Fourier Series and Transforms Revision Lecture

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The Basic Idea

Periodic signals can be written as a sum of sine and cosine waves:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

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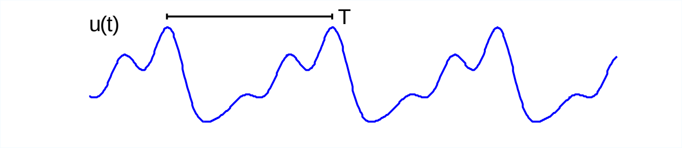

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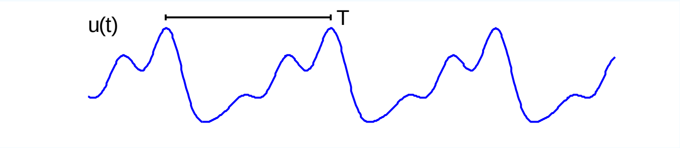

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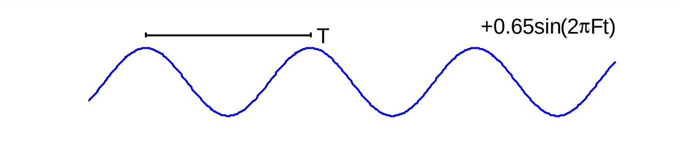

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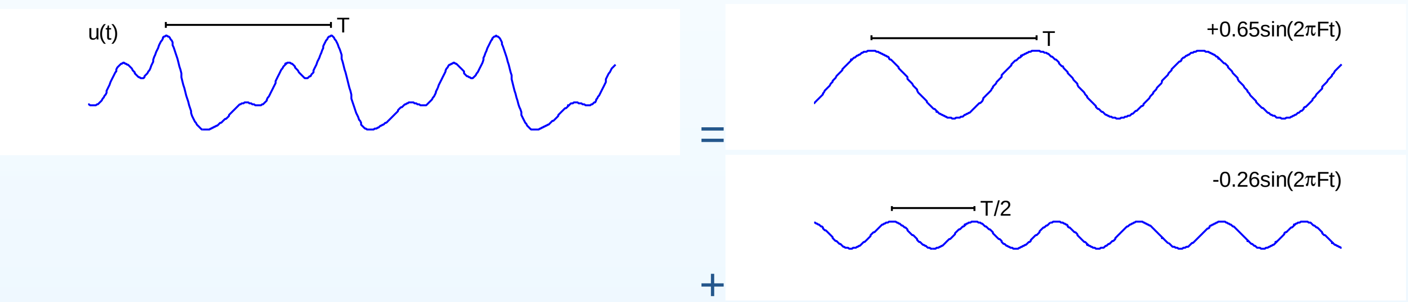

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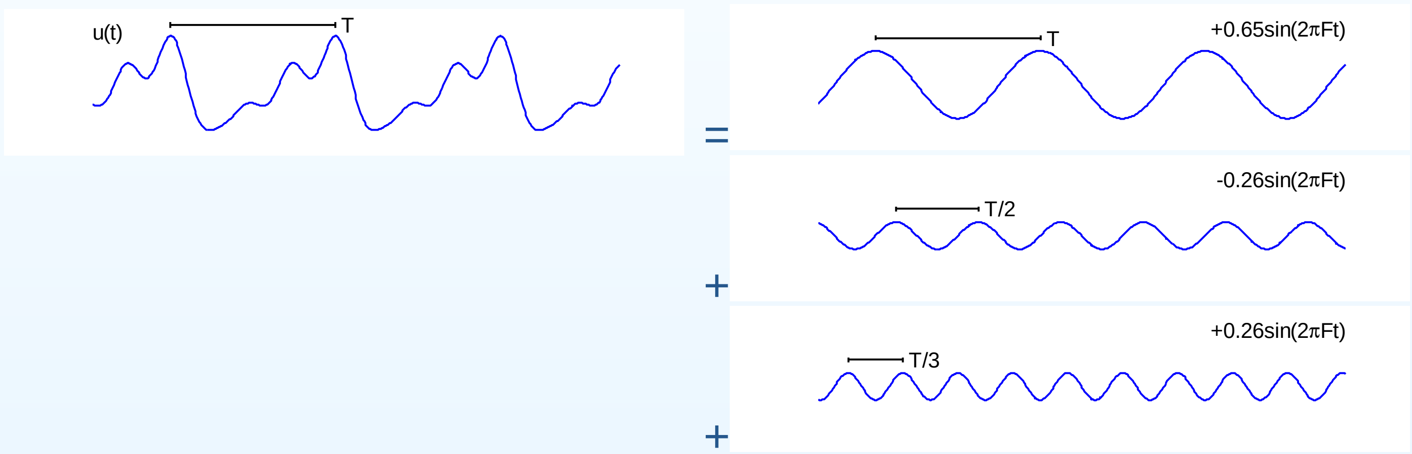

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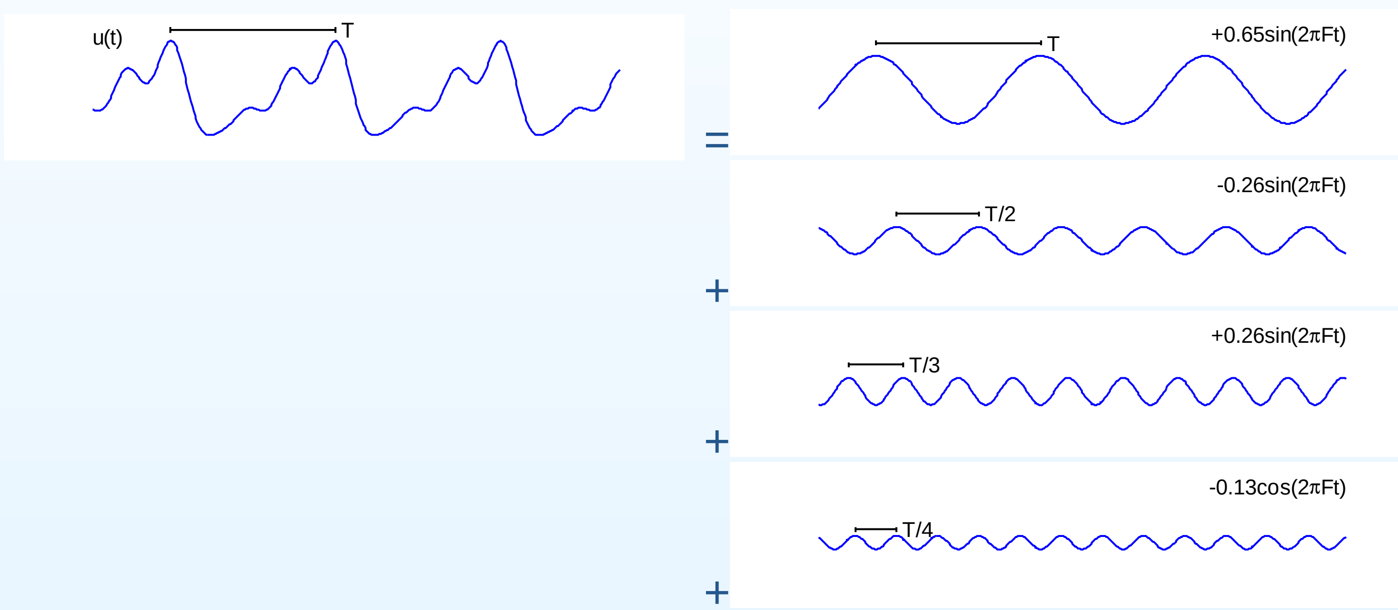

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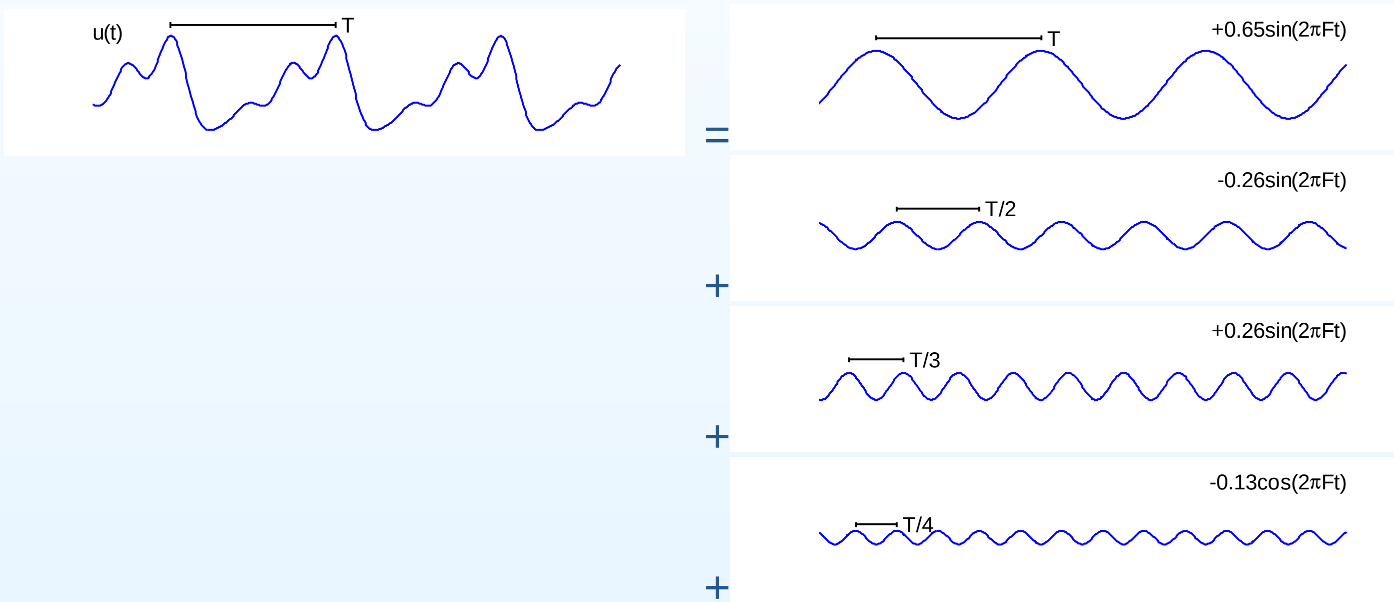

