Fourier Series and Transforms
Revision Lecture

- The Basic Idea
- Real vs Complex
- Series vs Transform
- Fourier Analysis
- Power Conservation
- Gibbs Phenomenon
- Coefficient Decay Rate
- Periodic Extension
- Dirac Delta Function
- Fourier Transform
- Convolution
- Correlation
Periodic signals can be written as a sum of sine and cosine waves:

\[ u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos 2\pi nFt + b_n \sin 2\pi nFt \right) \]

Fundamental Period: the smallest \( T > 0 \) for which \( u(t + T) = u(t) \).

Fundamental Frequency: \( F = \frac{1}{T} \). The \( n^{th} \) harmonic is at frequency \( nF \).

Some waveforms need infinitely many harmonics (countable infinity).
All the algebra is much easier if we use $e^{i\omega t}$ instead of $\cos \omega t$ and $\sin \omega t$

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi n Ft + b_n \sin 2\pi n Ft)$$

Substitute: $\cos \omega t = \frac{1}{2} e^{i\omega t} + \frac{1}{2} e^{-i\omega t}$ \hspace{1cm} $\sin \omega t = -\frac{i}{2} e^{i\omega t} + \frac{i}{2} e^{-i\omega t}$

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( \frac{1}{2} (a_n - ib_n) e^{i2\pi n Ft} + \frac{1}{2} (a_n + ib_n) e^{-i2\pi n Ft} \right)$$

$$= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n Ft}$$

- $U_{+n} = \frac{1}{2} (a_n - ib_n)$ and $U_{-n} = \frac{1}{2} (a_n + ib_n)$.
- $U_{+n}$ and $U_{-n}$ are complex conjugates.
- $U_{+n}$ is half the equivalent phasor in Analysis of Circuits.

Plot the magnitude spectrum and phase spectrum:
Fourier Series versus Fourier Transform

- **Periodic signals** → Fourier Series → Discrete spectrum

  \[ u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t} \]

- **Aperiodic signals** → Fourier Transform → Continuous Spectrum

  \[ u(t) = \int_{f=-\infty}^{\infty} U(f) e^{i2\pi f t} df \]

- Both types of spectrum are conjugate symmetric.
- If \( u(t) \) is periodic, its Fourier transform consists of Dirac \( \delta \) functions with amplitudes \( \{U_n\} \).
Fourier Series: \[ u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n Ft} \]

Fourier Analysis = “how do you work out the Fourier coefficients, \( U_n \)?”

Key idea: \[ \langle e^{i\omega t} \rangle = \langle \cos \omega t + i \sin \omega t \rangle = \begin{cases} 1 & \text{if } \omega = 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ \Rightarrow \text{Orthogonality: } \langle e^{i2\pi n Ft} \times e^{-i2\pi m Ft} \rangle = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases} \]

So, to find a particular coefficient, \( U_m \), we work out
\[ \langle u(t)e^{-i2\pi m Ft} \rangle = \langle \left( \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n Ft} \right) e^{-i2\pi m Ft} \rangle \]
\[ = \sum_{n=-\infty}^{\infty} U_n \langle e^{i2\pi n Ft} e^{-i2\pi m Ft} \rangle \]
\[ = U_m \quad \text{[since all other terms are zero]} \]

Calculate the average by integrating over any integer number of periods
\[ U_m = \langle u(t)e^{-i2\pi m Ft} \rangle = \frac{1}{T} \int_{t=0}^{T} u(t)e^{-i2\pi m Ft} dt \]

Notice the negative sign in Fourier analysis: in order to extract the term in the series containing \( e^{+i2\pi m Ft} \) we need to multiply by \( e^{-i2\pi m Ft} \).
Fourier Series: \( u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t} \)

Average power in \( u(t) \): \( P_u \triangleq \left\langle |u(t)|^2 \right\rangle = \frac{1}{T} \int_{0}^{T} u^2(t) dt \) \([u(t) \text{ real}]\)

Average power in Fourier component \( n \):
\[
\left\langle |U_n e^{i2\pi n F t}|^2 \right\rangle = \left\langle |U_n|^2 |e^{i2\pi n F t}|^2 \right\rangle = |U_n|^2
\]

Power conservation (Parseval’s Theorem):
\[
P_u = \left\langle |u(t)|^2 \right\rangle = \sum_{n=-\infty}^{\infty} |U_n|^2
\]

The average power in \( u(t) \) is equal to the sum of the average powers in all the Fourier components.

This is a consequence of orthogonality:
\[
\left\langle |u(t)|^2 \right\rangle = \left\langle \left( \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t} \right) \left( \sum_{m=-\infty}^{\infty} U^*_m e^{-i2\pi m F t} \right) \right\rangle
= \left\langle \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U_n U^*_m e^{i2\pi n F t} e^{-i2\pi m F t} \right\rangle
= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U_n U^*_m \left\langle e^{i2\pi n F t} e^{-i2\pi m F t} \right\rangle
= \sum_{n=-\infty}^{\infty} |U_n|^2
\]
Truncated Fourier Series: \( u_N(t) = \sum_{n=-N}^{N} U_n e^{i2\pi n Ft} \)

Approximation error: \( e_N(t) = u_N(t) - u(t) \)

Average error power \( P_{eN} = \sum_{|n|>N} |U_n|^2 \).

\( P_{eN} \to 0 \) monotonically as \( N \to \infty \).

