E1.10 Fourier Series and Transforms

Problem Sheet 4 - Solutions

1. $\int_{-\infty}^{\infty}\delta(t-3)t^3e^{-t}dt = \left[t^3e^{-t}\right]_{t=3} = 3^3e^{-3} = 27\times 0.498 = 1.344.$

2. (a) $\int_{-\infty}^{\infty}\delta(t-6)t^2\,dt = \left[t^2\right]_{t=6} = 36$

   (b) Substituting $t=3\tau$ gives $\int_{-\infty}^{\infty}\delta(3\tau-6)9\tau^2\,3d\tau = 27\int_{-\infty}^{\infty}\delta(3\,(\tau-2))\tau^2\,d\tau$ $= 27\int_{-\infty}^{\infty}\frac{1}{|3|}\delta(\tau-2)\tau^2\,d\tau = 9\left[\tau^2\right]_{\tau=2} = 36$. We here use the relation that $|c|\,\delta(cx)=\delta(x)$.

3. $2x^2\delta(8-2x) = 2x^2\delta\left(-2\,(x-4)\right) = \frac{2x^2}{|-2|}\delta\,(x-4) = x^2\delta\,(x-4) = 16\delta\,(x-4).$

4. (a) $V(f) = \int_{-\infty}^{\infty}e^{-|t|}e^{-i2\pi ft}dt = \int_{-\infty}^{0}e^{t}e^{-i2\pi ft}dt + \int_{0}^{\infty}e^{-t}e^{-i2\pi ft}dt$ $= \frac{1}{1-i2\pi f}\left[e^{(1-i2\pi f)t}\right]_{t=-\infty}^{0} + \frac{1}{-1-i2\pi f}\left[e^{(-1-i2\pi f)t}\right]_{t=0}^{\infty} = \frac{1}{1-i2\pi f} - \frac{1}{-1-i2\pi f} = \frac{2}{1+4\pi^2f^2}$. Notice that in the first step we split the integral up into the two ranges of t for which the quantity $|t|$ is equal to $-t$ and $+t$ respectively; this is necessary for any integral involving absolute values. Also notice that $e^{(a+bi)t}$ is zero at $t=+\infty$ if $a<0$ and zero at $t=-\infty$ if $a>0$.

   (b) If $v_1(t)=v(at)$ then $V_1(f) = \frac{1}{|a|}V\left(\frac{f}{a}\right) = \frac{2a^2}{a^2+4\pi^2f^2}$.

   If $v_2(t)=v(t-b)$ then $V_2(f) = e^{-i2\pi fb}V(f) = \frac{2e^{-i2\pi fb}}{1+4\pi^2f^2}$.

   If $w(t)=V(t)=\frac{2}{1+4\pi^2t^2}$ then $W(f)=v(-f)=e^{-|f|}$. However we want $v_3(t)=0.5w\left(\frac{t}{2\pi}\right)$ so $V_3(f) = 0.5\times 2\pi\times W(2\pi f) = \pi e^{-|2\pi f|}$.

5.

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\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} t^2e^{-|t|}e^{-i2\pi ft}dt \\
&= \int_{-\infty}^{0} t^2e^{t}e^{-i2\pi ft}dt + \int_{0}^{\infty} t^2e^{-t}e^{-i2\pi ft}dt \\
&= \int_{-\infty}^{0} t^2e^{(1-i2\pi f)t}dt + \int_{0}^{\infty} t^2e^{(-1-i2\pi f)t}dt \\
&= \left[\left((1-i2\pi f)^2t^2 - 2\,(1-i2\pi f)\,t+2\right)\frac{e^{(1-i2\pi f)t}}{(1-i2\pi f)^3}\right]_{t=-\infty}^{0} \\
&\quad + \left[\left((-1-i2\pi f)^2t^2 - 2\,(-1-i2\pi f)\,t+2\right)\frac{e^{(-1-i2\pi f)t}}{(-1-i2\pi f)^3}\right]_{t=0}^{\infty} \\
&= 2\left(\frac{1}{(1-i2\pi f)^3} - \frac{1}{(-1-i2\pi f)^3}\right) \\
&= \frac{4+48\pi^2f^2}{(1+4\pi^2f^2)^3}
\end{aligned}
$$

6. $X(f) = \int_{-\infty}^{\infty}x(t)e^{-i2\pi ft}dt = \int_{-\infty}^{\infty}\delta(t)e^{-i2\pi ft}dt = \left[e^{-i2\pi ft}\right]_{t=0}1$. Note that this is the same for all values of f and is called a “flat” or “white” spectrum. The inverse transform is

$$\delta(t) = \int_{-\infty}^{\infty}X(f)e^{i2\pi ft}df = \int_{-\infty}^{\infty}e^{i2\pi ft}df.$$

