MATHEMATICS 1A - SAMPLE QUESTIONS

Information for Candidates:

Calculators are not permitted in this exam.

1. a) i) How many roots does the following equation have, and what are they? [2 marks]

$$f(x) = x^2 – 4x + 4$$

ii) Give an example of a polynomial with real coefficients which has two imaginary roots and no other roots. [4 marks]

iii) Show that the following function $g(x)$ must have a real root in the range $–1 < x < 1$,

$$g(x) = x^3 – 3x + 1.$$

[6 marks]

b) Using the geometry of Fig 1.1 below, derive the trigonometric identity for $\sin(A+B)$. Use the notation where, for example, RN represents the length of the segment between points R and N. Note that MR and TP are parallel, the two angles indicates by * are equal and that all four angles indicated by the symbol • are equal. [6 marks]

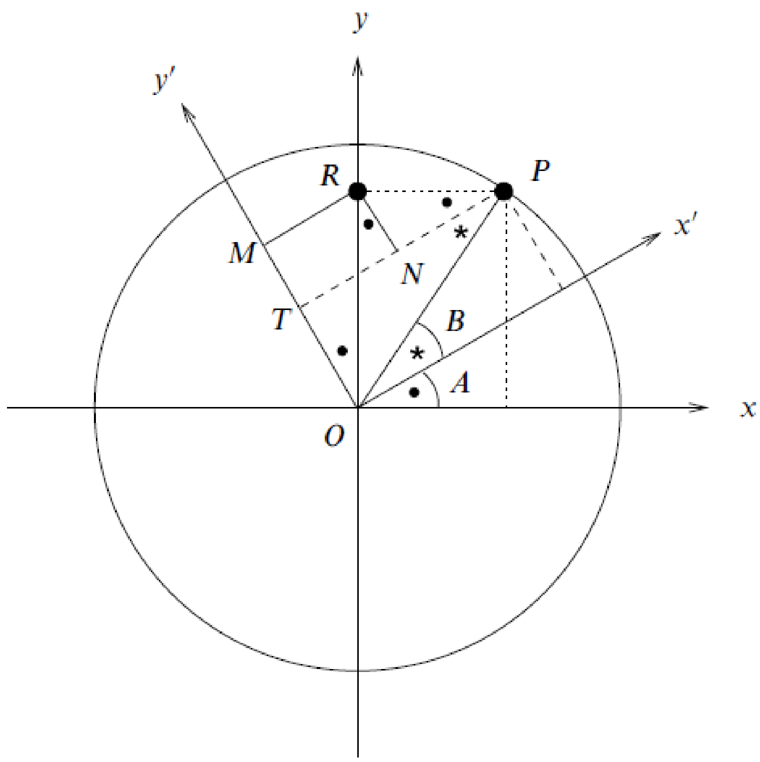

Figure 1.1

c) i) A certain complex number X has a modulus $|X| = 3$. Find the value of $|Y|$, where $Y = XX^*$. [4 marks]

ii) Given the following relation:

$$r_2 \exp(i\theta_2) = r_1 \exp(i\theta_1) + i r_1 \exp(i\theta_1)$$

find expressions for each of r_2 and θ_2 in terms of r_1 and θ_1. [4 marks]

iii) Find all the unique solutions for the equation $Z^5 = 1$ where Z is a complex number. [4 marks]

iv) Find the value of Y in the form $A + iB$ where $Y = \sqrt{X}$ and $X = 2 + i2\sqrt{3}$. [Hint: a calculator is not required.] [4 marks]

2. Let a be a positive real number and $g(x)$ a function defined for all real values of x as

$$g(x) = a^x + a^{-x}.$$

a) Show that, if $a \neq 1$, g is strictly increasing for $x > 0$ and strictly decreasing for $x < 0$. [5 marks]

b) Let $a = e$, sketch the graph of the function $f(x) = e^x + e^{-x}$, and the graph of the function $h(x) = \frac{1}{f(x)}$. Identify and classify the stationary points of $f(x)$ and $h(x)$. [12 marks]

c) Compute the integral $I(t) = \int_0^t h(x)dx$. [8 marks]

d) Find the limit of $I(t)$ for $t \to +\infty$ and give a geometric interpretation of the value of the integral. [8 marks]

