7: Fourier Transforms: Convolution and Parseval's Theorem

Question: What is the Fourier transform of $w(t) = u(t)v(t)$?

Let $u(t) = \int_{h=-\infty}^{+\infty} U(h)e^{i2\pi ht}dh$ and $v(t) = \int_{g=-\infty}^{+\infty} V(g)e^{i2\pi gt}dg$

[Note use of different dummy variables]

$$\begin{aligned}
w(t) &= u(t)v(t) \\
&= \int_{h=-\infty}^{+\infty} U(h)e^{i2\pi ht}dh \int_{g=-\infty}^{+\infty} V(g)e^{i2\pi gt}dg \\
&= \int_{h=-\infty}^{+\infty} U(h) \int_{g=-\infty}^{+\infty} V(g)e^{i2\pi(h+g)t}dg\,dh \qquad \text{[merge } e^{(\cdots)}\text{]}
\end{aligned}$$

Now we make a change of variable in the second integral: $g = f - h$

$$\begin{aligned}
&= \int_{h=-\infty}^{+\infty} U(h) \int_{f=-\infty}^{+\infty} V(f-h)e^{i2\pi ft}df\,dh \\
&= \int_{f=-\infty}^{\infty} \int_{h=-\infty}^{+\infty} U(h)V(f-h)e^{i2\pi ft}dh\,df \qquad \text{[swap } \textstyle\int\text{]} \\
&= \int_{f=-\infty}^{+\infty} W(f)e^{i2\pi ft}df
\end{aligned}$$

where $W(f) = \int_{h=-\infty}^{+\infty} U(h)V(f-h)dh$ $\int_{h=-\infty}^{+\infty} U(h)V(f-h)dh \triangleq U(f) * V(f)$

This is the *convolution* of the two spectra $U(f)$ and $V(f)$.

$$w(t) = u(t)v(t) \quad \Leftrightarrow \quad W(f) = U(f) * V(f)$$

Multiplication Example

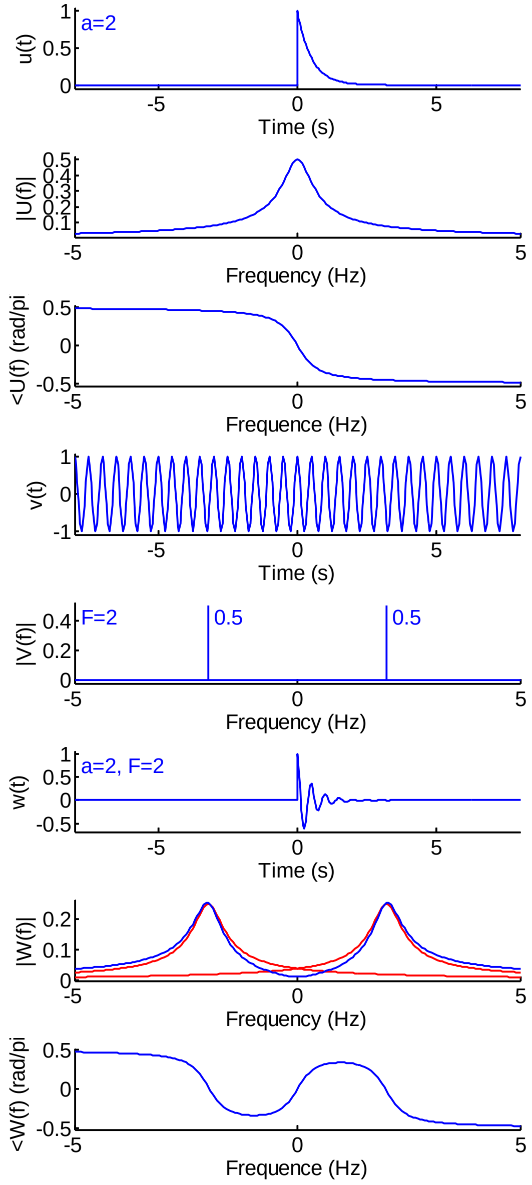
$$u(t) = \begin{cases} e^{-at} & t \ge 0 \\ 0 & t < 0 \end{cases}$$

$$U(f) = \frac{1}{a+i2\pi f}$$ [from before]

$$v(t) = \cos 2\pi F t$$
$$V(f) = 0.5\left(\delta(f+F) + \delta(f-F)\right)$$

$$w(t) = u(t)v(t)$$
$$W(f) = U(f) * V(f)$$
$$= \frac{0.5}{a+i2\pi(f+F)} + \frac{0.5}{a+i2\pi(f-F)}$$

If $V(f)$ consists entirely of Dirac impulses then $U(f) * V(f)$ just replaces each impulse with a complete copy of $U(f)$ scaled by the area of the impulse and shifted so that $0\,\mathrm{Hz}$ lies on the impulse. Then add the overlapping complex spectra.



