E1.10 Fourier Series and Transforms

Problem Sheet 3 - Solutions

1. (a) We have $u(t) = \cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$. So the fundamental period is $T = \pi$ and the fundamental frequency is $F = \frac{1}{T} = \frac{1}{\pi}$. The Fourier coefficients are $a_0 = 1$ and $a_1 = \frac{1}{2}$, so the complex Fourier coefficients are $U_0 = \frac{1}{2}$, $U_{-1} = U_1 = \frac{1}{4}$.

   (b) $P_u = \frac{1}{\pi}\int_0^\pi \cos^4 t\,dt = \frac{1}{32\pi}\left[12t + 8\sin 2t + \sin 4t\right]_0^\pi = \frac{1}{32\pi}(12\pi + 0 + 0) = \frac{3}{8}$.

   (c) $\sum_{n=-\infty}^{\infty} |U_n|^2 = \left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{3}{8}$. Also $\frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \frac{1}{4}\times 1^2 + \frac{1}{2}\times\left(\frac{1}{2}\right)^2 = \frac{3}{8}$. Note that the formula for Parseval's theorem is much more elegant and memorable when using complex Fourier coefficients.

2. (a) We have

$$\begin{aligned}
U_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)e^{-i2\pi nFt}dt \\
&= \frac{1}{1}\int_{-\frac{a}{2}}^{\frac{a}{2}} a^{-1}e^{-i2\pi nt}dt \\
&= \frac{i}{2an\pi}\left[e^{-i2\pi nt}\right]_{t=-\frac{a}{2}}^{\frac{a}{2}} \\
&= \frac{-i}{2an\pi}\left(e^{i\pi na} - e^{-i\pi na}\right) \\
&= \frac{\sin an\pi}{an\pi}
\end{aligned}$$

   Note that U_n is real-valued and even as expected since $u(t)$ is real-valued and even.

   (b) From the formula $U_0 = \left.\frac{\sin an\pi}{an\pi}\right|_{n=0}$ but this is not defined so we either determine U_0 directly from the original integral as $U_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)\,dt = 1$ or else as a limit: $U_0 = \lim_{n\to 0}\frac{\sin an\pi}{an\pi}$. We can find this limit using L'Hôpital's rule: $\lim_{n\to 0}\frac{\sin an\pi}{an\pi} = \left.\frac{a\pi\cos an\pi}{a\pi}\right|_{n=0} = 1$ or, equivalently, by using the small angle approximation, $\sin x \approx x$, which is exact for $x = 0$ and gives $U_0 = \lim_{n\to 0}\frac{\sin an\pi}{an\pi} = \frac{an\pi}{an\pi} = 1$. It is always true that $U_0 = \langle u(t)\rangle$ so since the average value of $u(t)$ is 1 for all values of a, it follows that U_0 will not depend on a.

   (c) We can calculate

$$\begin{aligned}
\left\langle |u(t)|^2 \right\rangle &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u^2(t)\,dt \\
&= \int_{-\frac{w}{2}}^{\frac{w}{2}} w^{-2}dt \\
&= \frac{1}{w}
\end{aligned}$$

   So, by Parseval's theorem, we know that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |U_n|^2 &= \sum_{n=-\infty}^{\infty}\left(\frac{\sin wn\pi}{wn\pi}\right)^2 \\
&= \left\langle |u(t)|^2 \right\rangle = \frac{1}{w}
\end{aligned}$$

3. (a) Expanding the product gives $x(t) = 6\cos 20\pi t + 4\cos 8\pi t\cos 20\pi t = 6\cos 20\pi t + 2\cos 12\pi t + 2\cos 28\pi t$. The fundamental frequency is the HCF of the frequencies of these three components (or, equivalently, of the original two components) and equals 2 Hz (or 4π rad/s). The three frequency components ar therefore at 5, 3 and 7 times the fundamental frequency giving the coefficient set: $X_{-7:+7} = [1, 0, 3, 0, 1, 0, 0, \underline{0}, 0, 0, 1, 0, 3, 0, 1]$. Note that since $x(t)$ is even, the coefficients are

are symmetrical around X_0 which is underlined.

(b) We can write $x(t) = u(t)v(t)$ where $u(t) = 6 + 4\cos 8\pi t$ and $v(t) = \cos 20\pi t$. Using the fundamental frequency of the output (i.e. 2 Hz), the coefficients of $u(t)$ and $v(t)$ are $U_{-2:2} = [2,\ 0,\ \underline{6},\ 0,\ 2]$ and $V_{-5:5} = [0.5,\ 0,\ 0,\ 0,\ 0,\ \underline{0},\ 0,\ 0,\ 0,\ 0,\ 0.5]$. To convolve these, we replace each non-zero entry in $V_{-5:5}$ with a complete copy of $U_{-2:2}$ scaled by the corresponding entry of $V_{-5:5}$. This gives the same coefficients as in the previous part.