3. The Fourier series for a real-valued periodic function, $u(t)$, with period $T = \frac{1}{F}$ is given by

$$u(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi nFt + b_n \sin 2\pi nFt.$$

a) By using the identities

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$
$$\sin\theta = \frac{-j}{2}\left(e^{j\theta} - e^{-j\theta}\right)$$

where $j = \sqrt{-1}$, show that this may also be written as

$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{j2\pi nFt}$$

and derive expressions for the complex coefficients, U_n, in terms of a_n and b_n. [7 marks]

b) Explain the relationship between the coefficients U_{-n} and U_{+n}. [3 marks]

c) Suppose $u(t)$ has period $T = 8$ and, over the interval $-4 \le t < 4$ is given by

$$u(t) = \begin{cases} 5 & \text{for } -1 \le t < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Determine the complex coefficients, U_n by evaluating the Fourier analysis integral

$$U_n = \frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} u(t)e^{-j2\pi nFt}dt.$$

[8 marks]

d) The function $v(t)$ is said to be "antiperiodic" if $v(t+\frac{1}{2}T) = -v(t)$.

Suppose that the antiperiodic function $v(t)$ has period $T = 8$ and satisfies $v(t) = u(t)$ over the range $-2 \le t < 2$.

Sketch dimensioned graphs of both $u(t)$ and $v(t)$ over the range $-10 \le t \le 10$. [5 marks]

i) Show that, by dividing the integration range into two halves, the Fourier analysis integral may be expressed as

$$V_n = \frac{1}{T}\int_{-\frac{1}{2}T}^{0} v(t)e^{-j2\pi nFt}dt + \frac{1}{T}\int_{-\frac{1}{2}T}^{0} v(t+\frac{1}{2}T)e^{-j2\pi nF(t+\frac{1}{2}T)}dt.$$

[5 marks]

ii) Show that $e^{-j2\pi nF\frac{1}{2}T} = 1$ if n is an even integer. [2 marks]

iii) Hence show that, if $v(t)$ is antiperiodic, its complex Fourier coefficients, V_n, are zero for even values of n. [3 marks]

MATHEMATICS 1A - SAMPLE QUESTIONS

********* **Solutions** *********

Information for Candidates:

Calculators are not permitted in this exam.

********* Questions and Solutions *********

1. a) i) How many roots does the following equation have, and what are they? [2 marks]

$$f(x) = x^2 – 4x + 4$$

$f(x)$ is a second order polynomial and so has 2 roots. The can be found using the quadratic equation or by factorizing as $f(x) = (x-2)(x-2)$. So the roots are identical: $x = \{2, 2\}$.

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ii) Give an example of a polynomial with real coefficients which has two imaginary roots and no other roots. [4 marks]

A polynomial with 2 roots must be a quadratic. The quadratic equation, $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, shows that for imaginary roots, the linear term must have a coefficient, b, of zero and that the other two coefficients must have the same sign. Thus the answer is any equation $ax^2 + c = 0$ where a and c have the same sign. For example $x^2 + 1 = 0$ has roots $x = \pm i$.

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iii) Show that the following function $g(x)$ must have a real root in the range $–1 < x < 1$,

$$g(x) = x^3 – 3x + 1.$$

[6 marks]

Differentiating gives $g'(x) = 3x^2 - 3$ and $g'(x) = 0$ for $x = \pm 1$. Therefore, the slope changes sign at these two values of x.

Evaluating $g(x)$ at these values gives $g(-1) = 3$ and $g(+1) = -1$. Since these have opposite signs and $g(x)$ is a continuous function, it must cross the x-axis between $x = -1$ and $x = +1$ and have a root at the corresponding value of x.

