5: Gibbs Phenomenon

Discontinuities

A function, $v(t)$, has a discontinuity of amplitude b at $t = a$ if

$$\lim_{e\to 0}\left(v(a+e) - v(a-e)\right) = b \neq 0$$

Conversely, $v(t)$, is continuous at $t = a$ if the limit, b, equals zero.

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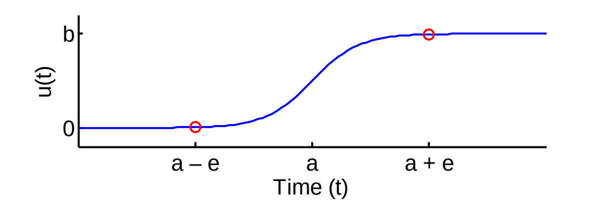
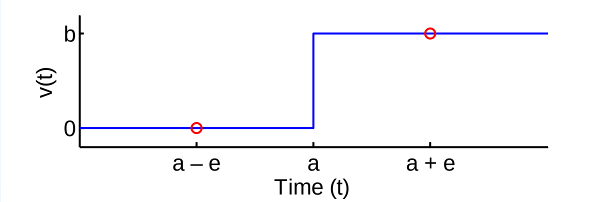


Continuous



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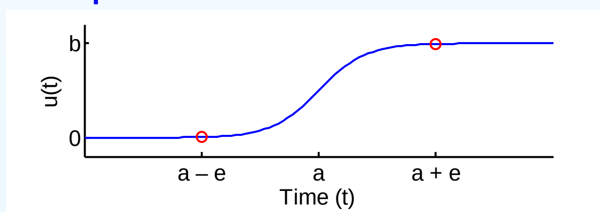

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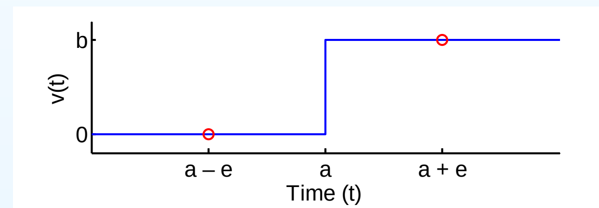
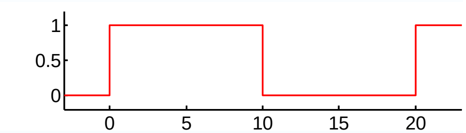


Discontinuous

We will see that if a periodic function, $v(t)$, is discontinuous, then its Fourier series behaves in a strange way.

Discontinuous Waveform

Pulse: $T = \frac{1}{F} = 20$, width=$\frac{1}{2}T$, height $A = 1$



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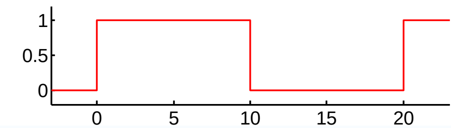

$$U_m = \frac{1}{T}\int_0^{0.5T} Ae^{-i2\pi mFt}dt$$

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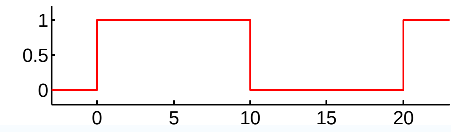

$$U_m = \frac{1}{T}\int_0^{0.5T} Ae^{-i2\pi mFt}dt$$
$$= \frac{i}{2\pi mFT}\left[e^{-i2\pi mFt}\right]_0^{0.5T}$$

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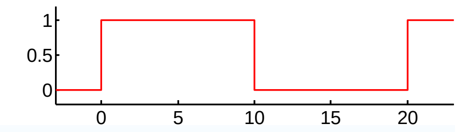

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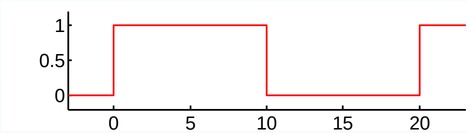

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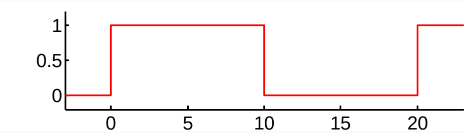

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&= \begin{cases} 0 & m \neq 0, \text{ even} \\ 0.5 & m = 0 \\ \frac{-i}{m\pi} & m \text{ odd} \end{cases}
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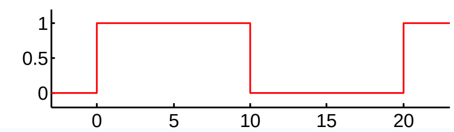

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So, $u(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin 2\pi Ft + \frac{1}{3}\sin 6\pi Ft + \frac{1}{5}\sin 10\pi Ft + \ldots\right)$

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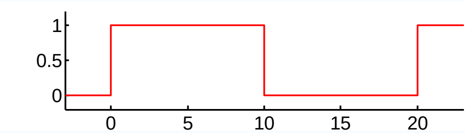

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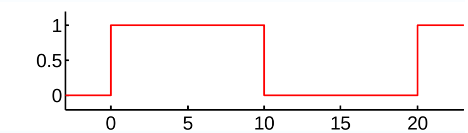

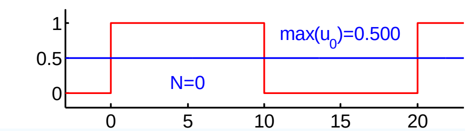

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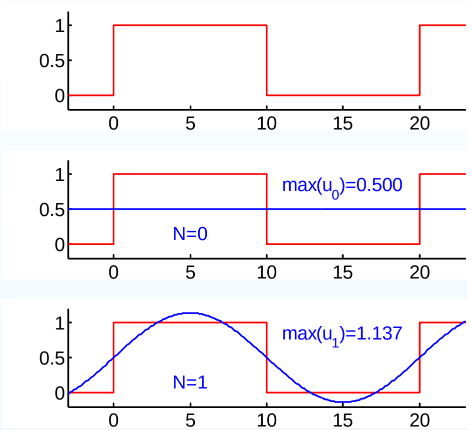

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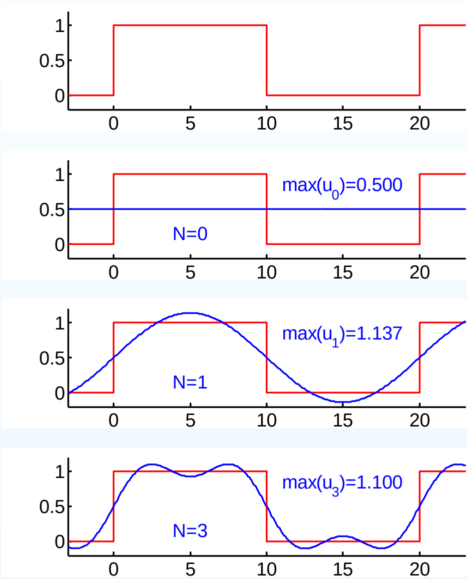