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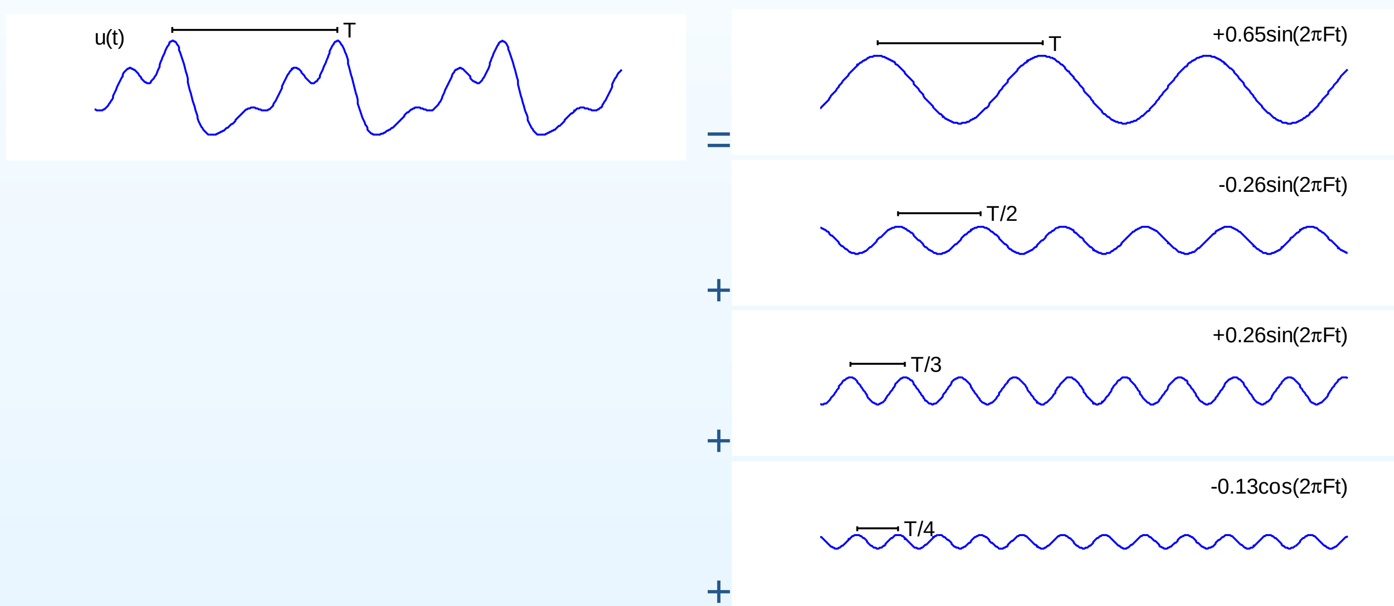

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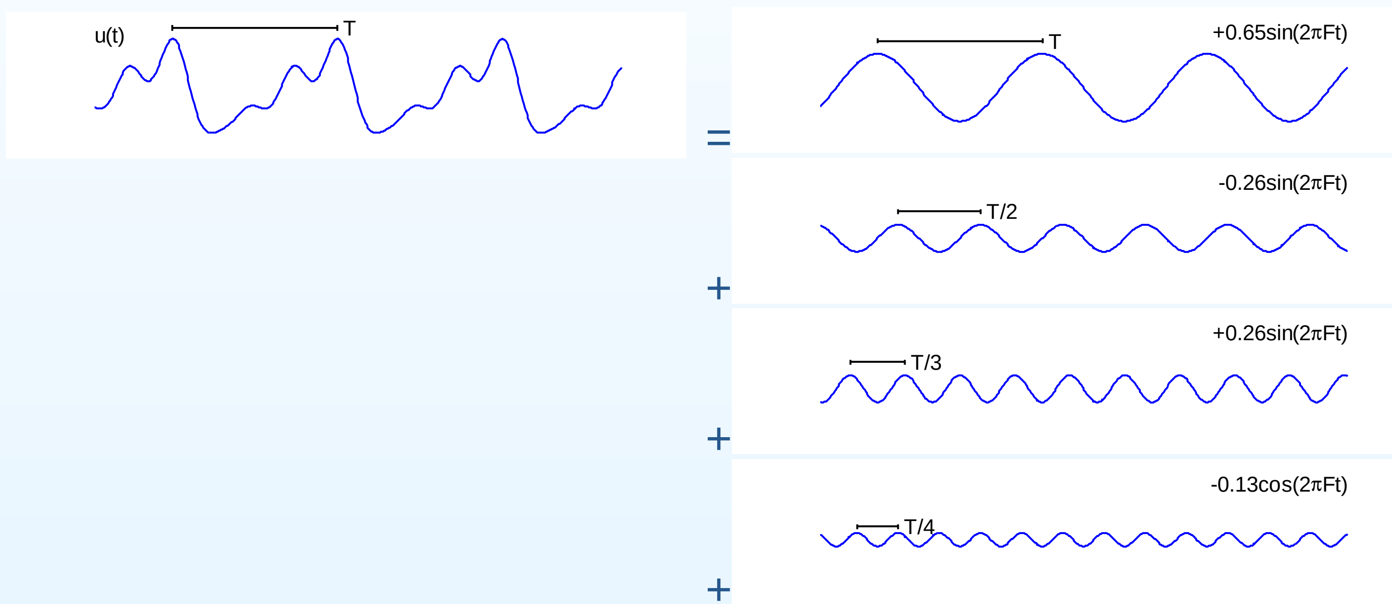

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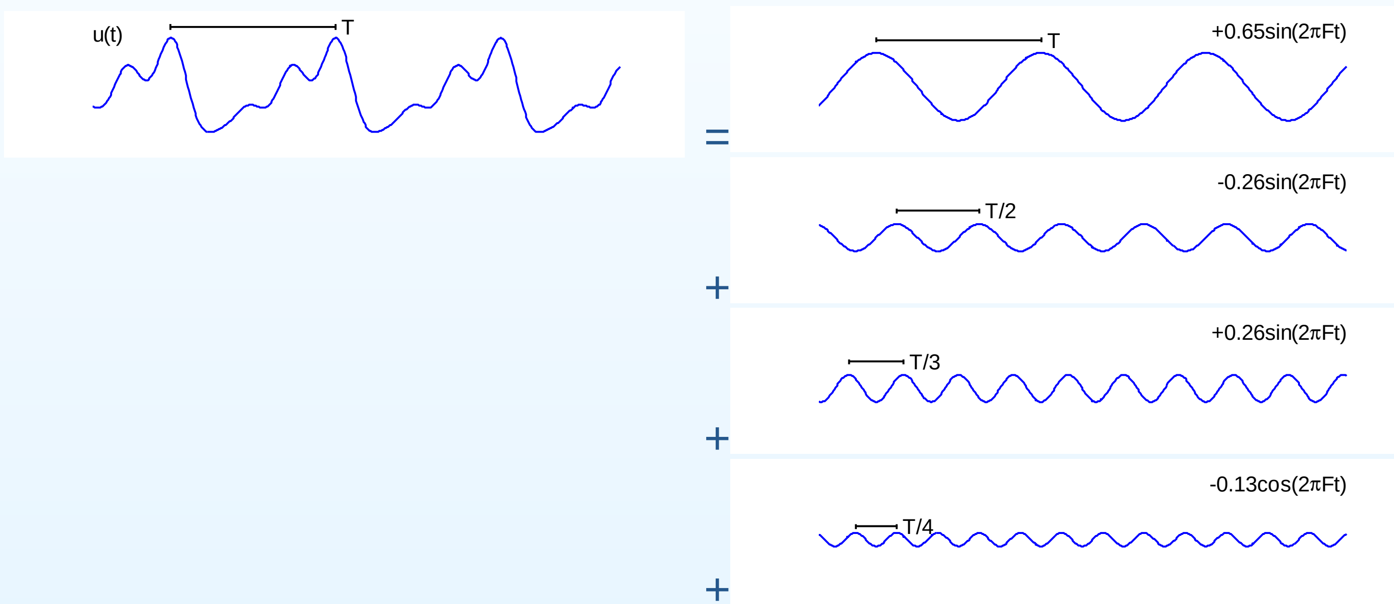

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Some waveforms need infinitely many harmonics (countable infinity).