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b) Using the geometry of Fig 1.1 below, derive the trigonometric identity for $\sin(A + B)$. Use the notation where, for example, RN represents the length of the segment between points R and N. Note that MR and TP are parallel, the two angles indicates by * are equal and that all four angles indicated by the symbol • are equal. [6 marks]

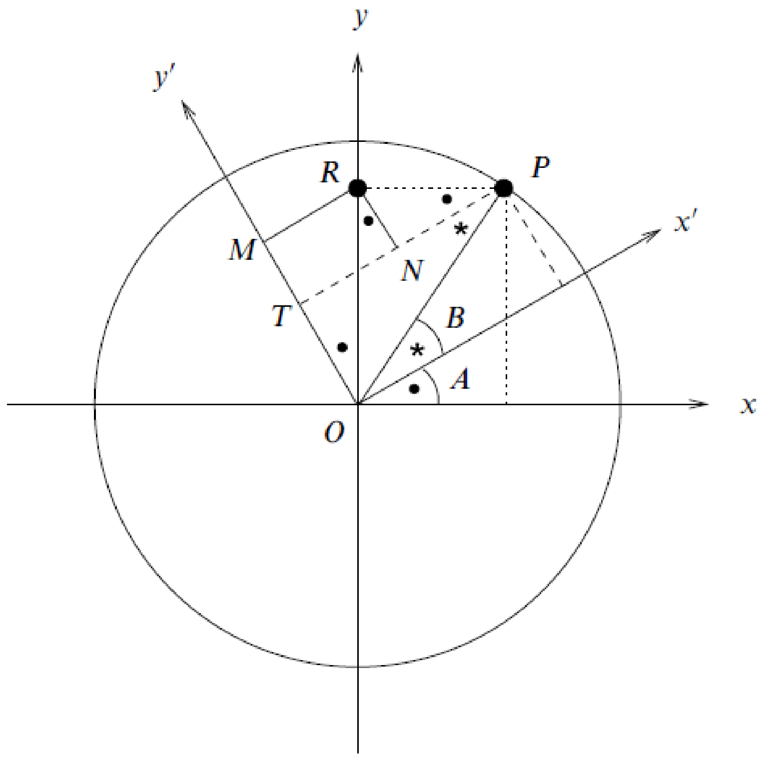

Figure 1.1

From the triangle OPT, $OP\cos B = TN + NP$ *and, since* $MR = TN$, $OP\cos B = MR + NP$. *We can see that* $MR = OR\sin A$ *and* $NP = RP\cos A$. *From triangle ORP, we get* $OR = OP\sin(A+B)$ *and* $RP = OP\cos(A+B)$. *Putting all this together gives*

$$OP\cos B = MR + NP = OR\sin A + RP\cos A = OP\left(\sin(A+B)\sin A + \cos(A+B)\cos A\right) \tag{1.1}$$

In a similar way, we can write

$$\begin{aligned} OP\sin B &= OT = OM - MT = OM - RN \\ &= OR\cos B - RP\sin A = OP\left(\sin(A+B)\cos A - \cos(A+B)\sin A\right) \end{aligned} \tag{1.2}$$

Now, divide (1.1) *and* (1.2) *by OP and then take* $\sin A \times (1.1) + \cos A \times (1.2)$ *to get*

$$\begin{aligned} \sin A\cos B + \cos A\sin B &= \sin A\left(\sin(A+B)\sin A + \cos(A+B)\cos A\right) \\ &\quad + \cos A\left(\sin(A+B)\cos A - \cos(A+B)\sin A\right) \\ &= \sin(A+B)\sin^2 A + \sin(A+B)\cos^2 A + \text{two terms that cancel} \\ &= \sin(A+B)\left(\sin^2 A + \cos^2 A\right) = \sin(A+B) \end{aligned}$$

c) i) A certain complex number X has a modulus $|X| = 3$. Find the value of $|Y|$, where $Y = XX^*$. [4 marks]

Since $Y = XX^*$, we can write $|Y| = |X||X^*| = |X|^2 = 3^2 = 9$. We have used the facts that $|XY| = |X| \times |Y|$ and that $|X^*| = |X|$.