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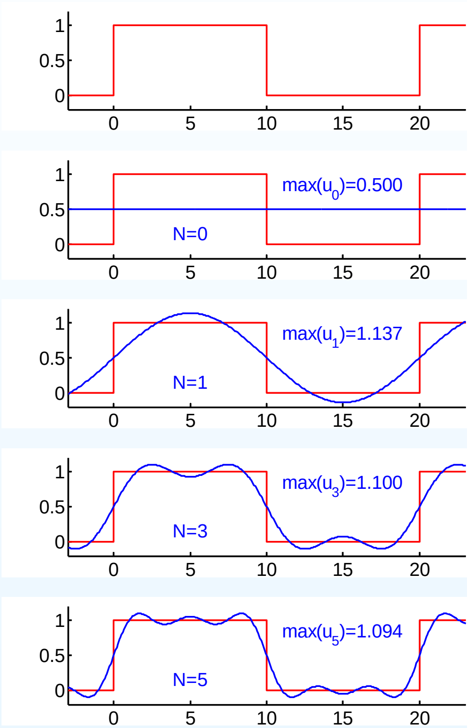

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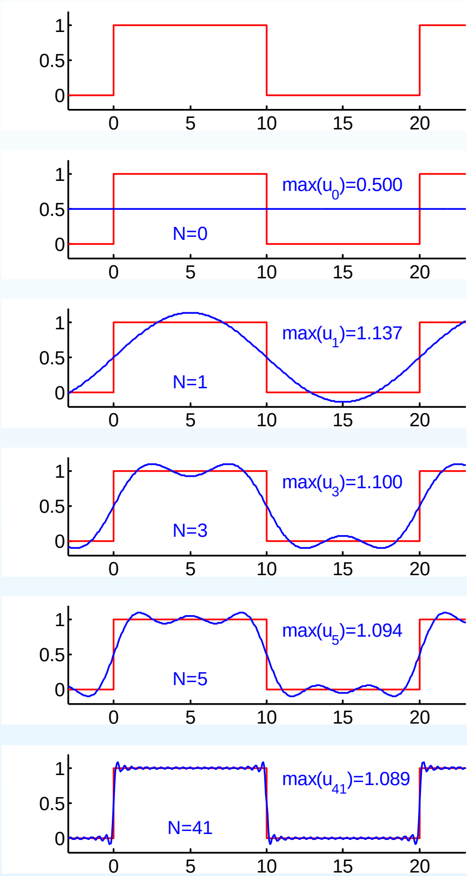

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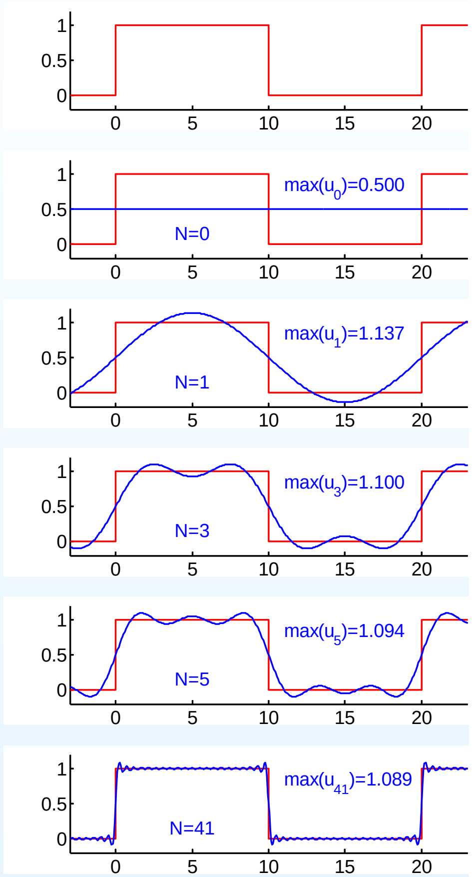

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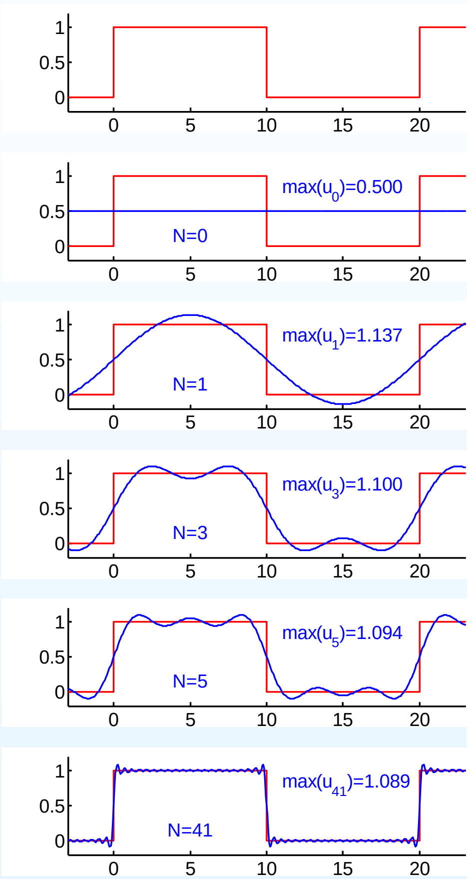

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$$\max_t u_N(t) \underset{N\to\infty}{\longrightarrow} \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt \approx 1.0895$$

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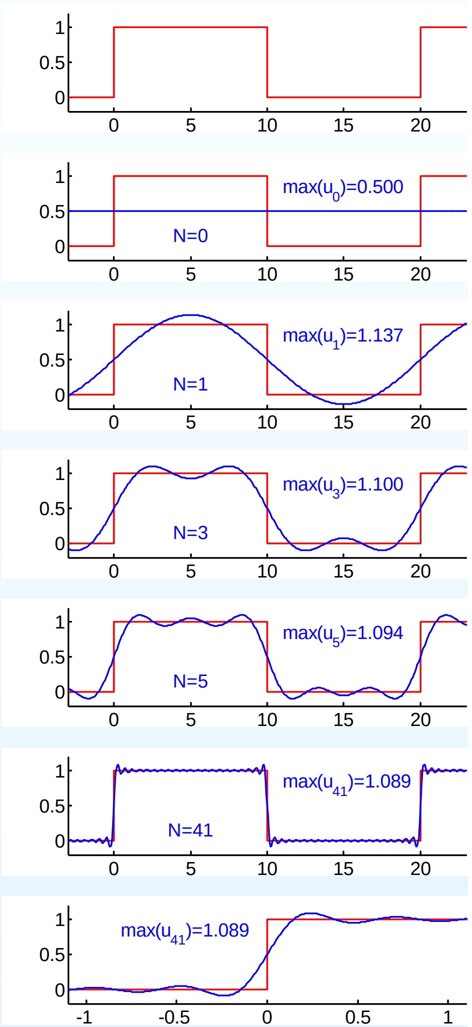