4. (a) The only non-zero coefficients are $U_{\pm 1} = 0.5$. (b) Convolving U_n with itself gives $V_{\pm 2} = 0.25$ and $V_0 = 0.25 + 0.25 = 0.5$. Thinverse Fourier transform gives $v(t) = \frac{1}{2}\cos 2t + \frac{1}{2}$ as required. (c) Convolving $V_n$with itself gives $W_{\pm 4} = 0.25^2 = 0.0625$, $W_{\pm 2} = 0.5 \times 0.25 + 0.25 \times 0.5 = 0.25$ and $W_0 = 0.25^2 + 0.5^2 + 0.25^2 = 0.375$. Taking the inverse Fourier transform gives the required answer.

5. (a) $U_{-1} = \frac{i}{2}$ and $U_1 = \frac{-i}{2}$. For V_n we write

$$
\begin{aligned}
V_0 &= \frac{1}{2\pi}\int_0^{\pi} e^{-i0t}dt = \frac{1}{2} \\
\text{for}\, n \neq 0:\ V_n &= \frac{1}{2\pi}\int_0^{\pi} e^{-int}dt \\
&= \frac{i}{2n\pi}\left[e^{-int}\right]_0^{\pi} \\
&= \frac{i}{2n\pi}\left(e^{-in\pi} - 1\right) \\
&= \frac{i}{2n\pi}\left((-1)^n - 1\right) \\
&= \begin{cases} \frac{-i}{n\pi} & n\,\text{odd} \\ 0 & n\,\text{even},\, n \neq 0 \\ \frac{1}{2} & n = 0 \end{cases}
\end{aligned}
$$

Note that, except for its DC component of $V_0 = \frac{1}{2}$, $v(t)$ is a real-valued, odd, anti-periodic function and therefore has purely imaginary coefficients with all even coefficients (except V_0) equal to zero.

(b) From the notes (slide 4-5) the convolution is defined by $W_n = U_n * V_n = V_n * U_n = \sum_{m=-\infty}^{\infty} V_{n-m}U_m$. Since $U_m = 0$ except for $m = \pm 1$, the infinite sum actually only has two non-zero terms and $W_n = U_1 V_{n-1} + U_{-1} V_{n+1} = \frac{i}{2}\left(V_{n+1} - V_{n-1}\right)$. If n is even, then $n+1$ and $n-1$ are both odd so, using the formula for V_n given above, $W_n = \frac{i}{2}\left(V_{n+1} - V_{n-1}\right) = \frac{i}{2}\left(\frac{-i}{(n+1)\pi} - \frac{-i}{(n-1)\pi}\right) = \frac{1}{2\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = \frac{1}{2\pi}\left(\frac{-2}{n^2-1}\right) = \frac{-1}{(n^2-1)\pi}$. If n is odd then $n+1$ and $n-1$ are both even and V_{n+1} and V_{n-1} are both zero unless $n+1$ or $n-1$ equals zero, i.e. unless $n = \pm 1$. So we have $W_1 = \frac{i}{2}\left(-V_0\right) = \frac{-i}{4}$ and $W_{-1} = \frac{i}{2}\left(V_0\right) = \frac{i}{4}$. We can combine all these results to give

$$
W_n = \begin{cases} 0 & n\,\text{odd},\, n \neq \pm 1 \\ \frac{-i}{4n} & n = \pm 1 \\ \frac{-1}{(n^2-1)\pi} & n\,\text{even} \end{cases}
$$

6. We have $u(0-) = u(1-) = 3$ but $u(0+) = -1$ so there is a discontinuity at $t = 0$. Therefore $u_N(0) \to \frac{3+(-1))}{2} = 1$. Notice that the actual value defined for $u(0) = 0$ has no affect on this answer. Due to Gibbs phenomenon, $u_N(t)$ will undershoot and overshoot the discontinuity by about 9% of the discontinuity height: $3 - (-1) = 4$. So $0.09 * 4 = 0.36$. So the maximum value of $u_N(t)$ will be 3.36 and the minimum value will be -1.36.

7. (a) $u(0) = 0$ but $u(1) = 1$ so the waveform has a discontinuity and the coefficients, U_n, will decrease $\propto |n|^{-1}$.

   (b) $u(0) = 0$ but $u(1) = 1$ so the waveform again has a discontinuity and the coefficients, U_n, will decrease $\propto |n|^{-1}$.

   (c) $u(0) = u(1) = 0$ but $u'(0) \neq u'(1)$ so coefficients, U_n, will decrease $\propto |n|^{-2}$.

(d) The first non-equal derivative is $u''(0) \neq u''(1)$ so coefficients, U_n, will decrease $\propto |n|^{-3}$.

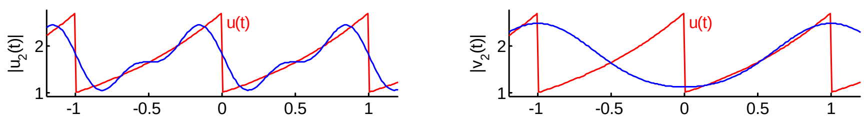
(e) $u(0) = u(1) = 1$ and $u'(0) = u'(1) = 1$. The first non-equal derivative is $-6 = u''(0) \neq u''(1) = 6$ so coefficients, U_n, will decrease $\propto |n|^{-3}$.