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ii) Given the following relation:

$$r_2 \exp(i\theta_2) = r_1 \exp(i\theta_1) + ir_1 \exp(i\theta_1)$$

find expressions for each of r_2 and θ_2 in terms of r_1 and θ_1. [4 marks]

To solve this algebraically, we can write

$$\begin{aligned} r_2 \exp(i\theta_2) &= (r_1 + ir_1) \exp(i\theta_1) \\ &= r_1 (1+i) \exp(i\theta_1) \\ &= r_1 \left(\sqrt{2} \exp(i\frac{\pi}{4}\right) \exp(i\theta_1) \\ &= \sqrt{2} r_1 \exp\left(i\left(\theta_1 + \frac{\pi}{4}\right)\right) \end{aligned}$$

From this we get $r_2 = \sqrt{2} r_1$ and $\theta_2 = \theta_1 + \frac{\pi}{4}$.

It is also possible to derive this graphically using geometry.

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iii) Find all the unique solutions for the equation $Z^5 = 1$ where Z is a complex number. [4 marks]

We have $z^5 = 1 = \exp(i2\pi k)$ for any integer k. So therefore

$$z = \sqrt[5]{1} = \exp\left(i\frac{2\pi}{5}k\right).$$

So taking $k = \{0, 1, 2, 3, 4\}$ gives $z = \left\{1, e^{i0.4\pi}, e^{i0.8\pi}, e^{i1.2\pi}, e^{i1.6\pi}\right\}$.

Any additional values of k just repeat these roots, e.g. $k = 5$ gives $e^{i2\pi} = 1$.

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iv) Find the value of Y in the form $A + iB$ where $Y = \sqrt{X}$ and $X = 2 + i2\sqrt{3}$. [Hint: a calculator is not required.] [4 marks]

We can write $X = 2\left(1 + i\sqrt{3}\right) = 2 \times 2\angle 60^\circ = 4e^{i\frac{\pi}{3}}$. It follows that $Y = \sqrt{X} = 2e^{i\frac{\pi}{6}} = 2\angle 30^\circ$. Converting this to rectangular form gives $Y = \sqrt{3} + i$.

There are two very frequently arising right-angled triangles that you should know by heart:

(i) the triangle with sides $\left\{1, 1, \sqrt{2}\right\}$ has angles $\{45^\circ, 45^\circ, 90^\circ\}$

(ii) the triangle with sides $\{1, \sqrt{3}, 2\}$ *has angles* $\{30°, 60°, 90°\}$.

In terms of complex numbers, these translate to $1+i=\sqrt{2}\angle 45°$, $1+i\sqrt{3}=2\angle 60°$ *and* $\sqrt{3}+i=2\angle 30°$.

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2. Let a be a positive real number and $g(x)$ a function defined for all real values of x as

$$g(x) = a^x + a^{-x}.$$

a) Show that, if $a \neq 1$, g is strictly increasing for $x > 0$ and strictly decreasing for $x < 0$. [5 marks]

The derivative of g is $g'(x) = a^x \ln(a) - a^{-x}\ln(a) = (\ln a)(a^x - a^{-x})$. *If* $a > 1$, *then* $\ln(a) > 0$ *and* $g'(x)$ *is positive when* $a^x - a^{-x} > 0$, *that is, when* $x > 0$. *If* $0 < a < 1$, *then* $\ln(a) < 0$ *and* $g'(x)$ *is positive when* $a^x - a^{-x} < 0$, *that is, when* $x > 0$. *Then, g is strictly increasing for* $x > 0$ *and strictly decreasing for* $x < 0$. *Note that g is even and then the analysis of the function can be limited to* $x > 0$ *(or* $x < 0$*). Note also that the question explicitly says that* $a > 0$ *so we need not consider negative values of a (which would result in* $g(x)$ *being complex-valued).*

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b) Let $a = e$, sketch the graph of the function $f(x) = e^x + e^{-x}$, and the graph of the function $h(x) = \frac{1}{f(x)}$. Identify and classify the stationary points of $f(x)$ and $h(x)$. [12 marks]

The function $f(x) = e^x + e^{-x} = 2\cosh(x)$ *is defined for all real values of x and is even. The graph can be obtained as a sum of the two known function* e^x *and* e^{-x}. *The first derivative is* $f'(x) = e^x - e^{-x}$, *whereas the second derivative is* $f''(x) = e^x + e^{-x}$. $f'(x) = 0$ *has the unique solution* $x = 0$. $f''(x)$ *is always positive, since it is the sum of two positive functions, thus the function is convex. Then 0 is a minimum. The function* $f(x)$ *does not cross the x axis and goes to* $+\infty$ *when x goes to* $\pm\infty$ *(there are no asymptotes).*