[Enlarged View: $u_{41}(t)$]

Gibbs Phenomenon

Truncated Fourier Series: $u_N(t) = \sum_{m=-N}^{N} U_m e^{i2\pi mFt}$

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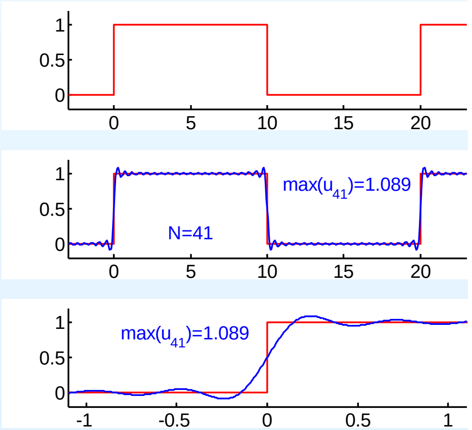

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(1) $u_N(a) \underset{N\to\infty}{\longrightarrow} \lim_{e\to 0} \frac{u(a-e)+u(a+e)}{2}$

(2) $u_N(t)$ has an overshoot of about 9% of b at the discontinuity.

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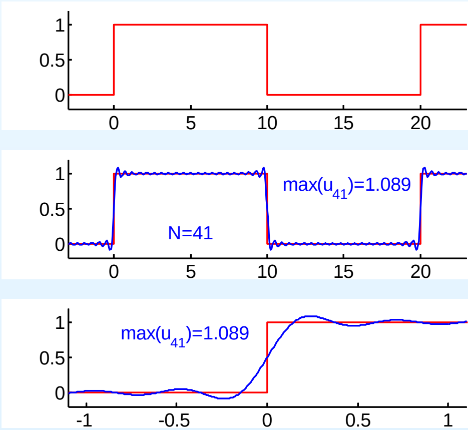

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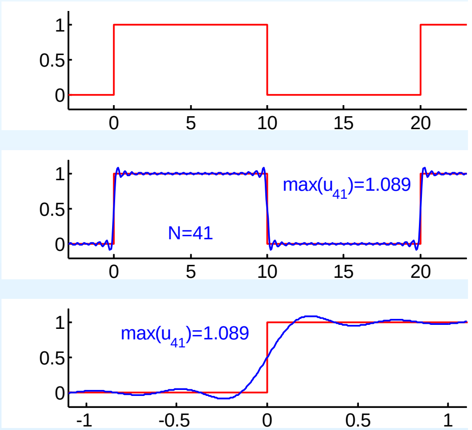

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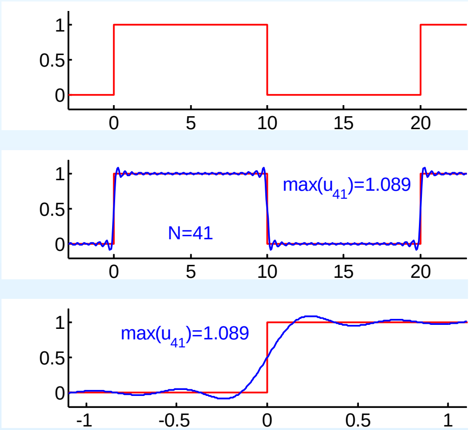

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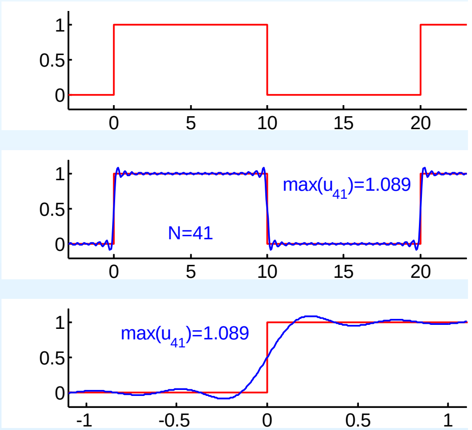

Gibbs Phenomenon

Truncated Fourier Series: $u_N(t) = \sum_{m=-N}^{N} U_m e^{i2\pi mFt}$

If $u(t)$ has a discontinuity of height b at $t = a$ then:

(1) $u_N(a) \underset{N\to\infty}{\longrightarrow} \lim_{e\to 0} \frac{u(a-e)+u(a+e)}{2}$

(2) $u_N(t)$ has an overshoot of about 9% of b at the discontinuity. For large N the overshoot moves closer to the discontinuity but does not get smaller (Gibbs phenomenon). In the limit the overshoot equals $\left(-\frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{\sin t}{t} dt\right) b \approx 0.0895b$.

(3) For large m, the coefficients, U_m decrease no faster than $|m|^{-1}$.

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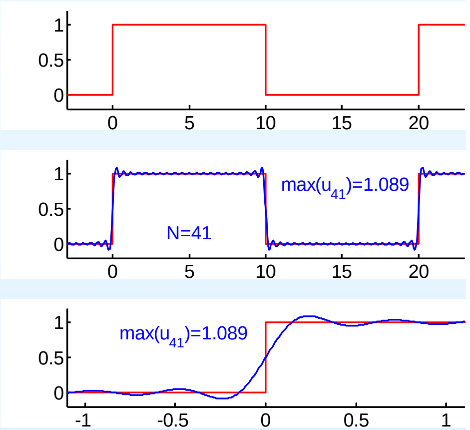

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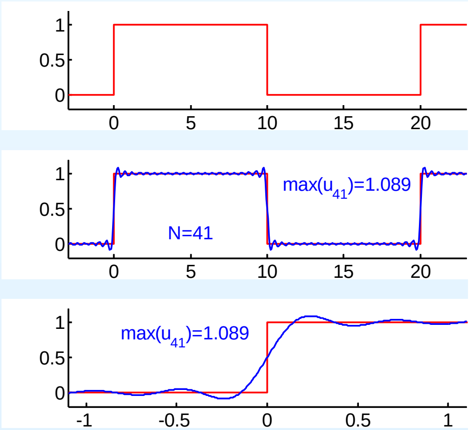
$$U_m = \begin{cases} 0 & m \neq 0, \text{ even} \\ 0.5 & m = 0 \\ \frac{-i}{m\pi} & m \text{ odd} \end{cases}$$