8. (a) $U_n = \frac{1}{T_u}\int_0^1 e^t e^{-i2\pi n F_u t}dt = \int_0^1 e^{(1-i2\pi n)t}dt = \frac{1}{1-i2\pi n}\left[e^{(1-i2\pi n)t}\right]_{t=0}^{1} = \frac{1}{1-i2\pi n}\left(e^{(1-i2\pi n)}-1\right)$ $= \frac{1}{1-i2\pi n}\left(e \times e^{-i2\pi n}-1\right) = \frac{1}{1-i2\pi n}(e-1) = \frac{e-1}{1-i2\pi n}$. Note that we use the fact that $e^{-i2\pi n} = 1$ for any integer n.

$V_n = \frac{1}{T_v}\int_{-1}^{1} e^{|t|}e^{-i2\pi n F_v t}dt = \frac{1}{2}\left(\int_{-1}^{0} e^{-t}e^{-i\pi n t}dt + \int_0^1 e^t e^{-i\pi n t}dt\right)$
$= \frac{1}{2}\left(\frac{1}{-1-i\pi n}\left(1-e^{-(-1-i\pi n)}\right) + \frac{1}{1-i\pi n}\left(e^{(1-i\pi n)}-1\right)\right)$
$= \frac{1}{2}\left(\frac{1}{-1-i\pi n}\left(1-e\times(-1)^n\right) + \frac{1}{1-i\pi n}\left(e\times(-1)^n-1\right)\right)$
$= \frac{(-1)^n e-1}{2}\left(\frac{1}{1+i\pi n}+\frac{1}{1-i\pi n}\right) = \frac{(-1)^n e-1}{2}\times\frac{2}{1+\pi^2 n^2} = \frac{(-1)^n e-1}{1+\pi^2 n^2}$. We see that this is real-symmetric (because $v(t)$ is real-symmetric) and that it decays $\propto n^{-2}$ because $v(t)$ is continuous but has gradient discontinuities at $t=0$ and $t=1$.

(b) $\langle u^2(t)\rangle = \frac{1}{T_u}\int_0^1 \left(e^t\right)^2 dt = \int_0^1 e^{2t}dt = \frac{1}{2}\left[e^{2t}\right]_{t=0}^{1} = \frac{e^2-1}{2} = 3.1945$.
$\langle v^2(t)\rangle = \langle u^2(t)\rangle = \frac{e^2-1}{2}$ since reflecting a waveform in time does not affect its power.
$\langle u_2^2(t)\rangle = \sum_{-2}^{2}|U_n|^2 = U_0^2 + 2|U_1|^2 + 2|U_2|^2$
$= 1.7183^2 + 2\left(0.2701^2 + 0.1363^2\right) = 2.9525 + 0.1459 + 0.0372 = 3.1355$.

$\langle v_2^2(t)\rangle = \sum_{-2}^{2}|V_n|^2 = V_0^2 + 2|V_1|^2 + 2|V_2|^2$
$= 1.7183^2 + 2\left(0.3421^2 + 0.0424^2\right) = 2.9525 + 0.2340 + 0.0036 = 3.1901$.



We see that, for the same number of harmonics, $v_2(t)$ fits the exponential much better than $u_2(t)$ over the range $0 \le t < 1$ and that it includes much more of the energy of $u(t)$.

(c) We can use Parseval's theorem to calculate the power of the error, $\left\langle\left(u(t)-u_2(t)\right)^2\right\rangle$. We know that $u(t) = \sum_{-\infty}^{+\infty} U_n e^{i2\pi nt}$ and that $u_2(t) = \sum_{-2}^{+2} U_n e^{i2\pi nt}$, so it follows that $u(t)-u_2(t) = \sum_{|n|>2} U_n e^{i2\pi nt}$. Applying Parseval's theorem to these threee expressions gives $\langle u^2(t)\rangle = \sum_{-\infty}^{+\infty}|U_n|^2$, $\langle u_2^2(t)\rangle = \sum_{-2}^{+2}|U_n|^2$ and $\left\langle\left(u(t)-u_2(t)\right)^2\right\rangle = \sum_{|n|>2}|U_n|^2$. By subtracting the first two of these equations, we can see that $\langle u^2(t)\rangle - \langle u_2^2(t)\rangle = \left\langle\left(u(t)-u_2(t)\right)^2\right\rangle$ and so, from part (b), $\left\langle\left(u(t)-u_2(t)\right)^2\right\rangle = \langle u^2(t)\rangle - \langle u_2^2(t)\rangle = 3.1945 - 3.1355 = 0.0590$. Likewise $\left\langle\left(v(t)-v_2(t)\right)^2\right\rangle = 3.1945 - 3.1901 = 0.0044$. Note that, for arbitrary functions $x(t)$ and $y(t)$ having the same period, the relationship $\left\langle\left(x(t)-y(t)\right)^2\right\rangle = \langle x^2(t)\rangle - \langle y^2(t)\rangle$ is only true if $\langle x(t)y(t)\rangle = 0$ or, equivalently, if they have non-overlapping Fourier series (i.e. X_n and Y_n are never both non-zero for any n).