Real versus Complex Fourier Series

All the algebra is much easier if we use $e^{i\omega t}$ instead of $\cos\omega t$ and $\sin\omega t$

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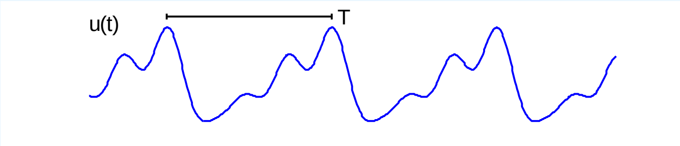

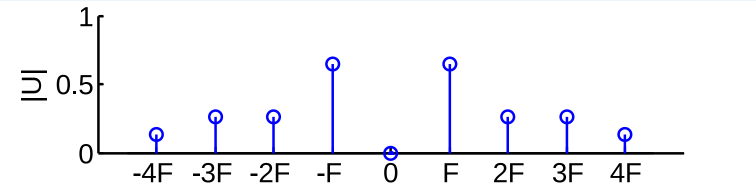

Plot the magnitude spectrum

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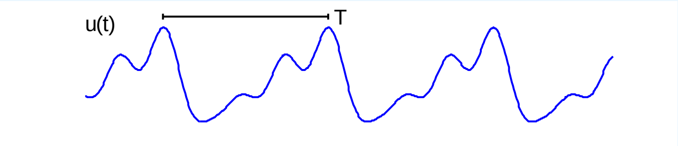
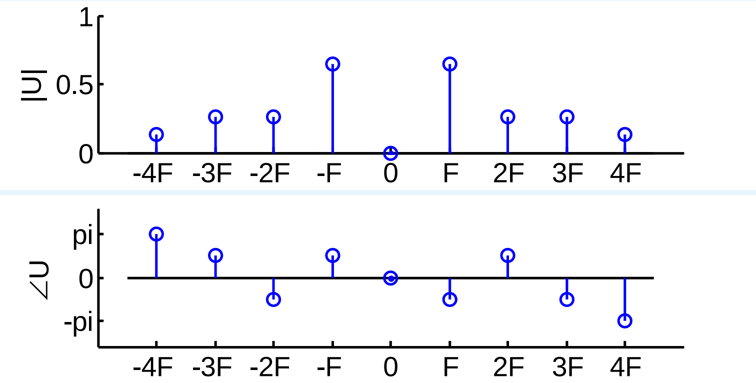
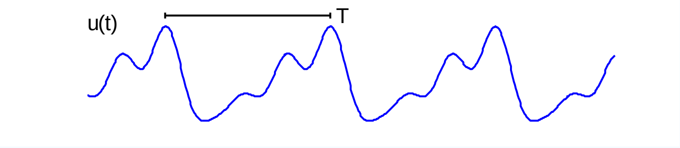


Plot the magnitude spectrum and phase spectrum:



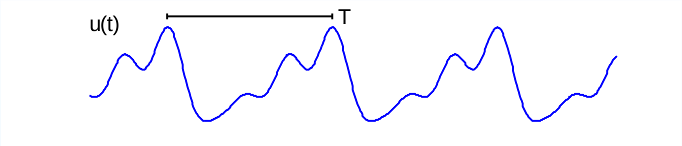
Fourier Series versus Fourier Transform

- Periodic signals



Fourier Series versus Fourier Transform

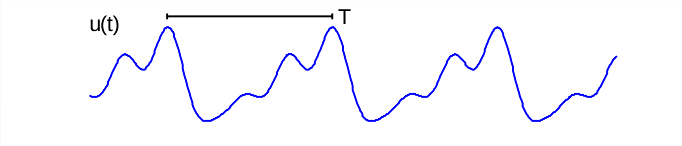
- Periodic signals $\rightarrow$ Fourier Series



$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$

Fourier Series versus Fourier Transform

- Periodic signals $\rightarrow$ Fourier Series$\rightarrow$ Discrete spectrum



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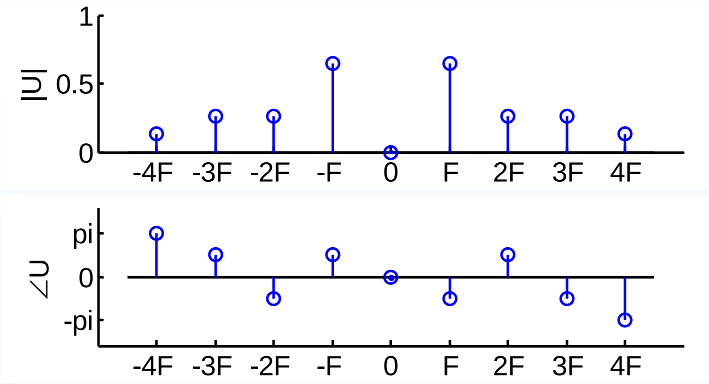

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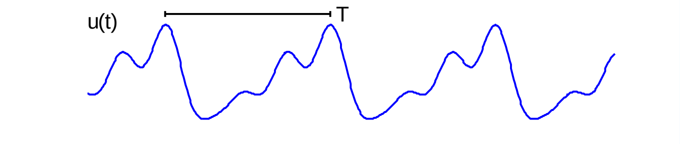

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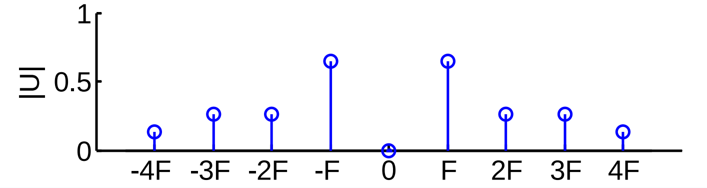
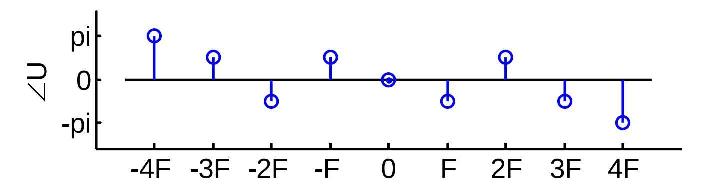
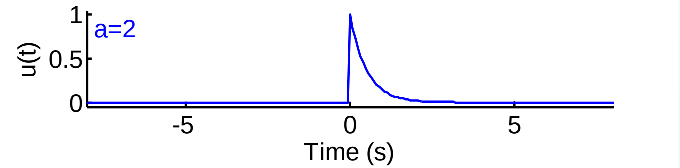




- Aperiodic signals



Fourier Series versus Fourier Transform

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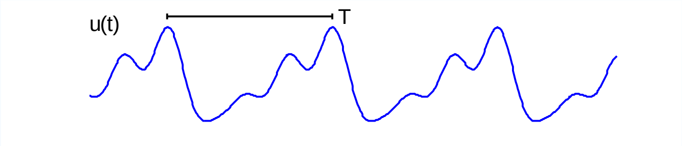

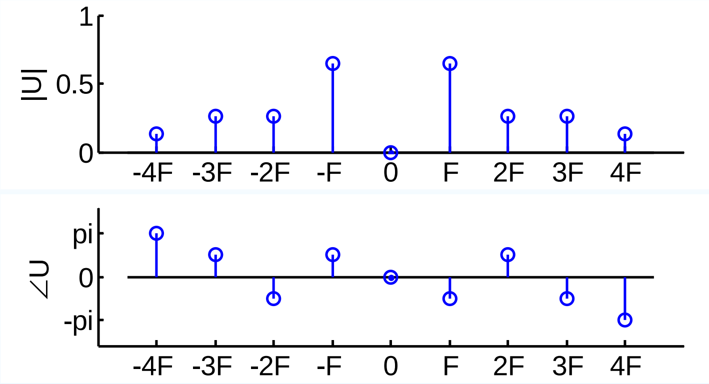

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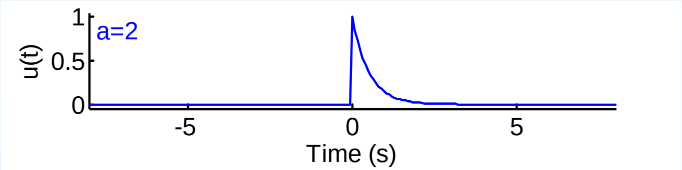
- Aperiodic signals $\rightarrow$ Fourier Transform



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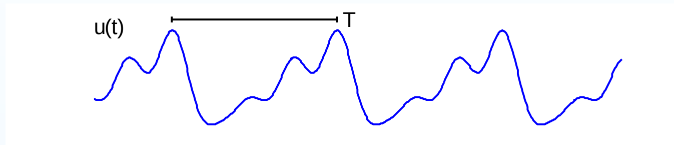
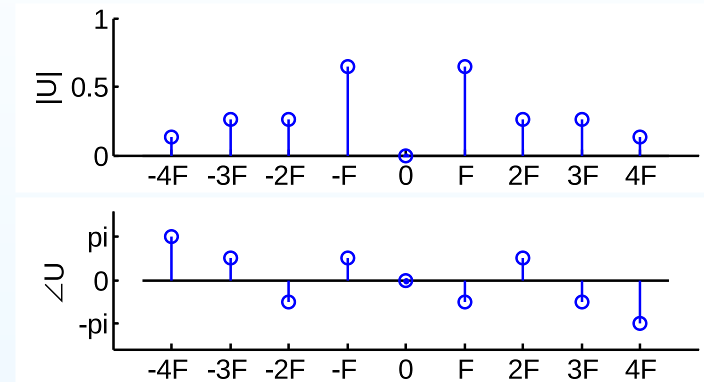
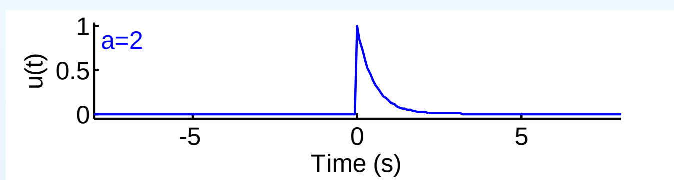
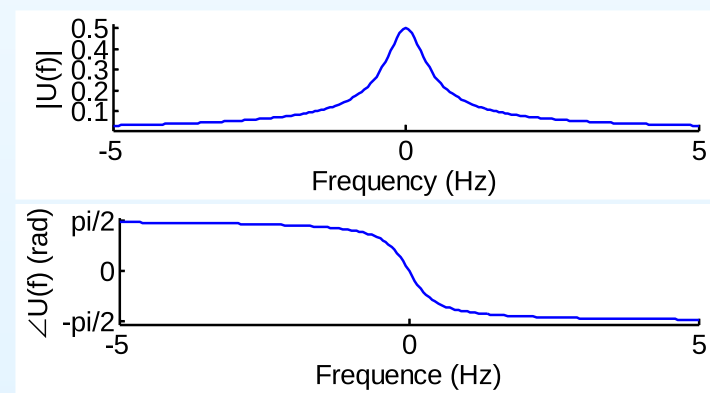




$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$

- Aperiodic signals $\rightarrow$ Fourier Transform$\rightarrow$ Continuous Spectrum





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Fourier Series versus Fourier Transform

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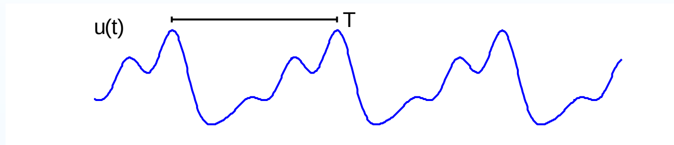