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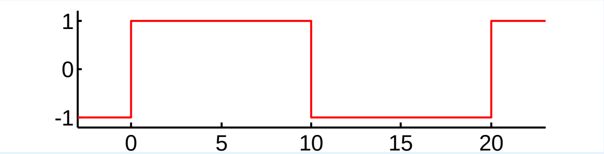

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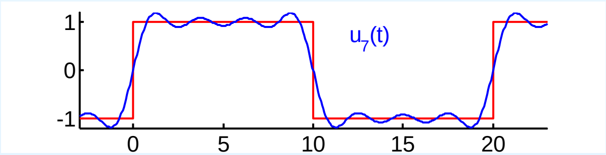

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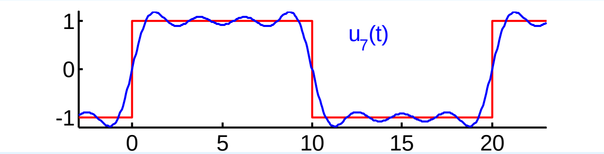

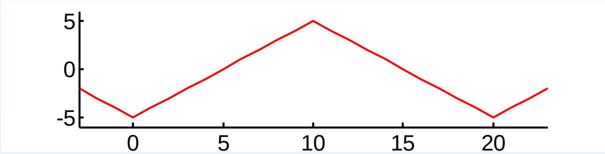

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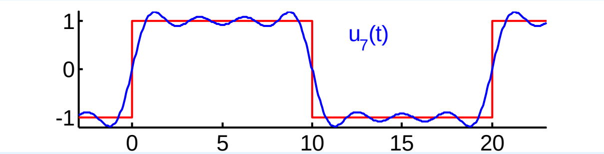

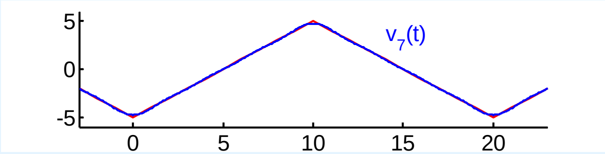

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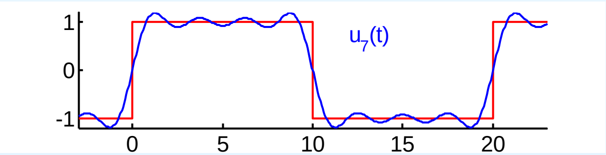

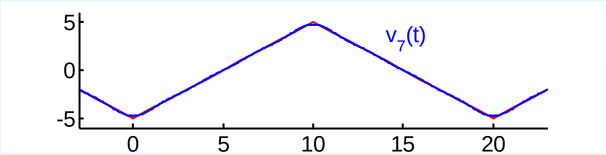

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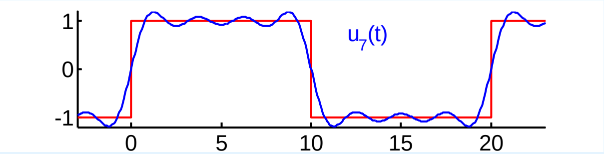

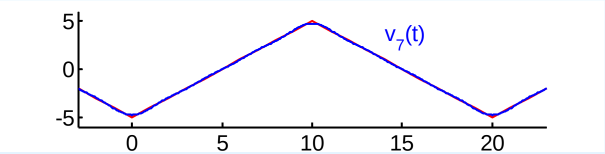

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Rate at which coefficients decrease with m

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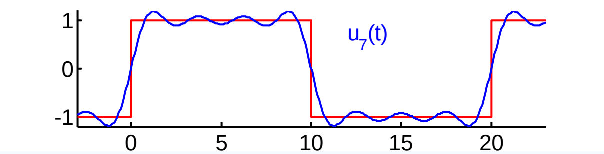

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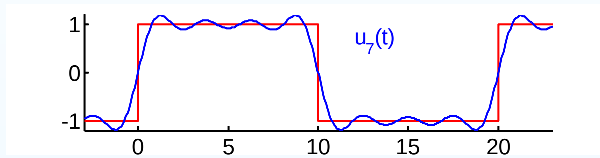

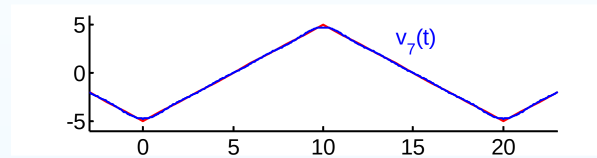

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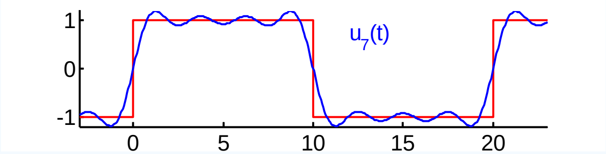

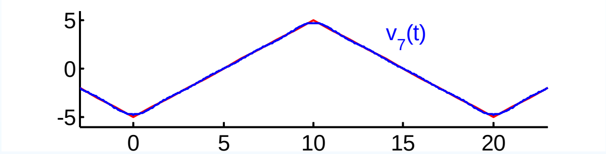

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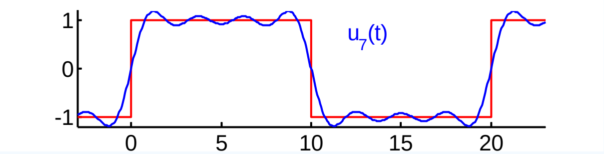

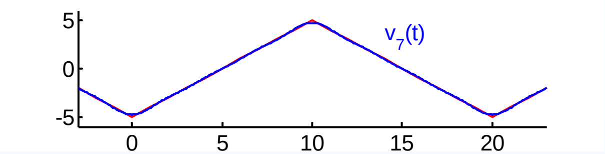

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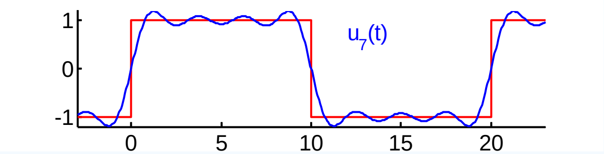