   If we now substitute $\tau = \frac{2\pi}{\alpha}t$, we obtain $\int_{-\infty}^{\infty}e^{i\alpha f\tau}df = \delta\left(\frac{\alpha}{2\pi}\tau\right) = \frac{2\pi}{|\alpha|}\delta(\tau)$. Alternatively, we could substitute $\nu = \frac{2\pi}{\alpha}f$ to obtain $\delta(t) = \frac{2\pi}{\alpha}\int_{f=-\infty}^{\infty}e^{i\alpha\nu t}d\nu$. The new limits (in terms of ν) are either $\nu=\mp\infty$ if $\alpha>0$ or else $\nu=\pm\infty$ if $\alpha<0$ and in the latter case we need to reverse the order of the limits and multiply by -1. Thus we end up with $\delta(t) = \frac{2\pi}{|\alpha|}\int_{f=-\infty}^{\infty}e^{i\alpha\nu t}d\nu$ which is the same result as before.

7. $X(f) = \int_{-\infty}^{\infty}10e^{-i2\pi ft}dt = 10\delta(f)$. This follows from the answer to question 6 with $\alpha=-2\pi$.

8. The Fourier transform of a periodic waveform is iust the complex Fourier series coefficients multiplied by delta functions at the appropriate positive and negative frequencies. So $X(f) = 6\delta(f+100) + 6\delta(f-100) + 4i\delta(f+200) - 4i\delta(f-200)$.

9. The complex Fourier series coefficients are $V_n = F\int_{-0.5T}^{0.5t}\delta(t)e^{-i2\pi Ft}dt = F\left[e^{-i2\pi Ft}\right]_{t=0} = F$ (i.e. the same for all n). In fact, $x(t)$ is equal to $v(t)$ but iust written in a different way. So, from the theorem on page 6-8 of the notes, $X(f) = \sum_{n=-\infty}^{\infty} X_n\delta(f-nF) = F\sum_{n=-\infty}^{\infty}\delta(f-nF)$. Thus the Fourier transform of an impulse train with spacing $\frac{1}{F}$ is another impulse train with spacing F.

10. (a) If $v(t) = X(t) = \cos 100t$, then $V(f) = \frac{1}{2}\delta(f+\frac{50}{\pi}) + \frac{1}{2}\delta(f-\frac{50}{\pi})$. So, from the duality theorem, $x(f) = V(-f)$, so $x(t) = \frac{1}{2}\delta(t+\frac{50}{\pi}) + \frac{1}{2}\delta(t-\frac{50}{\pi})$.
(b) $x(t) = \int_{-\infty}^{\infty}\cos(100f)\,e^{i2\pi ft}df = \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{i100f} + e^{-i100f}\right)e^{i2\pi ft}df$
$= \frac{1}{2}\int_{-\infty}^{\infty} e^{i\left(2\pi\left(t+\frac{50}{\pi}\right)\right)f}df + \frac{1}{2}\int_{-\infty}^{\infty} e^{i\left(2\pi\left(t-\frac{50}{\pi}\right)\right)f}df == \frac{1}{2}\delta(t+\frac{50}{\pi}) + \frac{1}{2}\delta(t-\frac{50}{\pi})$.

11. $X(f) = \int_{-0.5}^{0.5} e^{-i2\pi ft}dt = \frac{1}{-i2\pi f}\left[e^{-i2\pi ft}\right]_{t=-0.5}^{0.5} = \frac{1}{-i2\pi f}\times -2i\sin\pi f = \frac{\sin\pi f}{\pi f}$.

12. $X(f) = \int_0^{\infty} e^{-at}e^{-i2\pi ft}dt = \int_0^{\infty} e^{(-a-i2\pi f)t}dt = \frac{1}{-a-i2\pi f}\left[e^{(-a-i2\pi f)t}\right]_{t=0}^{\infty} = \frac{-1}{-a-i2\pi f} = \frac{1}{a+i2\pi f}$. Note that the value of $e^{(-a-i2\pi f)t}$ is zero at $t=\infty$ provided that $a>0$.