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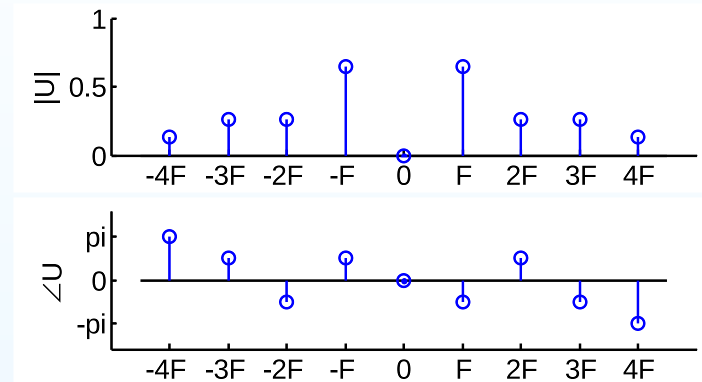

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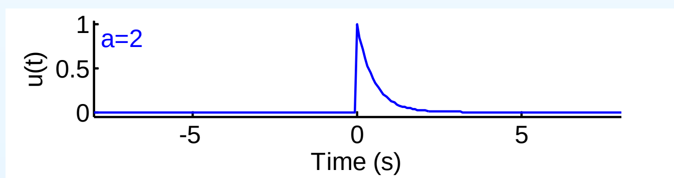

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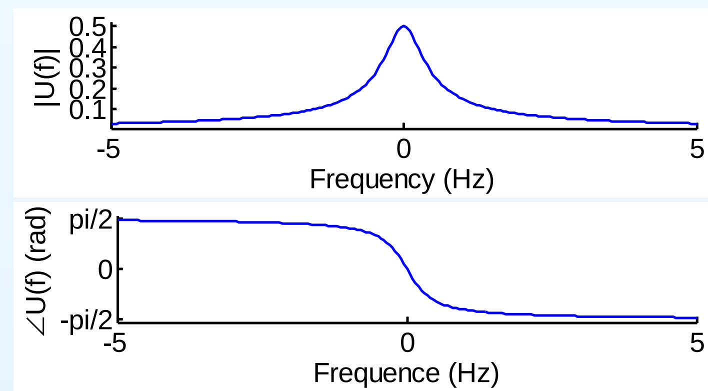

- Both types of spectrum are conjugate symmetric.
- If $u(t)$ is periodic, its Fourier transform consists of Dirac δ functions with amplitudes $\{U_n\}$.

Fourier Analysis

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Fourier Analysis = "how do you work out the Fourier coefficients, U_n ?"

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Key idea: $\langle e^{i\omega t}\rangle = \langle \cos\omega t + i \sin\omega t\rangle = \begin{cases} 1 & \text{if } \omega = 0 \\ 0 & \text{otherwise} \end{cases}$

Fourier Analysis

Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

Fourier Analysis = "how do you work out the Fourier coefficients, U_n ?"

Key idea: $\langle e^{i\omega t} \rangle = \langle \cos \omega t + i \sin \omega t \rangle = \begin{cases} 1 & \text{if } \omega = 0 \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow$ Orthogonality: $\langle e^{i2\pi nFt} \times e^{-i2\pi mFt} \rangle = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$

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Notice the negative sign in Fourier analysis: in order to extract the term in the series containing $e^{+i2\pi mFt}$ we need to multiply by $e^{-i2\pi mFt}$.

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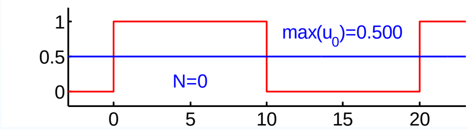

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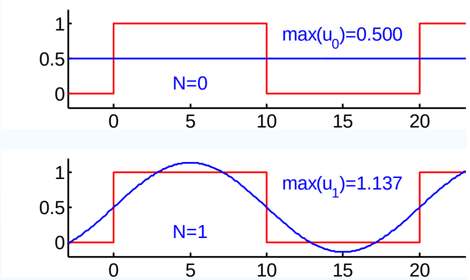

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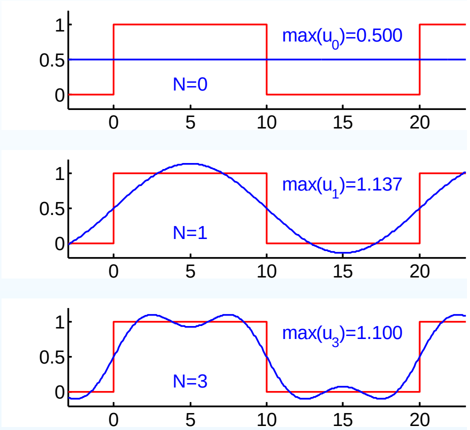

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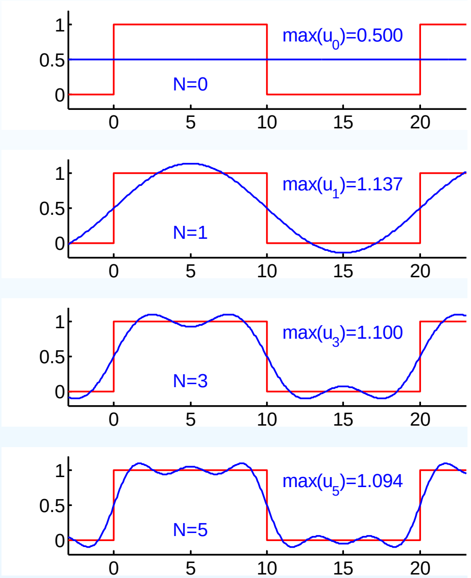

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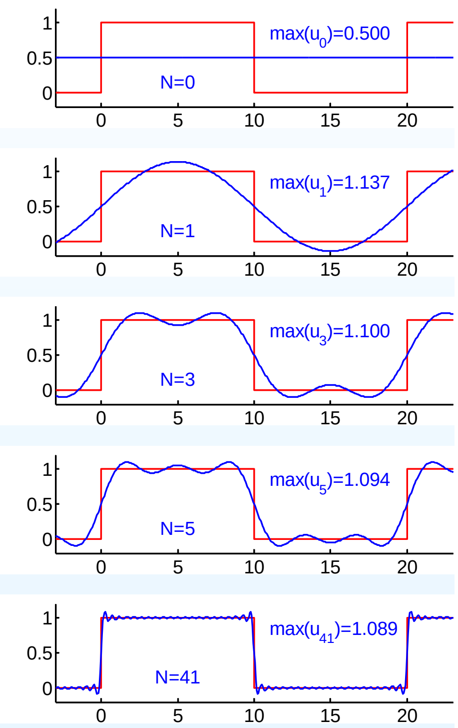

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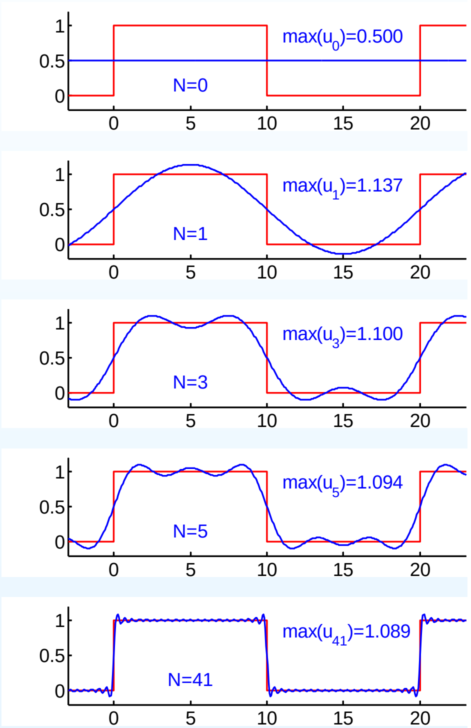

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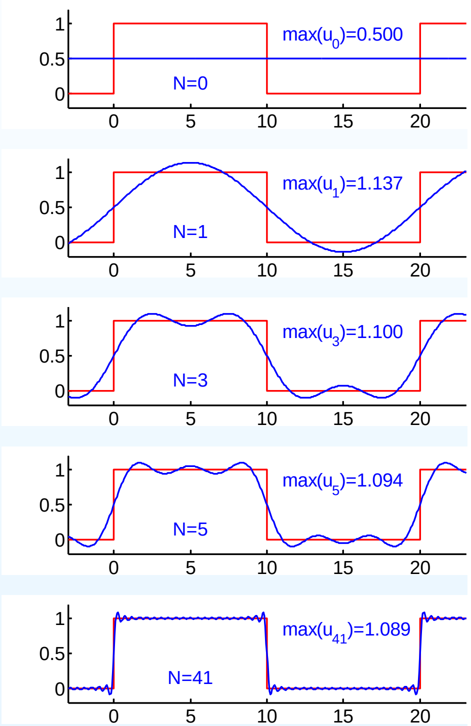

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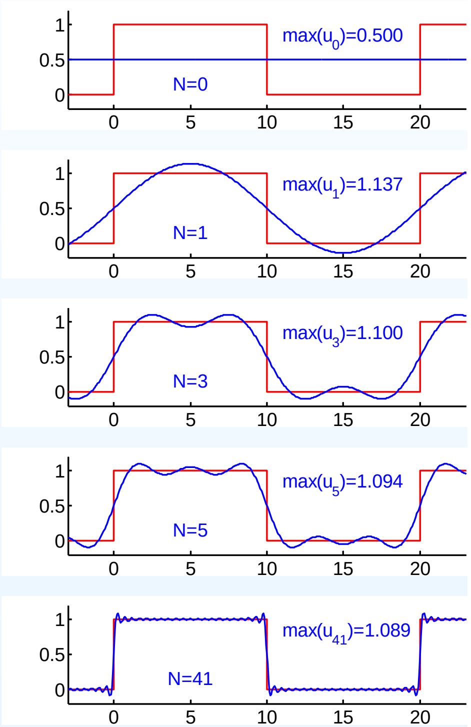

