E1.10 Fourier Series and Transforms

Problem Sheet 3 (Lectures 4, 5)

Key: [A]= easy ... [E]=hard

Questions from RBH textbook: 12.19, 12.23, 12.25.

1. [C] (a) Determine the fundamental frequency, the Fourier coefficients and the complex Fourier coefficients of $u(t) = \cos^2 t$.

   (b) Determine the power, $P_u = \left\langle u^2(t) \right\rangle$ where $\langle \cdots \rangle$ denotes the time average. Hint: $\cos^4 t = \frac{1}{8}\cos 4t + \frac{1}{2}\cos 2t + \frac{3}{8}$.

   (c) Show that Parseval's theorem applies: $P_u = \sum_{n=-\infty}^{\infty} |U_n|^2 = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$.

2. [C] The even function $u(t)$ with period $T = 1$ is defined in the region $|t| \le \frac{1}{2}$ by $u(t) = \begin{cases} a^{-1} & |t| \le \frac{a}{2} \\ 0 & |t| > \frac{a}{2} \end{cases}$ where $0 < a < 1$.

   (a) Determine the complex Fourier coefficients, U_n.

   (b) Explain why U_0 does not depend on a.

   (c) Show that $\sum_{n=-\infty}^{\infty}\left(\frac{\sin an\pi}{an\pi}\right)^2 = \frac{1}{a}$.

3. [C] Determine the fundamental frequency and the complex Fourier Series coefficients of $x(t) = (6 + 4\cos 8\pi t)\cos 20\pi t$ in two ways: (a) by expanding our the product using trigonometrical formulae and (b) by convolving the Fourier coefficients of the two factors.

4. [C] (a) Give the complex Fourier coefficients, U_n, if $u(t) = \cos t$. (b) Show, by using the convolution theorem, that $v(t) = u^2(t) = \frac{1}{2}\cos 2t + \frac{1}{2}$. (c) Show, by using the convolution theorem again that $w(t) = v^2(t) = u^4(t) = \frac{1}{8}\cos 4t + \frac{1}{2}\cos 2t + \frac{3}{8}$.

5. [C] Suppose $u(t) = \sin t$ and $v(t) = \begin{cases} 1 & 0 \le t < \pi \\ 0 & \pi \le t < 2\pi \end{cases}$ both with period $T = 2\pi$.

   (a) Determine the complex Fourier coefficients U_n and V_n.

   (b) If $w(t) = u(t)v(t)$, determine $W_n = U_n * V_n$ by convolving U_n and V_n.

6. [B] The waveform $u(t)$ has period $T = 1$ and equals $u(t) = 4t - 1$ for $0 \le t < 1$. If $u_N(t) = \sum_{n=-N}^{N} U_n e^{i2\pi nt}$ estimate, for large N, the minimum value and maximum value of $u_N(t)$ and also the value of $u_N(0)$.

7. [B] The waveform $u(t)$ has period $T = 1$. Estimate how rapidly U_n will decrease with $|n|$ when $u(t)$ in the range $0 \le t < 1$ is given by

   (a) t, (b) t^2, (c) $t\,(1-t)$, (d) $t^2\,(1-t)^2$, (e) $2t^3 - 3t^2 + t + 1$

8. [C] The waveform $u(t)$ has period $T_u = 1$ and satisfies $u(t) = \exp t$ for $0 \le t < 1$. The waveform $v(t)$ has period $T_v = 2$ and satisfies $v(t) = \exp|t|$ for $-1 \le t < 1$.

   (a) Find expressions for the complex Fourier coefficients U_n and V_n.
   (b) Calculate the average powers $\left\langle u^2(t) \right\rangle$ and $\left\langle v^2(t) \right\rangle$ and also those of $\left\langle u_2^2(t) \right\rangle$ and $\left\langle v_2^2(t) \right\rangle$ where $u_N(t)$ is the waveform formed by summing harmonics $-N$ to $+N$.

   (c) Determine the average error powers $\left\langle (u(t) - u_2(t))^2 \right\rangle$ and $\left\langle (v(t) - v_2(t))^2 \right\rangle$.