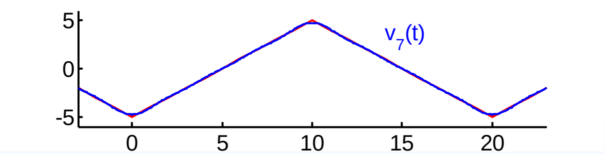

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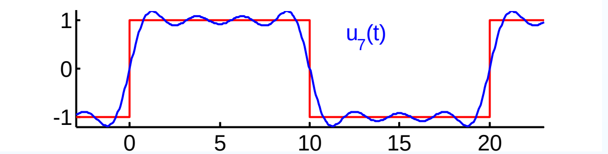

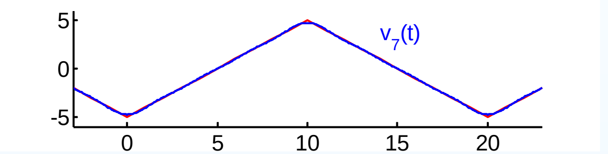

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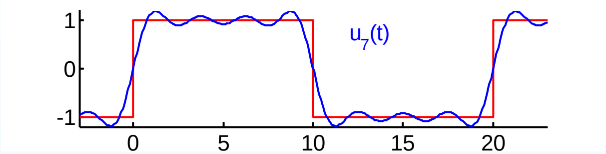

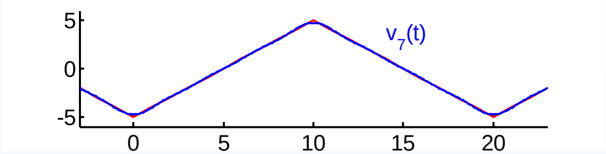

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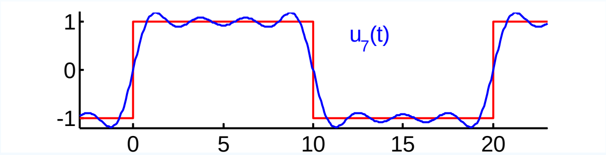
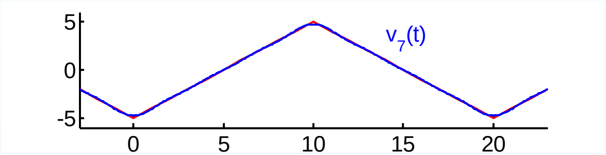
  For large $|m|$, U_m decreases as $|m|^{-(n+1)}$

Rate at which coefficients decrease with m

Square wave: $U_m = \frac{-2i}{\pi} m^{-1}$ for odd m (0 for even m)

Triangle wave: $V_m = \frac{-1}{\pi^2 F} m^{-2}$ for odd m (0 for even m)





Integrating $u(t)$ multiplies the U_m by $\frac{-i}{2\pi F} \times m^{-1} \Rightarrow$ they decrease faster.

The rate at which the coefficients, U_m, decrease with m depends on the lowest derivative that has a discontinuity:

- Discontinuity in $u(t)$ itself (e.g. square wave)
  For large $|m|$, U_m decreases as $|m|^{-1}$
- Discontinuity in $u'(t)$ (e.g. triangle wave)
  For large $|m|$, U_m decreases as $|m|^{-2}$
- Discontinuity in $u^{(n)}(t)$
  For large $|m|$, U_m decreases as $|m|^{-(n+1)}$
- No discontinuous derivatives
  For large $|m|$, U_m decreases faster than any power (e.g. $e^{-|m|}$)

Differentiation

Integration multiplies U_m by $\frac{-i}{2\pi m F}$.

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$$W_m = \begin{cases} 0 & m = 0 \\ i2\pi mFU_m & m \neq 0 \end{cases}.$$

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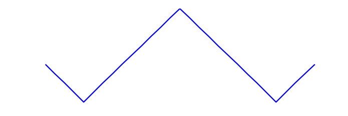

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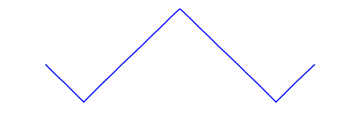

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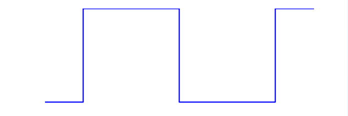
$\xrightarrow{\frac{d}{dt}}$



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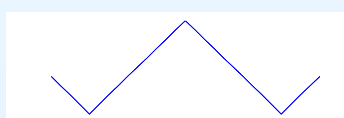

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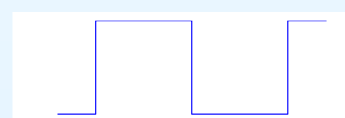

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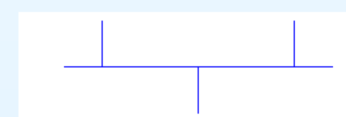

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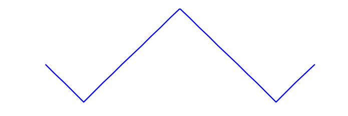

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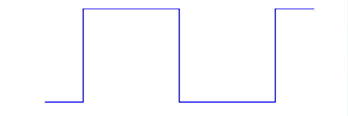

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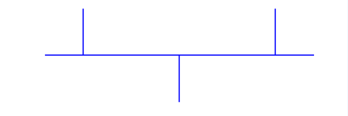

$U_m \propto |m|^{-2}$ $\quad$ $U_m \propto |m|^{-1}$ $\quad$ $U_m \propto |m|^{-0}$

Differentiation makes waveforms spikier and less smooth.

Periodic Extension

Suppose $y(t)$ is only defined over a finite interval (a, b).

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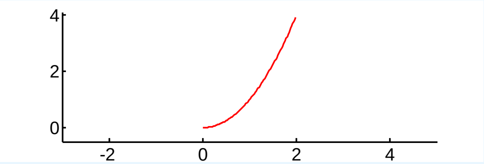

$y(t)$

Periodic Extension

Suppose $y(t)$ is only defined over a finite interval (a, b).