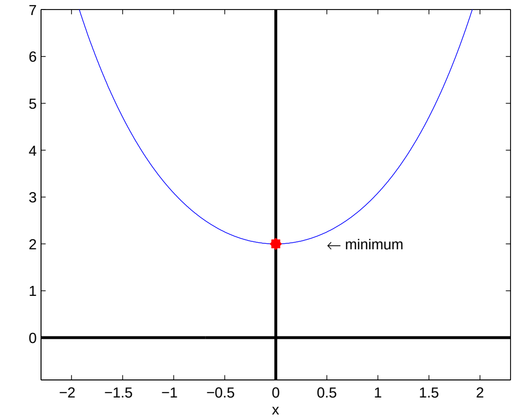

Figure S1.1 The function $f(x)$

The graph of the function $h(x) = \frac{1}{f(x)}$ can be deduced from the graph of $f(x)$. $h(x)$ is defined for all real values of x and it is always positive. The first derivative is $\frac{-(e^x - e^{-x})}{(e^x+e^{-x})^2}$, whereas the second derivative is $\frac{e^{2x}+e^{-2x}-6}{(e^x+e^{-x})^3}$. $f'(x) = 0$ has the unique solution $x = 0$. The functions $f(x)$ and $h(x)$ have increases and decreases "swapped", hence 0 is a maximum. We study the second derivative to determine the inflection points, namely we solve the equation $e^{2x}+e^{-2x}-6 = 0$. Let $z = e^{2x}$, then $x = \ln(\sqrt{3 \pm 2\sqrt{2}}) = \ln(\sqrt{2} \pm 1)$ are the inflection points.

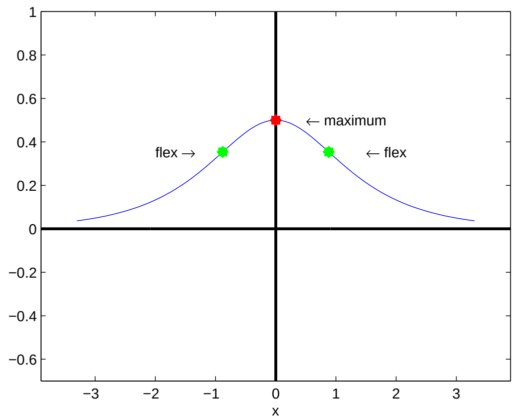

Figure S1.2 The function $h(x)$

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c) Compute the integral $I(t) = \int_0^t h(x)dx$. [8 marks]

$I(t) = \int_0^t h(x)dx = \int_0^t \frac{1}{e^x+e^{-x}}dx = \int_0^t \frac{e^x}{e^{2x}+1}dx$. Let $y = e^x$, then

$$I(t) = \int_1^{e^t} \frac{dy}{y^2+1}dx = [\arctan(y)]_1^{e^t} = \arctan(e^t) - \frac{\pi}{4}.$$

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d) Find the limit of $I(t)$ for $t \to +\infty$ and give a geometric interpretation of the value of the integral. [8 marks]

The value of $I(t)$ as $t \to +\infty$ is $\frac{\pi}{4}$. This value is the area under the curve $h(x)$ for $x > 0$. Note that the area is bounded even though the considered region is open.

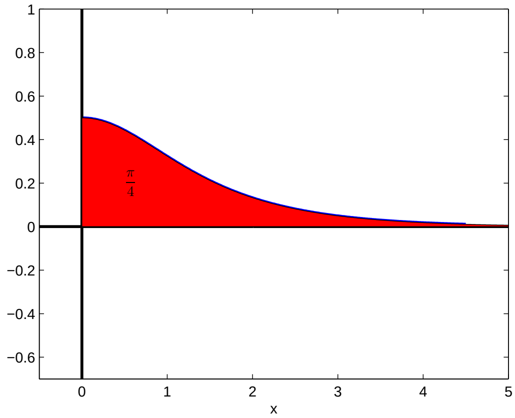

Figure S1.1 Area under the curve $h(x)$ for $x > 0$.