Convolution Theorem:

$$w(t) = u(t)v(t) \quad \Leftrightarrow \quad W(f) = U(f) * V(f)$$
$$w(t) = u(t) * v(t) \quad \Leftrightarrow \quad W(f) = U(f)V(f)$$

Convolution in the time domain is equivalent to multiplication in the frequency domain and vice versa.

Proof of second line:

Given $u(t)$, $v(t)$ and $w(t)$ satisfying

$$w(t) = u(t)v(t) \quad \Leftrightarrow \quad W(f) = U(f) * V(f)$$

define dual waveforms $x(t)$, $y(t)$ and $z(t)$ as follows:

$$\begin{aligned} x(t) = U(t) \quad &\Leftrightarrow \quad X(f) = u(-f) && \text{[duality]} \\ y(t) = V(t) \quad &\Leftrightarrow \quad Y(f) = v(-f) \\ z(t) = W(t) \quad &\Leftrightarrow \quad Z(f) = w(-f) \end{aligned}$$

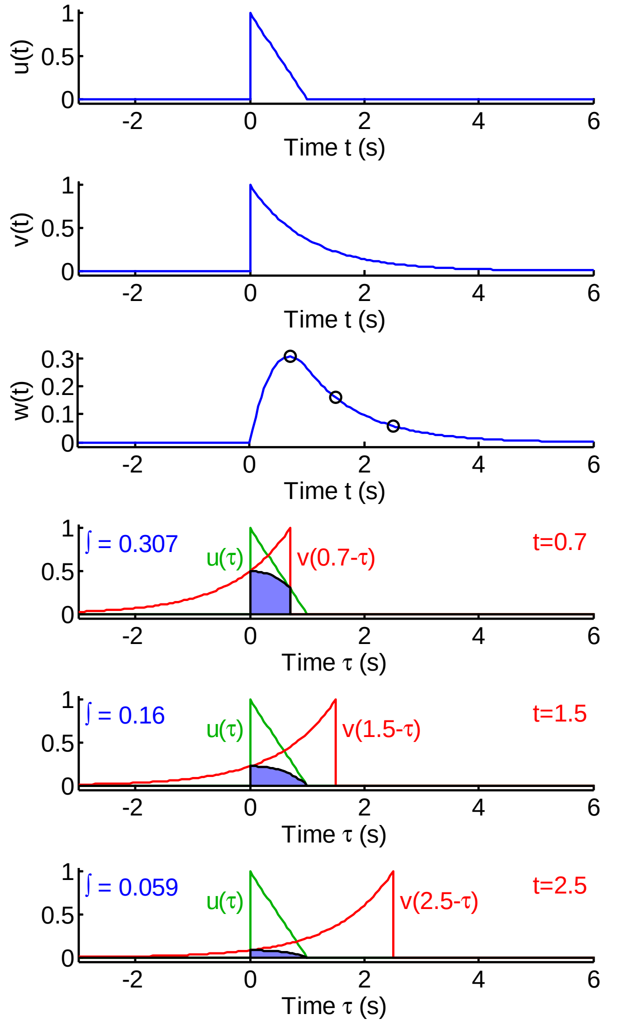
Now the convolution property becomes:

$$\begin{aligned} w(-f) = u(-f)v(-f) \quad &\Leftrightarrow \quad W(t) = U(t) * V(t) && [\text{sub } t \leftrightarrow \pm f] \\ Z(f) = X(f)Y(f) \quad &\Leftrightarrow \quad z(t) = x(t) * y(t) && \text{[duality]} \end{aligned}$$

$$u(t) = \begin{cases} 1-t & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$v(t) = \begin{cases} e^{-t} & t \ge 0 \\ 0 & t < 0 \end{cases}$$

$$\begin{aligned} w(t) &= u(t) * v(t) \\ &= \int_{-\infty}^{\infty} u(\tau)v(t-\tau)d\tau \\ &= \int_0^{\min(t,1)} (1-\tau)e^{\tau-t}d\tau \\ &= \left[(2-\tau)\, e^{\tau-t}\right]_{\tau=0}^{\min(t,1)} \\ &= \begin{cases} 0 & t<0 \\ 2-t-2e^{-t} & 0 \le t < 1 \\ (e-2)\, e^{-t} & t \ge 1 \end{cases} \end{aligned}$$



Note how $v(t-\tau)$ is time-reversed (because of the $-\tau$) and time-shifted to put the time origin at $\tau = t$.