You have two reasonable choices to make a periodic version:

(a) $T = b - a, \qquad u(t) = y(t) \text{ for } a \le t < b$

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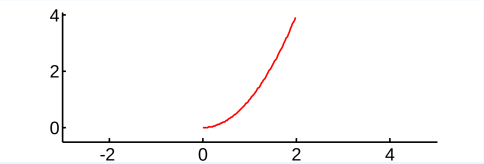

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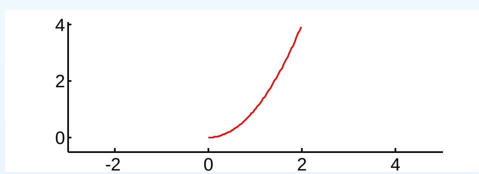

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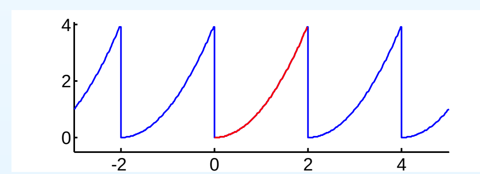

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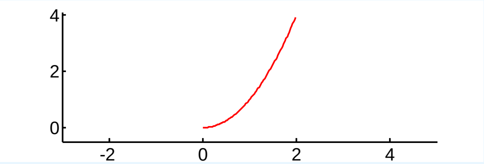

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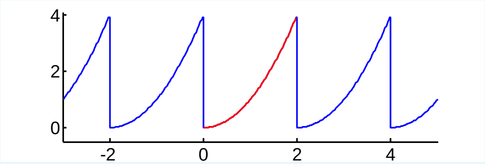

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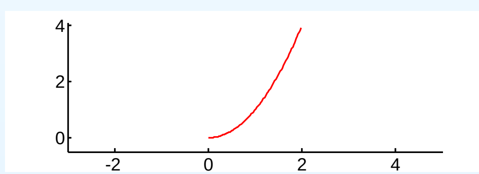

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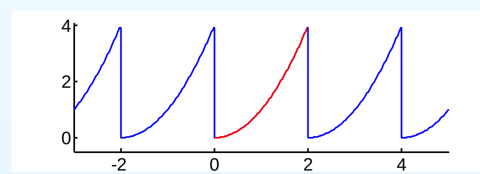
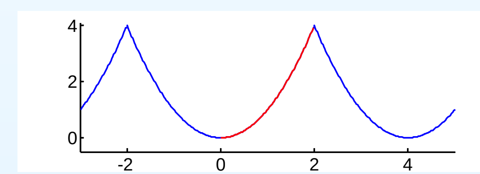


(a) $T = 2$



(b) $T = 4$

Periodic Extension

Suppose $y(t)$ is only defined over a finite interval (a, b).

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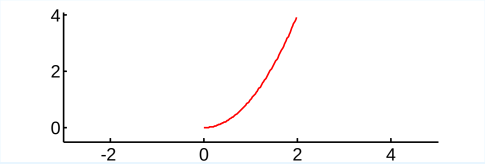

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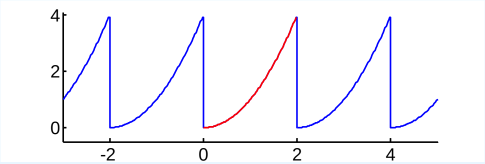
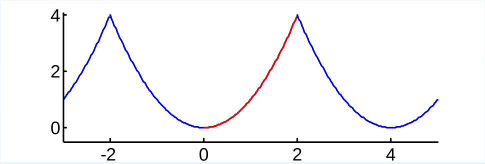


(a) $T = 2$



(b) $T = 4$

Option (b) has twice the period

Periodic Extension

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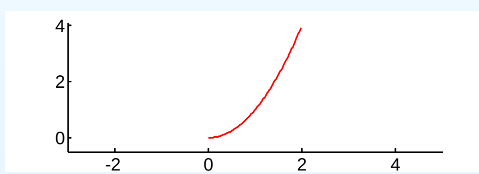
(b) $T = 2(b-a),\ u(t) = \begin{cases} y(t) & a \le t \le b \\ y(2b - t) & b \le t \le 2b - a \end{cases}$

Example:

$y(t) = t^2$ for $0 \le t < 2$



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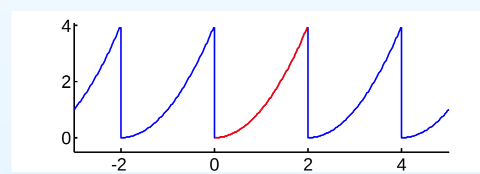
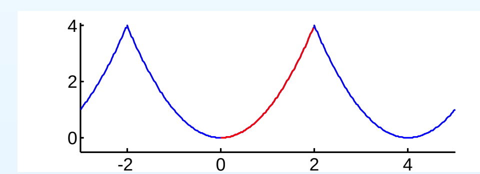


(a) $T = 2$



(b) $T = 4$

Option (b) has twice the period, no discontinuities

Periodic Extension

Suppose $y(t)$ is only defined over a finite interval (a, b).

You have two reasonable choices to make a periodic version:

(a) $T = b - a, \qquad u(t) = y(t) \text{ for } a \le t < b$

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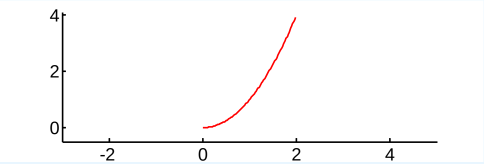
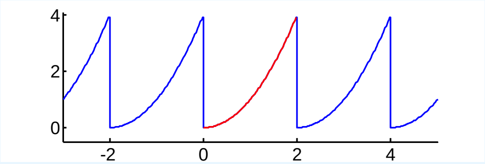
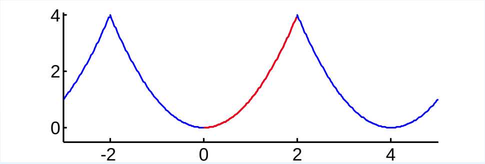


$y(t)$



(a) $T = 2$



(b) $T = 4$

Option (b) has twice the period, no discontinuities, no Gibbs phenomenon overshoots

Periodic Extension

Suppose $y(t)$ is only defined over a finite interval $(a,\ b)$.