3. The Fourier series for a real-valued periodic function, $u(t)$, with period $T = \frac{1}{F}$ is given by

$$u(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi nFt + b_n \sin 2\pi nFt.$$

a) By using the identities

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$
$$\sin\theta = \frac{-j}{2}\left(e^{j\theta} - e^{-j\theta}\right)$$

where $j = \sqrt{-1}$, show that this may also be written as

$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{j2\pi nFt}$$

and derive expressions for the complex coefficients, U_n, in terms of a_n and b_n.
[7 marks]

By making the given substitution, we get

$$\begin{aligned}
u(t) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi nFt + b_n \sin 2\pi nFt \\
&= \frac{1}{2}a_0 + \frac{1}{2}\sum_{n=1}^{\infty} a_n \left(e^{j2\pi nFt} + e^{-j2\pi nFt}\right) - jb_n\left(e^{j2\pi nFt} - e^{-j2\pi nFt}\right) \\
&= \frac{1}{2}a_0 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n - jb_n)\, e^{j2\pi nFt} + (a_n + jb_n)\, e^{-j2\pi nFt} \\
&= \frac{1}{2}\sum_{n=-\infty}^{\infty} U_n e^{j2\pi nFt}
\end{aligned}$$

.

where

$$U_n = \begin{cases} a_n - jb_n & n > 0 \\ a_0 & n = 0 \\ a_{-n} + jb_{-n} & n < 0 \end{cases}.$$

Alternatively, if we define $b_0 = 0$ we can write this as a single expression:

$$U_n = a_{|n|} - jb_{|n|}\,\mathrm{sgn}(n)$$

where the sign function is

$$\mathrm{sgn}(n) = \begin{cases} +1 & n > 0 \\ 0 & n = 0 \\ -1 & n < 0 \end{cases}.$$

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b) Explain the relationship between the coefficients U_{-n} and U_{+n}. [3 marks]

If n is positive, then $U_n = a_n - jb_n$ whereas $U_{-n} = a_n + jb_n$. Since the coefficients a_n and b_n are real-valued, U_{-n} is the complex conjugate of U_n.

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c) Suppose $u(t)$ has period $T = 8$ and, over the interval $-4 \le t < 4$ is given by

$$u(t) = \begin{cases} 5 & \text{for } -1 \le t < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Determine the complex coefficients, U_n by evaluating the Fourier analysis integral

$$U_n = \frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} u(t)e^{-j2\pi nFt}dt.$$

[8 marks]

$$\begin{aligned} U_n &= \frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} u(t)e^{-j2\pi nFt}dt \\ &= \frac{1}{T}\int_{-1}^{1} 5e^{-j2\pi nFt}dt \\ &= \frac{5}{-j2\pi nFT}\left[e^{-j2\pi nFt}\right]_{-1}^{1} \\ &= \frac{5}{-j2\pi n}\left(e^{-j2\pi nF} - e^{j2\pi nF}\right) \\ &= \frac{5}{-j2\pi n}\left(-2j\sin 2\pi nF\right) \\ &= \frac{5\sin 2\pi nF}{\pi n} \end{aligned}$$

Note that in the fourth line we use the relationship $FT = 1$. In the fifth line we use $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$ which can easily be deduced from $e^{j\theta} = \cos\theta + j\sin\theta$ but is also worth remembering in its own right. Note too that $\int e^{\alpha t}dt = \frac{1}{\alpha}e^{\alpha t}$ even if α is complex.

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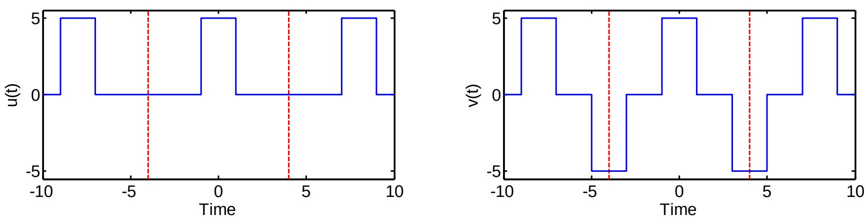
d) The function $v(t)$ is said to be "antiperiodic" if $v(t+\frac{1}{2}T) = -v(t)$.