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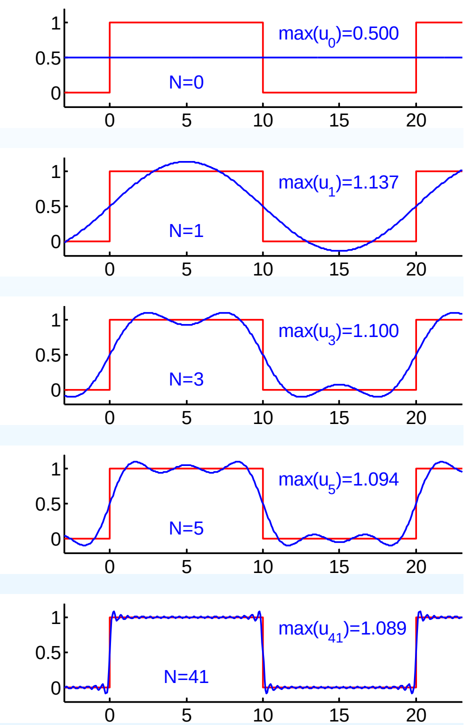

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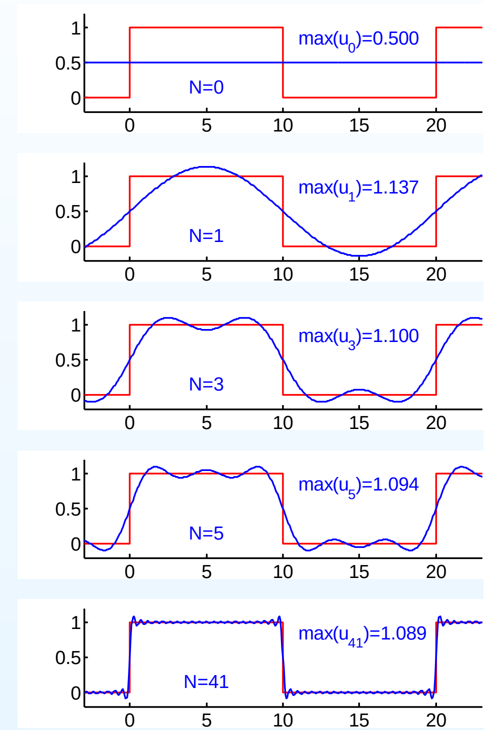

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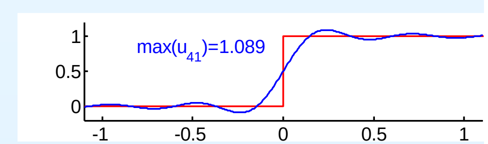

[Enlarged View: $u_{41}(t)$]

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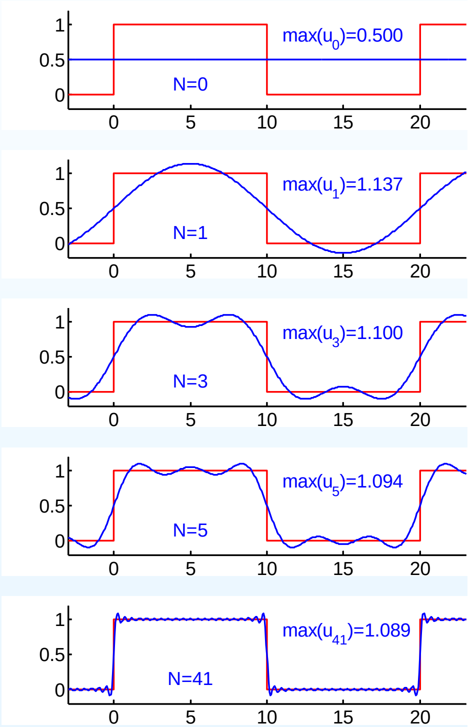

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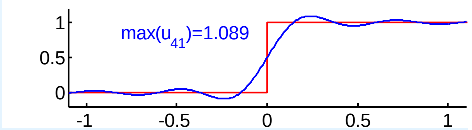

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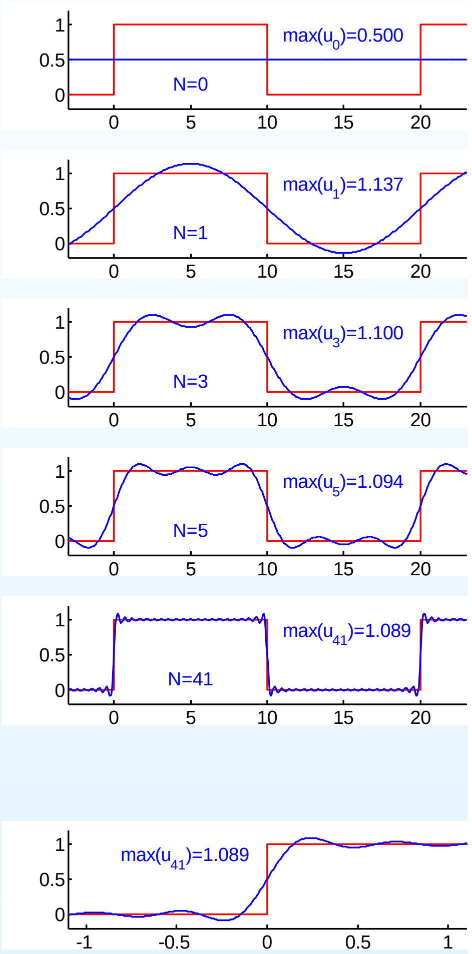
- For large N, the overshoots move closer to the discontinuity but do not decrease in size.



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$\frac{d^k u(t)}{dt^k}$ is the lowest derivative with a discontinuity

$\Rightarrow$ $|U_n|$ is $O\left(\frac{1}{n^{k+1}}\right)$ for large $|n|$

Coefficient Decay Rate

Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

Integration:

$$v(t) = \int_0^t u(\tau)d\tau \quad \Rightarrow \quad V_n = \frac{1}{i2\pi nF} U_n$$

provided $U_0 = V_0 = 0$.

Differentiation:

$$w(t) = \frac{du(t)}{dt} \qquad \Rightarrow \quad W_n = i2\pi nF \times U_n$$

provided $w(t)$ satisfies the Dirichlet conditions.

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$\quad \Rightarrow \quad |U_n|$ is $O\left(\frac{1}{n^{k+1}}\right)$ for large $|n|$

If the coefficients, U_n, decrease rapidly then only a few terms are needed for a good approximation.

Periodic Extension

If $u(t)$ is only defined over a finite range, $[0,\ B]$, we can make it periodic by defining $u(t \pm B) = u(t)$.

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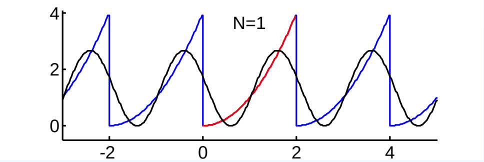

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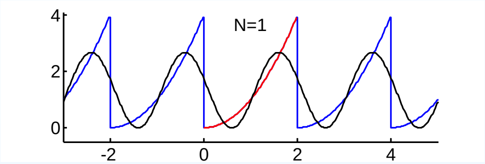

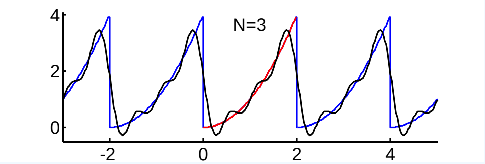

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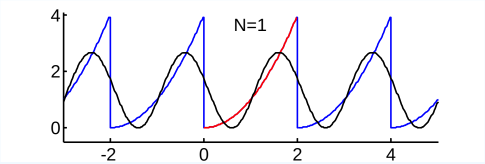

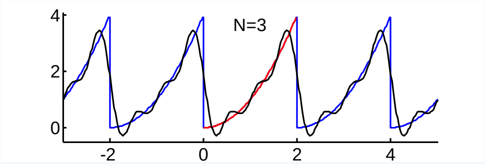

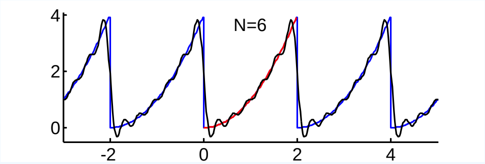

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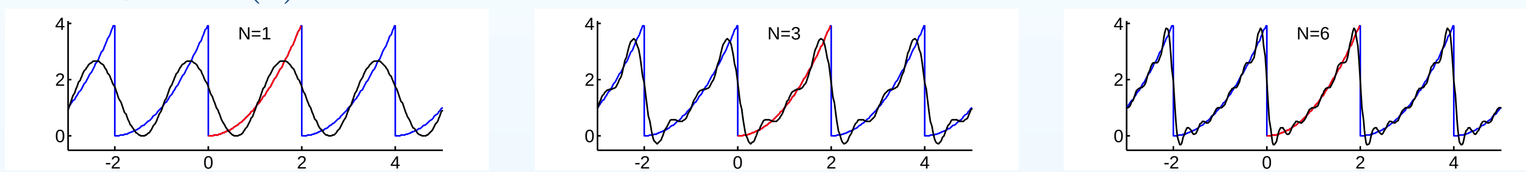

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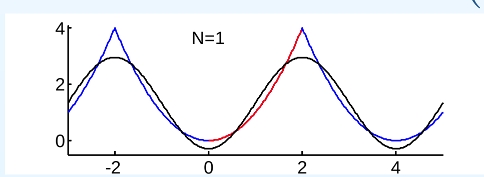

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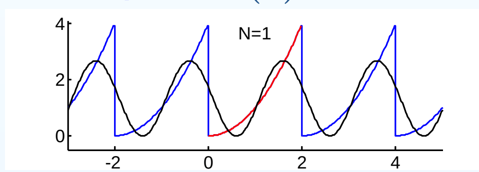