You have two reasonable choices to make a periodic version:

(a) $T = b - a, \qquad u(t) = y(t)$ for $a \le t < b$

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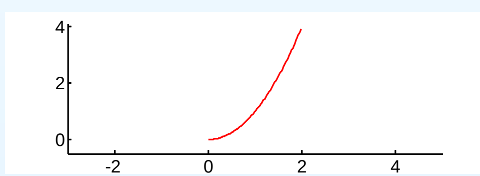
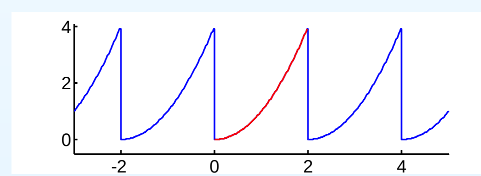
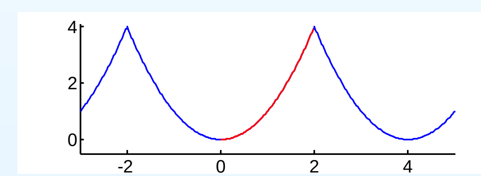
$y(t) = t^2$ for $0 \le t < 2$



$y(t)$



(a) $T = 2$

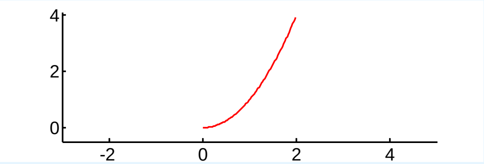


(b) $T = 4$

Option (b) has twice the period, no discontinuities, no Gibbs phenomenon overshoots and if $y(t)$ is continuous the coefficients decrease at least as fast as $|m|^{-2}$.

t^2 Periodic Extension: Method (a)

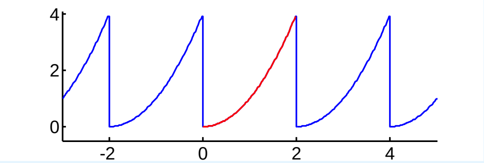
$y(t) = t^2$ for $0 \leq t < 2$



t^2 Periodic Extension: Method (a)

$y(t) = t^2$ for $0 \le t < 2$

Method (a): $T = \frac{1}{F} = 2$

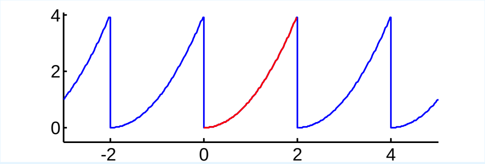


t^2 Periodic Extension: Method (a)

$y(t) = t^2$ for $0 \leq t < 2$

Method (a): $T = \frac{1}{F} = 2$

$$U_m = \frac{1}{T}\int_0^T t^2 e^{-i2\pi mFt}dt$$



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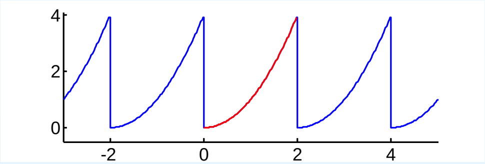
$$= \frac{1}{T}\left[\frac{t^2 e^{-i2\pi mFt}}{-i2\pi mF} - \frac{2te^{-i2\pi mFt}}{(-i2\pi mF)^2} + \frac{2e^{-i2\pi mFt}}{(-i2\pi mF)^3}\right]_0^T$$



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$y(t) = t^2$ for $0 \le t < 2$

Method (a): $T = \frac{1}{F} = 2$

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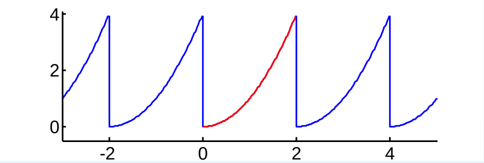
Substitute $e^{-i2\pi mF0} = e^{-i2\pi mFT} = 1$ [for integer m]



t^2 Periodic Extension: Method (a)

$y(t) = t^2$ for $0 \le t < 2$

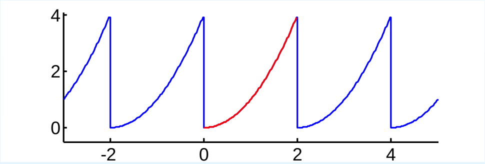
Method (a): $T = \frac{1}{F} = 2$

$$U_m = \frac{1}{T}\int_0^T t^2 e^{-i2\pi mFt}dt$$

$$= \frac{1}{T}\left[\frac{t^2 e^{-i2\pi mFt}}{-i2\pi mF} - \frac{2te^{-i2\pi mFt}}{(-i2\pi mF)^2} + \frac{2e^{-i2\pi mFt}}{(-i2\pi mF)^3}\right]_0^T$$

Substitute $e^{-i2\pi mF0} = e^{-i2\pi mFT} = 1$ [for integer m]

$$= \frac{1}{T}\left[\frac{T^2}{-i2\pi mF} - \frac{2T}{(-i2\pi mF)^2}\right]$$



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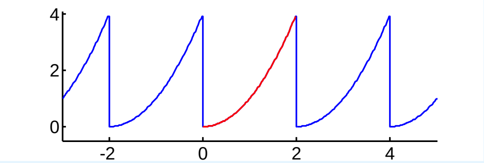

t^2 Periodic Extension: Method (a)

$y(t) = t^2$ for $0 \le t < 2$

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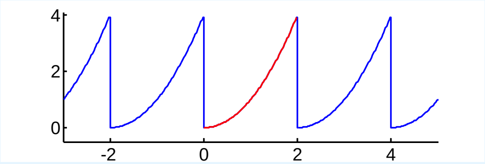

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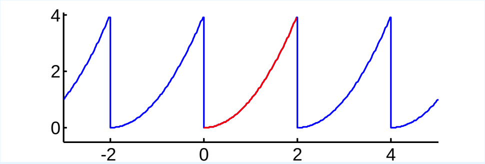

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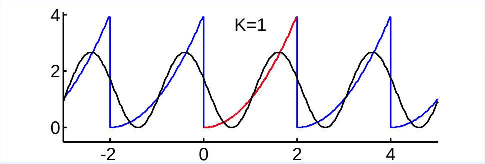

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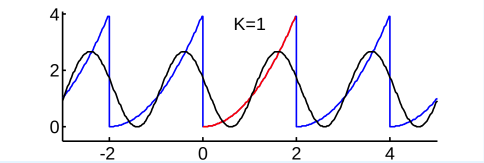

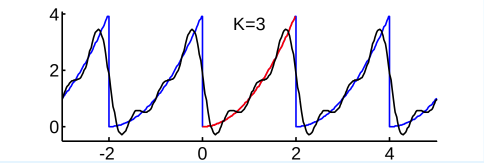

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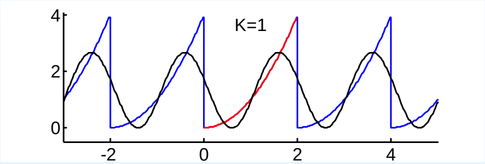