Convolution Properties

Convloution: $w(t) = u(t) * v(t) \triangleq \int_{-\infty}^{\infty} u(\tau)v(t-\tau)d\tau$

Convolution behaves algebraically like multiplication:

1) Commutative: $u(t) * v(t) = v(t) * u(t)$

2) Associative:
$$u(t) * v(t) * w(t) = (u(t) * v(t)) * w(t) = u(t) * (v(t) * w(t))$$

3) Distributive over addition:
$$w(t) * (u(t) + v(t)) = w(t) * u(t) + w(t) * v(t)$$

4) Identity Element or "1": $u(t) * \delta(t) = \delta(t) * u(t) = u(t)$

5) Bilinear: $(au(t)) * (bv(t)) = ab\,(u(t) * v(t))$

Proof: In the frequency domain, convolution is multiplication.

Also, if $u(t) * v(t) = w(t)$, then

6) Time Shifting: $u(t+a) * v(t+b) = w(t+a+b)$

7) Time Scaling: $u(at) * v(at) = \frac{1}{|a|} w(at)$

How to recognise a convolution integral:
the arguments of $u(\cdots)$ and $v(\cdots)$ sum to a constant.

Parseval's Theorem

Lemma:

$$X(f) = \delta(f-g) \quad \Rightarrow \quad x(t) = \int \delta(f-g) e^{i2\pi ft} df = e^{i2\pi gt}$$

$$\Rightarrow \quad X(f) = \int e^{i2\pi gt} e^{-i2\pi ft} dt = \int e^{i2\pi (g-f)t} dt = \delta(g-f)$$

Parseval's Theorem: $\int_{t=-\infty}^{\infty} u^*(t)v(t)dt = \int_{f=-\infty}^{+\infty} U^*(f)V(f)df$

Proof:

Let $u(t) = \int_{f=-\infty}^{+\infty} U(f)e^{i2\pi ft}df$ and $v(t) = \int_{g=-\infty}^{+\infty} V(g)e^{i2\pi gt}dg$

[Note use of different dummy variables]

Now multiply $u^*(t) = u(t)$ and $v(t)$ together and integrate over time:

$$\begin{aligned}
&\int_{t=-\infty}^{\infty} u^*(t)v(t)dt \\
&\quad = \int_{t=-\infty}^{\infty} \int_{f=-\infty}^{+\infty} U^*(f)e^{-i2\pi ft}df \int_{g=-\infty}^{+\infty} V(g)e^{i2\pi gt}dgdt \\
&\quad = \int_{f=-\infty}^{+\infty} U^*(f) \int_{g=-\infty}^{+\infty} V(g) \int_{t=-\infty}^{\infty} e^{i2\pi(g-f)t}dtdgdf \\
&\quad = \int_{f=-\infty}^{+\infty} U^*(f) \int_{g=-\infty}^{+\infty} V(g)\delta(g-f)dgdf \qquad \text{[lemma]} \\
&\quad = \int_{f=-\infty}^{+\infty} U^*(f)V(f)df
\end{aligned}$$

Energy Conservation

Parseval's Theorem: $\int_{t=-\infty}^{\infty} u^*(t)v(t)dt = \int_{f=-\infty}^{+\infty} U^*(f)V(f)df$

For the special case $v(t) = u(t)$, Parseval's theorem becomes:

$$\int_{t=-\infty}^{\infty} u^*(t)u(t)dt = \int_{f=-\infty}^{+\infty} U^*(f)U(f)df$$

$$\Rightarrow \quad E_u = \int_{t=-\infty}^{\infty} |u(t)|^2 dt = \int_{f=-\infty}^{+\infty} |U(f)|^2 df$$

Energy Conservation: The energy in $u(t)$ equals the energy in $U(f)$.