13. (a) $x(t) = \cos^2(1000t) = 0.5 + 0.5\cos(2000t)$. The gains at these component frequencies are $\frac{Y}{X}(i0) = 2$ and $\frac{Y}{X}(i2000) = \frac{2}{1+2i} = 0.4 - 0.8i$. It follows (from phasors) that

$$y(t) = 1 + 0.2\cos(2000t) + 0.4\sin(2000t).$$

The Fourier transforms are $X(f) = 0.5\delta(f) + 0.25\delta\left(f+\frac{1000}{\pi}\right) + 0.25\delta\left(f-\frac{1000}{\pi}\right)$ and $Y(f) = \delta(f) + (0.1+0.2i)\,\delta\left(f+\frac{1000}{\pi}\right) + (0.1-0.2i)\,\delta\left(f-\frac{1000}{\pi}\right)$. Note that the positive frequency term, $\delta\left(f-\frac{1000}{\pi}\right)$, is multiplied by $\frac{Y}{X}(i2\pi f)$ while the negative frequency term, $\delta\left(f+\frac{1000}{\pi}\right)$, is multiplied by its complex conjugate, $\frac{Y}{X}(-i2\pi f)$.
(b) From question 12 we know that $X(f) = \frac{1}{i2\pi f+500}$. So it follows that

$$Y(f) = X(f)\times\frac{Y}{X}(i2\pi f) = \frac{1}{i2\pi f+500}\times\frac{2000}{i2\pi f+1000} = \frac{2000}{(i2\pi f+500)(i2\pi f+1000)}$$

We can put the given expression over a common denominator: $\frac{c}{i2\pi f+500} + \frac{d}{i2\pi f+1000} = \frac{i2\pi f(c+d)+1000c+500d}{(i2\pi f+500)(i2\pi f+1000)}$.
Equating the numerator to 2000 gives $c=4$ and $d=-4$. Hence $y(t) = \begin{cases}4\left(e^{-500t}-e^{-1000t}\right) & t\ge 0\\ 0 & t<0\end{cases}$.

14. $y(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$. The integrand is only non-zero when the arguments of both top-hat functions lie in the range ± 0.5. Thus we must have $-0.5<\tau<0.5$ and also
$-0.5 < t-\tau < 0.5 \Leftrightarrow t-0.5 < \tau < t+0.5$.
We can therefore write $y(t) = \int_{\max(-0.5,\,t-0.5)}^{\min(0.5,\,t+0.5)} d\tau = \begin{cases}\int_{-0.5}^{t+0.5} d\tau & t<0\\ \int_{t-0.5}^{0.5} d\tau & t\ge 0\end{cases}$. The integration range is empty if $|t|>1$ and so we can write $y(t) = \begin{cases}1+t & t<0\\ 1-t & t\ge 0\end{cases}$ which also equals $y(t) = \begin{cases}1-|t| & |t|\le 1\\ 0 & |t|>1\end{cases}$ as requested.
From the convolution theorem, $Y(f) = X^2(f) = \frac{\sin^2\pi f}{\pi^2 f^2}$.

15. [B] An "energy signal" has finite energy: $\int_{-\infty}^{\infty}|x(t)|^2\,dt < \infty$. A "power signal" has infinite energy but finite power: $\lim_{A,B\to\infty}\frac{1}{B-A}\int_{-A}^{B}|x(t)|^2\,dt < \infty$. The answers are therefore (a) P, (b) P, (c) N, (d) N, (e) N, (f) N, (g) E, (h) E, (i) P, (i) E, (i) P. The final example has zero average power but is not an energy signal because it has infinite energy.

16. (a) We substitute $\omega = 2\pi f$ to obtain:

$$\begin{aligned}\widetilde{X}(\omega) &= \frac{1}{1+\omega^2} + 2i\left(\delta(\frac{\omega}{2\pi}+4) - \delta(\frac{\omega}{2\pi}-4)\right) \\ &= \frac{1}{1+\omega^2} + 2i\left(\delta(\frac{\omega+8\pi}{2\pi}) - \delta(\frac{\omega-8\pi}{2\pi})\right) \\ &= \frac{1}{1+\omega^2} + 4\pi i\left(\delta(\omega+8\pi) - \delta(\omega-8\pi)\right).\end{aligned}$$

The final line is obtained using the scaling formula for delta functions: $|c|\,\delta(cx) = \delta(x)$. Thus we see that in the angular-frequency version of the Fourier transform, any continuous functions of f remain the same amplitude but delta functions are multiplied by 2π. The inverse transform is given by $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{X}(\omega)e^{i\omega t}d\omega$; this can be obtained by changing the variable in the normal inverse transform from f to ω.

(b) $\widehat{X}(\omega)$ is exactly the same as $\widetilde{X}(\omega)$ but divided by $\sqrt{2\pi}$. So

$$\widetilde{X}(\omega) = \frac{1}{\sqrt{2\pi}\,(1+\omega^2)} + \sqrt{8\pi}i\left(\delta(\omega+8\pi) - \delta(\omega-8\pi)\right).$$

The inverse transform is the same as in the previous part but multiplied by $\sqrt{2\pi}$, i.e.

$$x(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{X}(\omega)e^{i\omega t}d\omega.$$