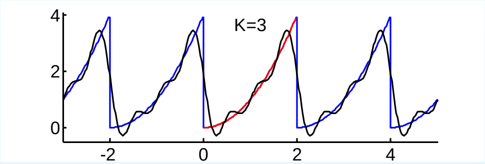

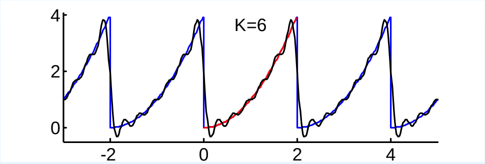

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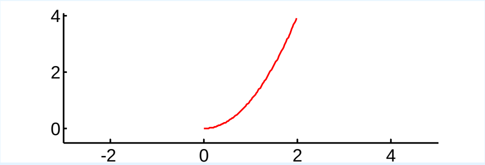

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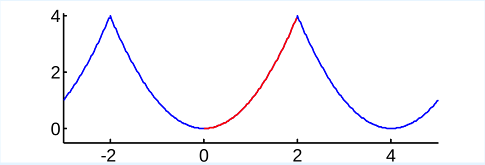

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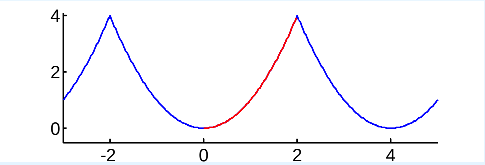

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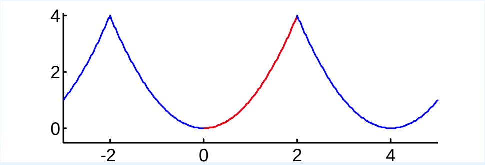

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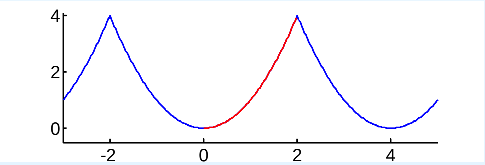

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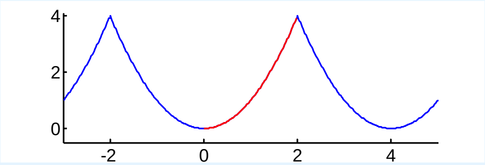

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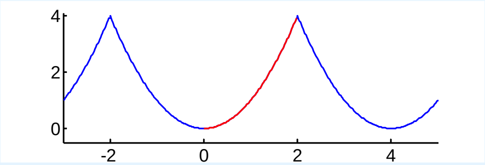

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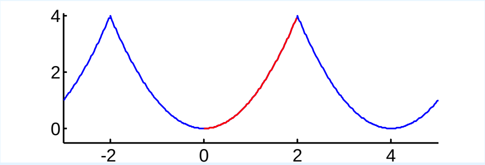

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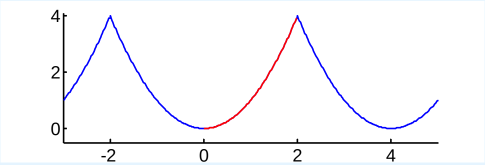

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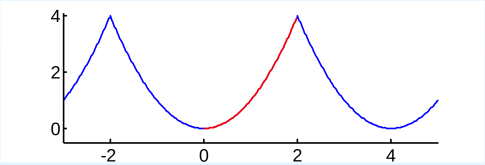

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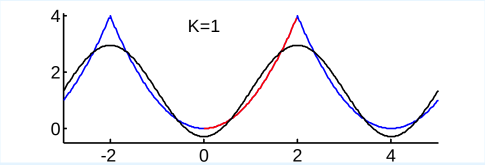

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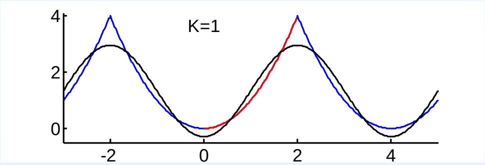

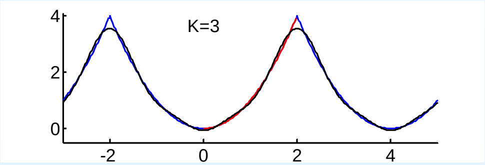

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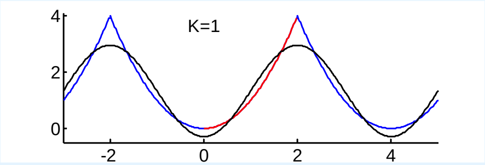

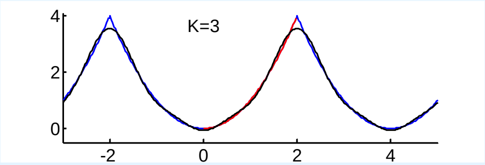

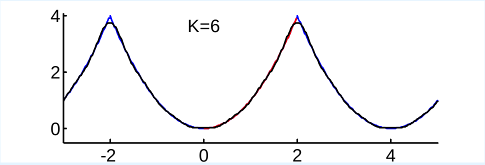

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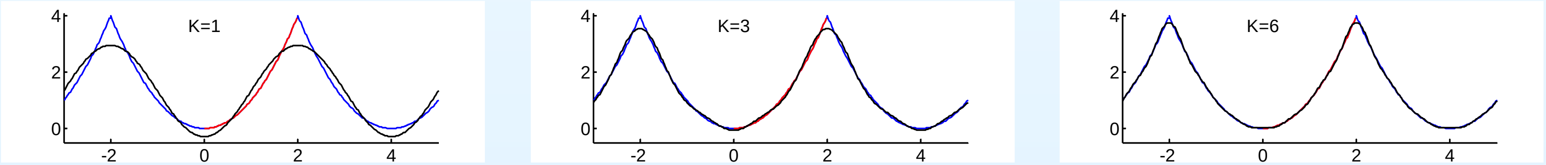
$$= \frac{1}{T}\left[\frac{t^2e^{-i2\pi mFt}}{-i2\pi mF} - \frac{2te^{-i2\pi mFt}}{(-i2\pi mF)^2} + \frac{2e^{-i2\pi mFt}}{(-i2\pi mF)^3}\right]_{-0.5T}^{0.5T}$$

Substitute $e^{\pm i\pi mFT} = e^{\pm i\pi m} = (-1)^m$ [for integer m]

$$= \frac{(-1)^m}{T}\left[\frac{-2T}{(-i2\pi mF)^2}\right]$$ [all even powers of t cancel out]

$$= \frac{(-1)^m T^2}{2\pi^2 m^2} = \frac{(-1)^m 8}{\pi^2 m^2}$$



$U_{0:3} = [1.333,\ -0.811,\ 0.203,\ -0.090]$ [$u(t)$ real+even $\Rightarrow U_m$ real]

Convergence is noticeably faster than for method (a)

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For further details see RHB Chapter 12.4, 12.5, 12.6