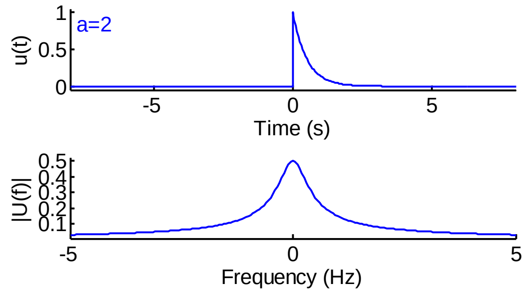
Example:

$$u(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \Rightarrow \quad E_u = \int |u(t)|^2 dt = \left[\frac{-e^{-2at}}{2a}\right]_0^{\infty} = \frac{1}{2a}$$

$U(f) = \frac{1}{a+i2\pi f}$ [from before]

$$\Rightarrow \quad \int |U(f)|^2 df = \int \frac{df}{a^2+4\pi^2 f^2}$$

$$= \left[\frac{\tan^{-1}\left(\frac{2\pi f}{a}\right)}{2\pi a}\right]_{-\infty}^{\infty} = \frac{\pi}{2\pi a} = \frac{1}{2a}$$



Energy Spectrum

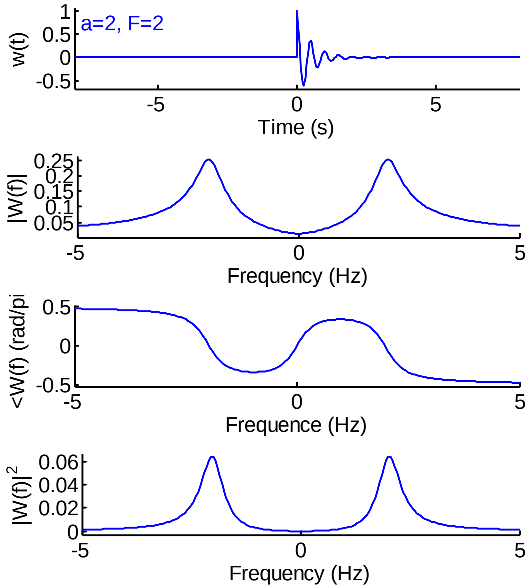
Example from before:

$$w(t) = \begin{cases} e^{-at}\cos 2\pi F t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$W(f) = \frac{0.5}{a+i2\pi(f+F)} + \frac{0.5}{a+i2\pi(f-F)}$$

$$= \frac{a+i2\pi f}{a^2+i4\pi a f - 4\pi^2(f^2-F^2)}$$

$$|W(f)|^2 = \frac{a^2+4\pi^2 f^2}{(a^2-4\pi^2(f^2-F^2))^2+16\pi^2a^2f^2}$$



Energy Spectrum

- The units of $|W(f)|^2$ are "*energy per Hz*" so that its integral, $E_w = \int_{-\infty}^{\infty} |W(f)|^2 \, df$, has units of energy.
- The quantity $|W(f)|^2$ is called the *energy spectral density* of $w(t)$ at frequency f and its graph is the *energy spectrum* of $w(t)$. It shows how the energy of $w(t)$ is distributed over frequencies.
- If you divide $|W(f)|^2$ by the total energy, E_w, the result is non-negative and integrates to unity like a probability distribution.

- Convolution:
  - $u(t) * v(t) \triangleq \int_{-\infty}^{\infty} u(\tau)v(t-\tau)d\tau$
    - ▷ Arguments of $u(\cdots)$ and $v(\cdots)$ sum to t
  - Acts like multiplication + time scaling/shifting formulae
- Convolution Theorem: multiplication ↔ convolution
  - $w(t) = u(t)v(t) \quad \Leftrightarrow \quad W(f) = U(f) * V(f)$
  - $w(t) = u(t) * v(t) \quad \Leftrightarrow \quad W(f) = U(f)V(f)$
- Parseval's Theorem: $\int_{t=-\infty}^{\infty} u^*(t)v(t)dt = \int_{f=-\infty}^{+\infty} U^*(f)V(f)df$
- Energy Spectrum:
  - Energy spectral density: $|U(f)|^2$ (energy/Hz)
  - Parseval: $E_u = \int |u(t)|^2 dt = \int |U(f)|^2 df$

For further details see RHB Chapter 13.1