued coefficients of any overlapping harmonics.

Two polynomials: $u(x) = U_3x^3 + U_2x^2 + U_1x + U_0$

$v(x) = \qquad V_2x^2 + V_1x + V_0$

Now multiply the two polynomials together:

$$\begin{aligned} w(x) &= u(x)v(x) \\ &= U_3V_2x^5 + (U_3V_1 + U_2V_2)\,x^4 + (U_3V_0 + U_2V_1 + U_1V_2)\,x^3 \\ &\quad + (U_2V_0 + U_1V_1 + U_0V_2)\,x^2 + (U_1V_0 + U_0V_1)\,x + U_0V_0 \end{aligned}$$

The coefficient of x^r consists of all the coefficient pair from U and V where the subscripts add up to r. For example, for $r = 3$:

$$W_3 = U_3V_0 + U_2V_1 + U_1V_2 = \sum_{m=0}^{2} U_{3-m}V_m$$

If all the missing coefficients are assumed to be zero, we can write

$$W_r = \sum_{m=-\infty}^{\infty} U_{r-m}V_m \triangleq U_r * V_r$$

So, to multiply two polynomials, you convolve their coefficient sequences.

Actually, the complex Fourier Series is iust a polynomial:

$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt} = \sum_{n=-\infty}^{\infty} U_n \left(e^{i2\pi Ft}\right)^n$$

Summary

- Parseval's Theorem: $\langle u^*(t)v(t)\rangle = \sum_{n=-\infty}^{\infty} U_n^* V_n$
  - Power Conservation: $\left\langle |u(t)|^2 \right\rangle = \sum_{n=-\infty}^{\infty} |U_n|^2$
  - or in terms of a_n and b_n:
    $$\left\langle |u(t)|^2 \right\rangle = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$
- Linearity: $w(t) = au(t) + bv(t) \Leftrightarrow W_n = aU_n + bV_n$
- Product of signals $\Leftrightarrow$ Convolution of complex Fourier coefficients:
  $w(t) = u(t)v(t) \Leftrightarrow W_n = U_n * V_n \triangleq \sum_{m=-\infty}^{\infty} U_{n-m}V_m$
- Convolution acts like multiplication:
  - Commutative: $U * V = V * U$
  - Associative: $U * V * W$ is unambiguous
  - Distributes over addition: $U * (V + W) = U * V + U * W$
  - Has an identity: $I_r = 1$ if $r = 0$ and $= 0$ otherwise
- Polynomial multiplication $\Leftrightarrow$ convolution of coefficients

For further details see RHB Chapter 12.8.