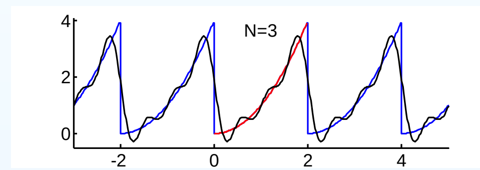

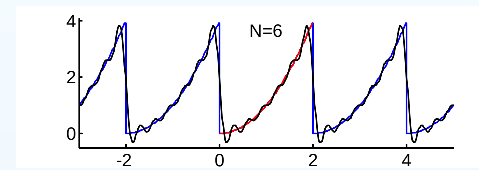

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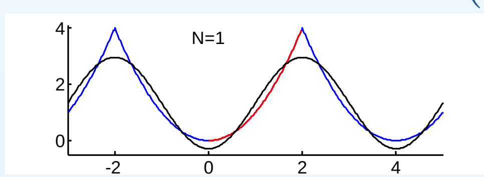

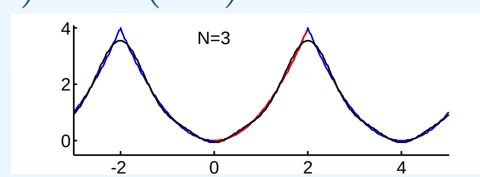

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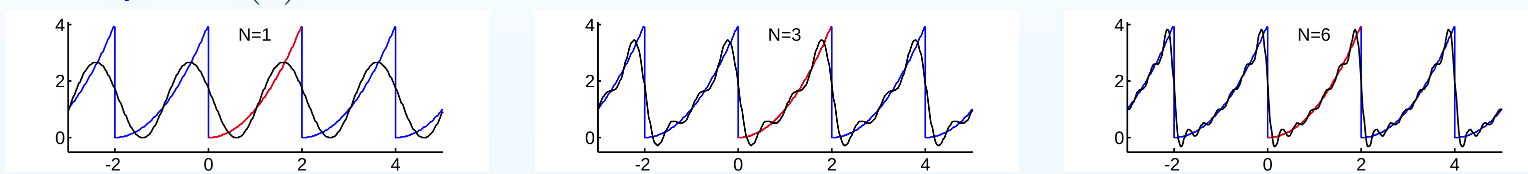

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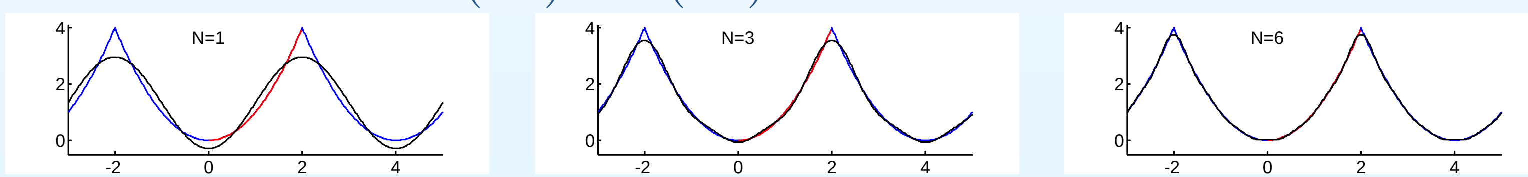

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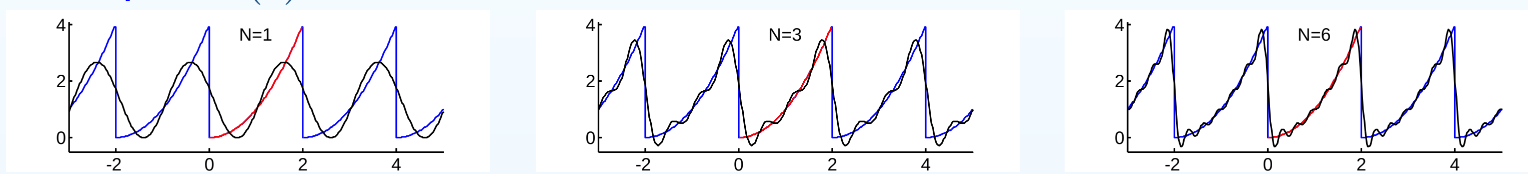

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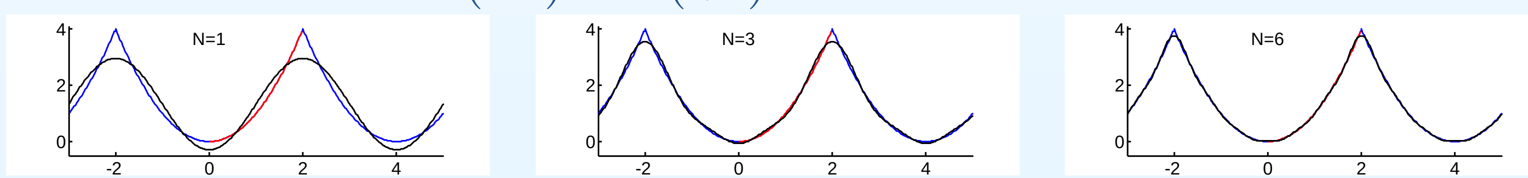

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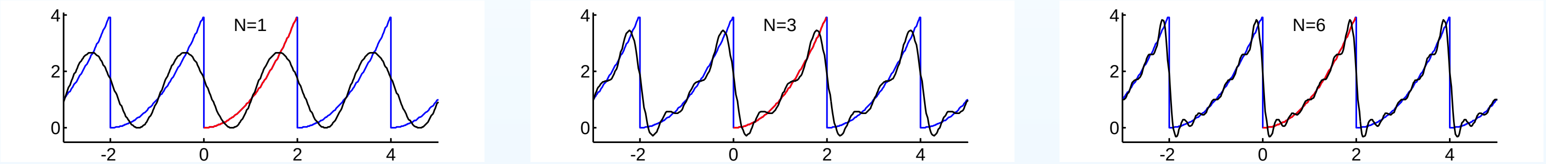

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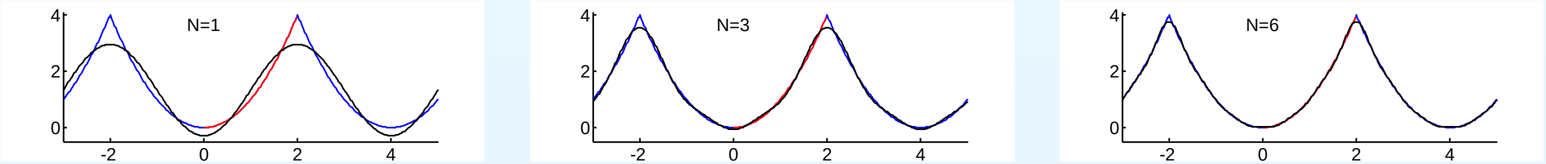
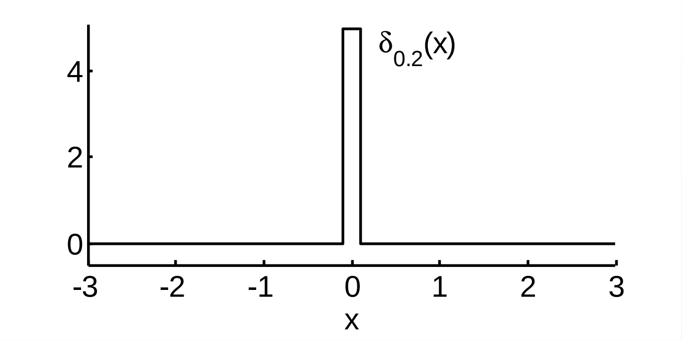


- Symmetry around $t = 0$ means coefficients are real-valued and symmetric ($U_{-n} = U_n^* = U_n$).
- Still have a first-derivative discontinuity at $t = B$ but now we have no Gibbs phenomenon and coefficients $\propto n^{-2}$ instead of $\propto n^{-1}$ so approximation error power decreases more quickly.

Dirac Delta Function

$\delta(x)$ is the limiting case as $w \to 0$ of a pulse w wide and $\frac{1}{w}$ high



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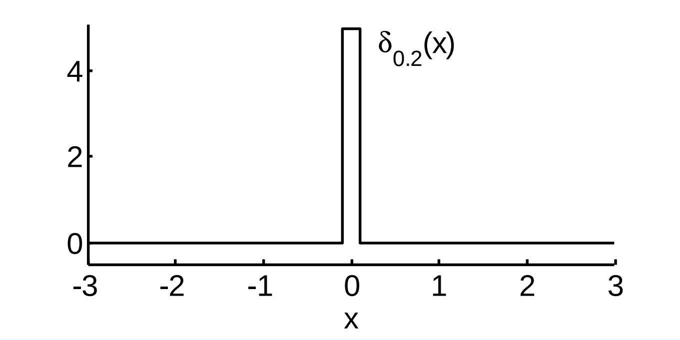

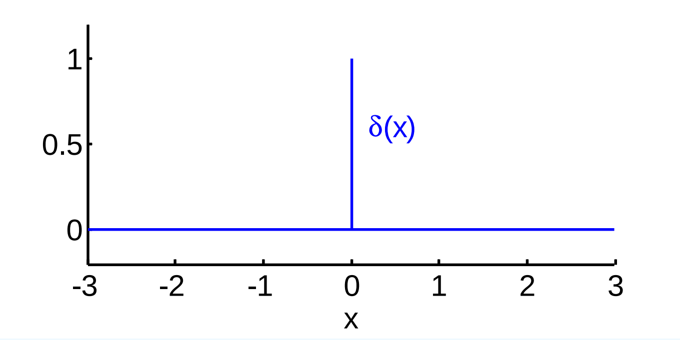

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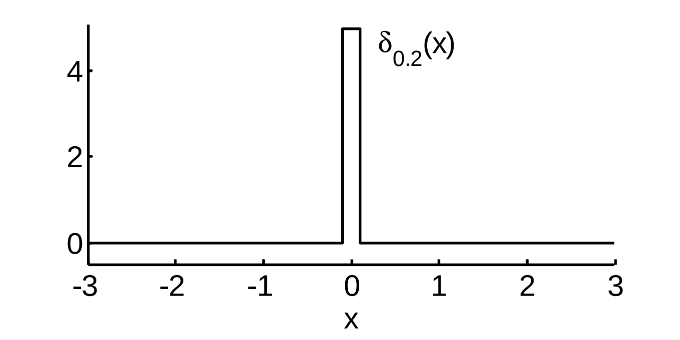