Suppose that the antiperiodic function $v(t)$ has period $T = 8$ and satisfies $v(t) = u(t)$ over the range $-2 \le t < 2$.

Sketch dimensioned graphs of both $u(t)$ and $v(t)$ over the range $-10 \le t \le 10$.

[5 marks]

Graphs of $u(t)$ and $v(t)$ are shown below:



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i) Show that, by dividing the integration range into two halves, the Fourier analysis integral may be expressed as

$$V_n = \frac{1}{T}\int_{-\frac{1}{2}T}^{0} v(t)e^{-j2\pi nFt}dt + \frac{1}{T}\int_{-\frac{1}{2}T}^{0} v(t+\frac{1}{2}T)e^{-j2\pi nF(t+\frac{1}{2}T)}dt.$$

[5 marks]

We can split up the integral into two halves and then make the substitution $t = \tau + \frac{1}{2}T$ in the second one:

$$\begin{aligned}
V_n &= \frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} v(t)e^{-j2\pi nFt}dt \\
&= \frac{1}{T}\int_{-\frac{1}{2}T}^{0} v(t)e^{-j2\pi nFt}dt + \frac{1}{T}\int_{0}^{\frac{1}{2}T} v(t)e^{-j2\pi nFt}dt \\
&= \frac{1}{T}\int_{-\frac{1}{2}T}^{0} v(t)e^{-j2\pi nFt}dt + \frac{1}{T}\int_{-\frac{1}{2}T}^{0} v(\tau+\frac{1}{2}T)e^{-j2\pi nF(\tau+\frac{1}{2}T)}d\tau \\
&= \frac{1}{T}\int_{-\frac{1}{2}T}^{0} v(t)e^{-j2\pi nFt}dt + \frac{1}{T}\int_{-\frac{1}{2}T}^{0} v(t+\frac{1}{2}T)e^{-j2\pi nF(t+\frac{1}{2}T)}dt
\end{aligned}$$

where in the last line we have made the substitution $\tau = t$.

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ii) Show that $e^{-j2\pi nF\frac{1}{2}T} = 1$ if n is an even integer. [2 marks]

$$\begin{aligned}
e^{-j2\pi nF\frac{1}{2}T} &= e^{-j2\pi n\frac{1}{2}} \\
&= e^{-j\pi n} \\
&= \cos n\pi - j\sin n\pi.
\end{aligned}$$

In the first line we made use of the fact that $FT = 1$. If n is an even integer, then $n\pi$ is a multiple of 2π and so $\cos n\pi = 1$ and $\sin n\pi = 0$. Hence $e^{-j2\pi nF\frac{1}{2}T} = 1$.

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iii) Hence show that, if $v(t)$ is antiperiodic, its complex Fourier coefficients, V_n, are zero for even values of n. [3 marks]

From part d)i), we have

$$
\begin{aligned}
V_n &= \frac{1}{T}\int_{-\frac{1}{2}T}^{0} v(t)e^{-j2\pi nFt}dt + \frac{1}{T}\int_{-\frac{1}{2}T}^{0} v(t+\frac{1}{2}T)e^{-j2\pi nF(t+\frac{1}{2}T)}dt \\
&= \frac{1}{T}\int_{-\frac{1}{2}T}^{0} v(t)e^{-j2\pi nFt}dt - \frac{1}{T}\int_{-\frac{1}{2}T}^{0} v(t)e^{-j2\pi nF\tau}e^{-j2\pi nF\frac{1}{2}T}dt \\
&= \frac{1}{T}\int_{-\frac{1}{2}T}^{0} v(t)e^{-j2\pi nFt}dt - \frac{1}{T}\int_{-\frac{1}{2}T}^{0} v(t)e^{-j2\pi nFt}dt \\
&= 0
\end{aligned}
$$

where in the second and third lines respectively we have used the relationships $v(t+\frac{1}{2}T) = -v(t)$ and $e^{-j2\pi nF\frac{1}{2}T} = 1$.