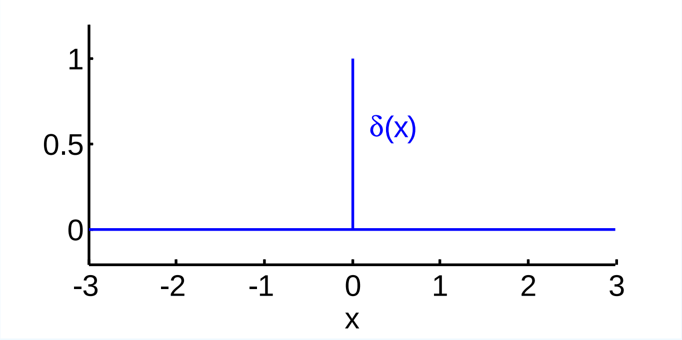

- Area: $\int_{-\infty}^{\infty} \delta(x) dx = 1$

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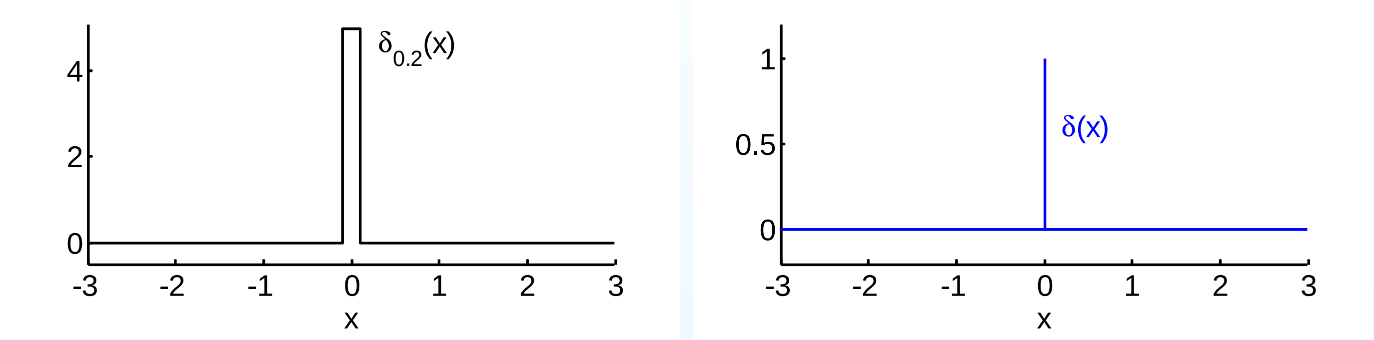

- **Area:** $\int_{-\infty}^{\infty} \delta(x)dx = 1$
- **Scaling:** $\delta(cx) = \frac{1}{|c|}\delta(x)$

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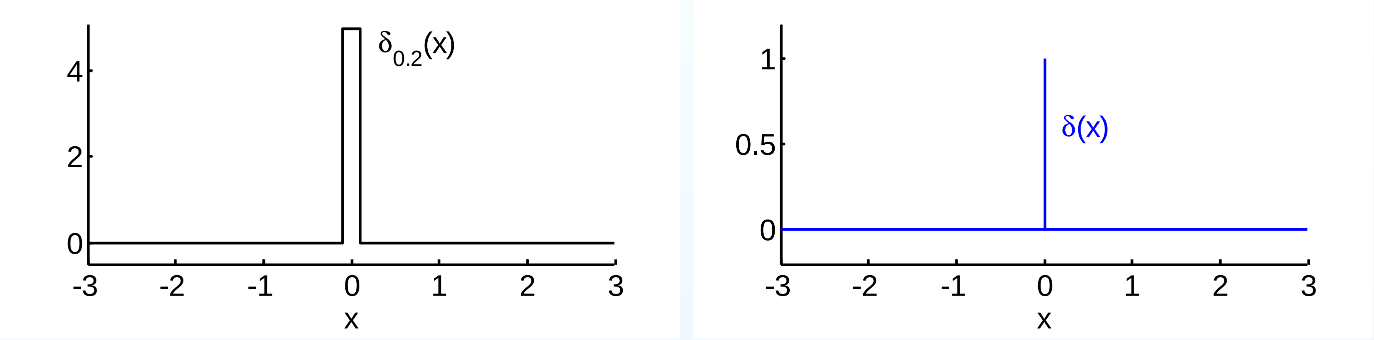

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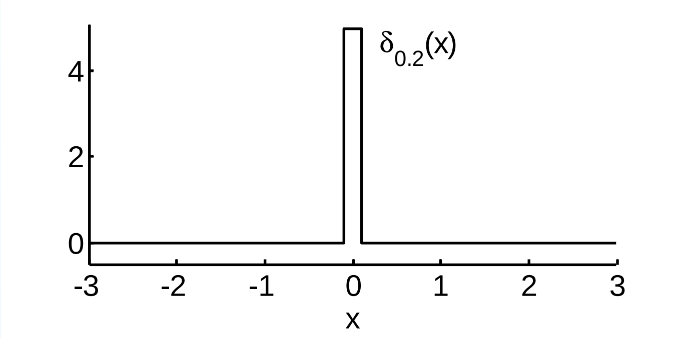

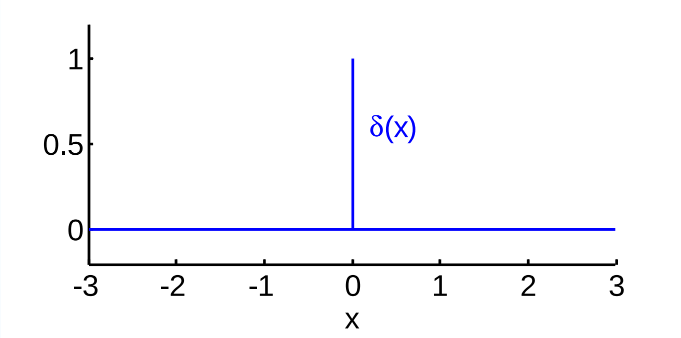

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- Multiplication: $f(x) \times \delta(x-a) = f(a) \times \delta(x-a)$

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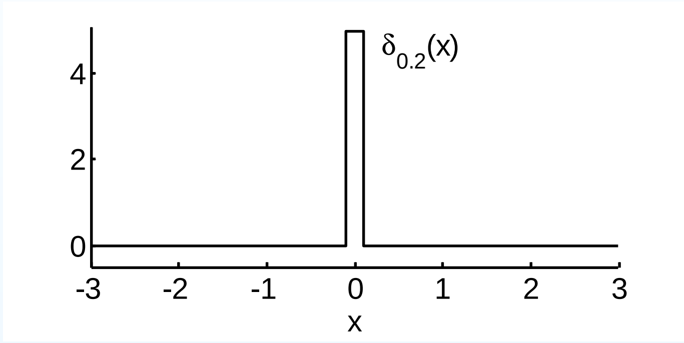

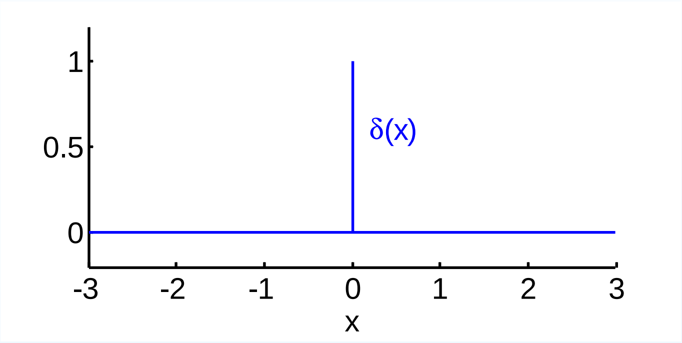

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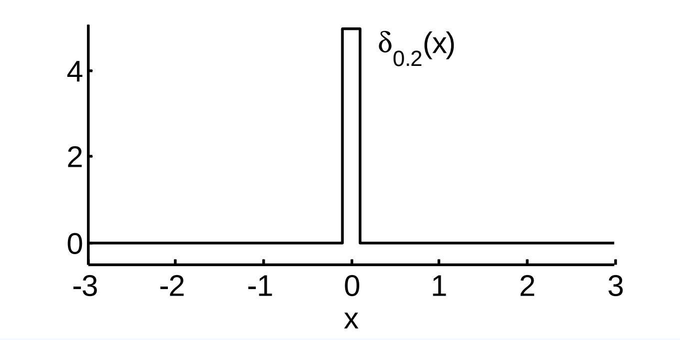

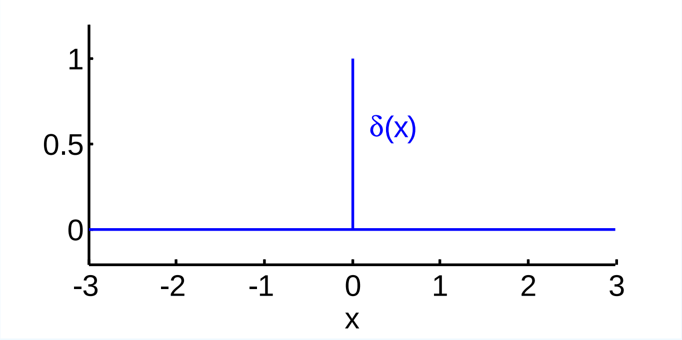

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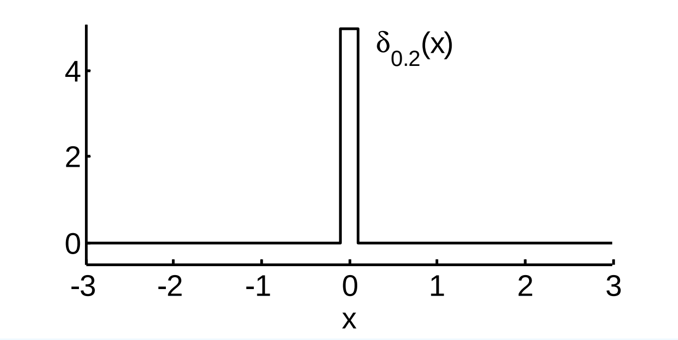

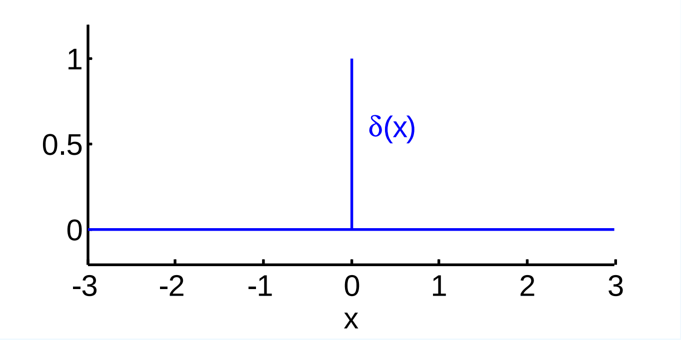

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- We plot $h\delta(x)$ as a pulse of height $|h|$ (instead of its true height of ∞)

Fourier Transform

Fourier Transform: $u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df$

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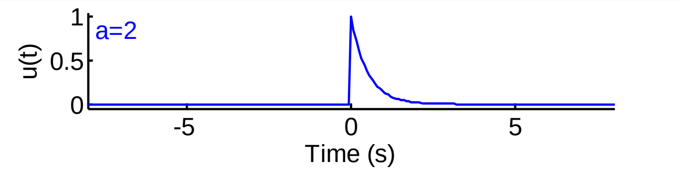
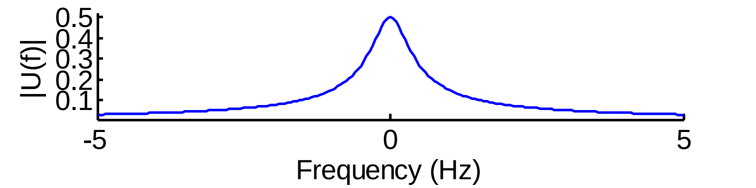
  - Complex-valued spectrum, $U(f)$, decays to zero as $f \to \pm\infty$





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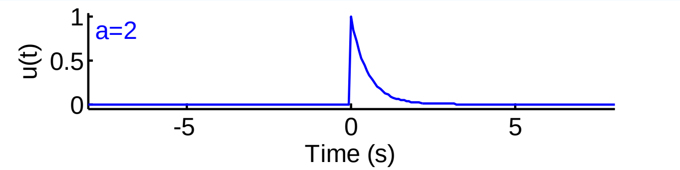

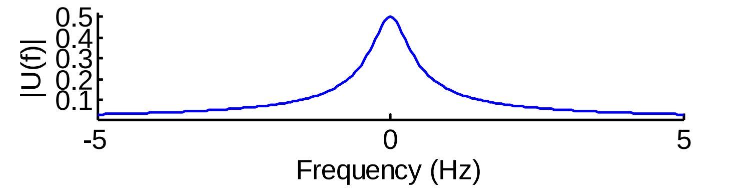

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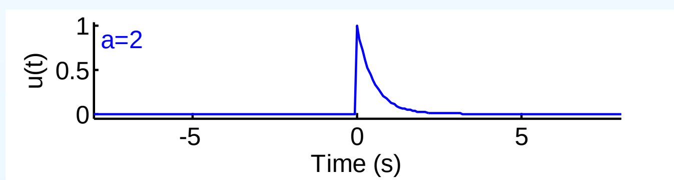
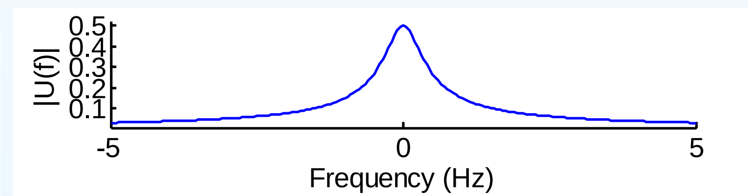
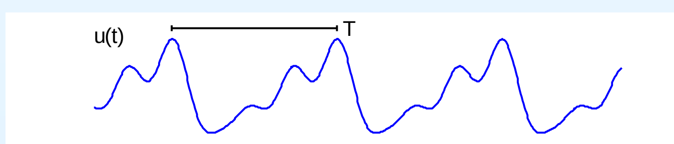
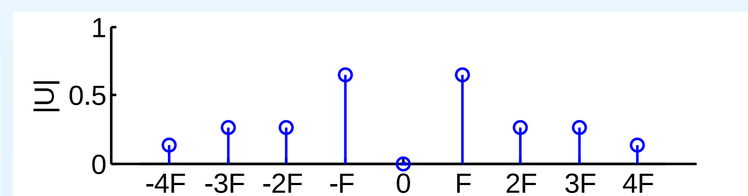




- Periodic Signals $\to$ Dirac δ functions at harmonics.
  Same complex-valued amplitudes as U_n from Fourier Series





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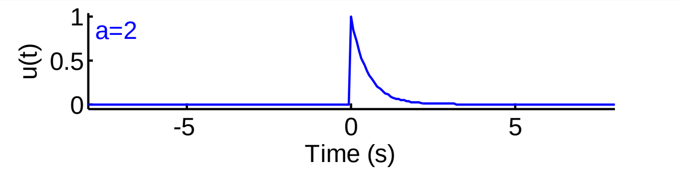
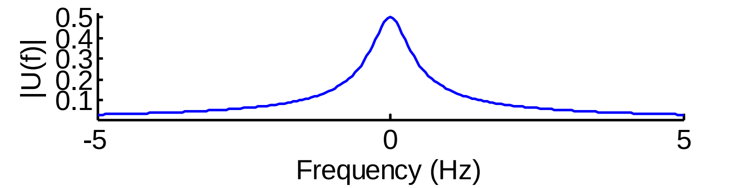
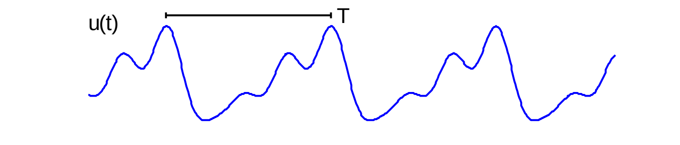
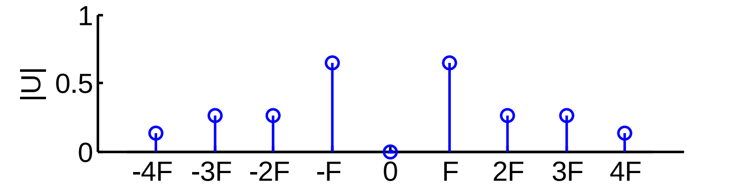




- Periodic Signals $\to$ Dirac δ functions at harmonics.
  Same complex-valued amplitudes as U_n from Fourier Series





  - $E_u = \infty$ but ave power is $P_u = \left\langle |u(t)|^2 \right\rangle = \sum_{n=-\infty}^{\infty} |U_n|^2$

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is equivalent to convolution in the other domain:

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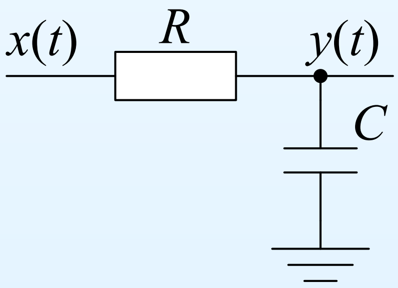

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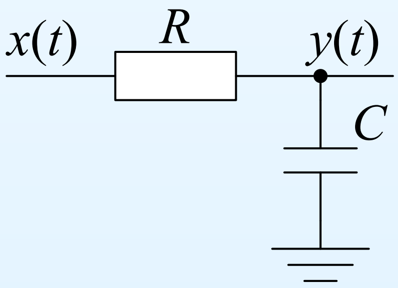
$$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}} \text{ for } t \geq 0$$



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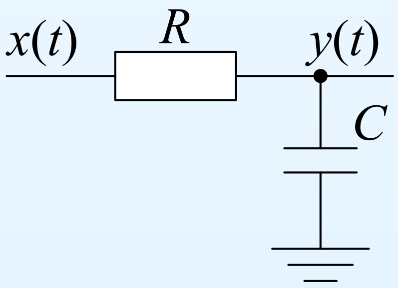

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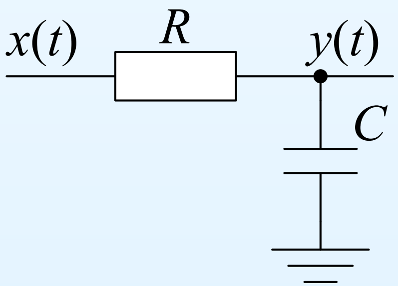

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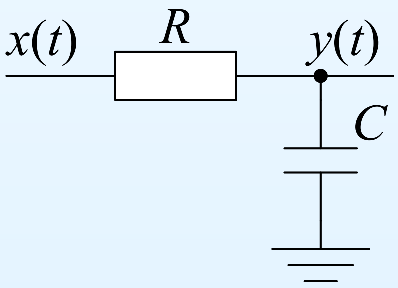

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- Auto-correlation is the cross-correlation of a signal with itself: used to find the period of a signal (i.e. the time shift where it matches itself).