**Gibbs phenomenon**

If \( u(t_0) \) has a discontinuity of height \( h \) then:

- \( u_N(t_0) \to \) the midpoint of the discontinuity as \( N \to \infty \).
- \( u_N(t) \) overshoots by \( \approx \pm 9\% \times h \) at \( t \approx t_0 \pm \frac{T}{2N+1} \).
- For large \( N \), the overshoots move closer to the discontinuity but do not decrease in size.
Coefficient Decay Rate

Fourier Series: \( u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t} \)

Integration:
\[
v(t) = \int_{0}^{t} u(\tau) d\tau \quad \Rightarrow \quad V_n = \frac{1}{i2\pi n F} U_n
\]
provided \( U_0 = V_0 = 0 \).

Differentiation:
\[
w(t) = \frac{du(t)}{dt} \quad \Rightarrow \quad W_n = i2\pi n F \times U_n
\]
provided \( w(t) \) satisfies the Dirichlet conditions.

Coefficient Decay Rate:
\( u(t) \) has a discontinuity \( \Rightarrow \ |U_n| \) is \( O \left( \frac{1}{n} \right) \) for large \( |n| \)
\[
\frac{d^k u(t)}{dt^k} \quad \text{is the lowest derivative with a discontinuity}\n\]
\[\Rightarrow \quad |U_n| \) is \( O \left( \frac{1}{n^{k+1}} \right) \) for large \( |n| \)

If the coefficients, \( U_n \), decrease rapidly then only a few terms are needed for a good approximation.
If \( u(t) \) is only defined over a finite range, \([0, B]\), we can make it periodic by defining \( u(t \pm B) = u(t) \).

- Coefficients are given by \( U_n = \frac{1}{B} \int_0^B u(t) e^{-i2\pi nFt} dt \).

**Example:** \( u(t) = t^2 \) for \( 0 \leq t < 2 \)

Symmetric extension:

- To avoid a discontinuity at \( t = T \), we can instead make the period \( 2B \) and define \( u(-t) = u(+t) \).

- Symmetry around \( t = 0 \) means coefficients are real-valued and symmetric \( (U_{-n} = U_n^* = U_n) \).
- Still have a first-derivative discontinuity at \( t = B \) but now we have no Gibbs phenomenon and coefficients \( \propto n^{-2} \) instead of \( \propto n^{-1} \) so approximation error power decreases more quickly.
The Dirac Delta Function

\( \delta(x) \) is the limiting case as \( w \to 0 \) of a pulse \( w \) wide and \( \frac{1}{w} \) high. It is an infinitely thin, infinitely high pulse at \( x = 0 \) with unit area.

- **Area**: \( \int_{-\infty}^{\infty} \delta(x) \, dx = 1 \)
  
- **Scaling**: \( \delta(cx) = \frac{1}{|c|} \delta(x) \)
  
- **Shifting**: \( \delta(x - a) \) is a pulse at \( x = a \) and is zero everywhere else
  
- **Multiplication**: \( f(x) \times \delta(x - a) = f(a) \times \delta(x - a) \)
  
- **Integration**: \( \int_{-\infty}^{\infty} f(x) \times \delta(x - a) \, dx = f(a) \)
  
- **Fourier Transform**: \( u(t) = \delta(t) \iff U(f) = 1 \)
  
- **We plot** \( h\delta(x) \) as a pulse of height \( |h| \) (instead of its true height of \( \infty \)
Fourier Transform

**Fourier Transform:**

\[ u(t) = \int_{-\infty}^{\infty} U(f)e^{i2\pi ft} df \]

\[ U(f) = \int_{-\infty}^{\infty} u(t)e^{-i2\pi ft} dt \]

- An “Energy Signal” has finite energy \( \iff E_u = \int_{-\infty}^{\infty} |u(t)|^2 dt < \infty \)
  - Complex-valued spectrum, \( U(f) \), decays to zero as \( f \to \pm \infty \)
  - **Energy Conservation:** \( E_u = E_U \) where \( E_U = \int_{-\infty}^{\infty} |U(f)|^2 df \)

- **Periodic Signals \( \rightarrow \) Dirac \( \delta \) functions at harmonics.**
  - Same complex-valued amplitudes as \( U_n \) from Fourier Series

  \( E_u = \infty \) but ave power is \( P_u = \left\langle |u(t)|^2 \right\rangle = \sum_{n=-\infty}^{\infty} |U_n|^2 \)
Convolution: \[ w(t) = u(t) * v(t) \Leftrightarrow w(t) = \int_{-\infty}^{\infty} u(\tau)v(t - \tau)d\tau \]

[In the integral, the arguments of \( u(\ ) \) and \( v(\ ) \) add up to \( t \)]

* acts algebraically like \( \times \): Commutative, Associative, Distributive over \( + \).
Identity element is \( \delta(t) \): \( u(t) \ast \delta(t) = u(t) \)

**Multiplication** in either the time or frequency domain
is equivalent to **convolution** in the other domain:
\[
\begin{align*}
\text{Time Domain:} & \quad w(t) = u(t) \ast v(t) \Leftrightarrow W(f) = U(f)V(f) \\
\text{Frequency Domain:} & \quad y(t) = u(t)v(t) \Leftrightarrow Y(f) = U(f) \ast V(f)
\end{align*}
\]

**Example application:**
- **Impulse Response:** \([\Delta \quad y(t) \text{ for } x(t) = \delta(t)]\)
  \[ h(t) = \frac{1}{RC}e^{-\frac{t}{RC}} \text{ for } t \geq 0 \]
- **Frequency Response:** \( H(f) = \frac{1}{1 + i2\pi fRC} \)
- **Convolution:** \( y(t) = h(t) \ast x(t) \)
- **Multiplication:** \( Y(f) = H(f)X(f) \)
Cross-correlation:
\[ w(t) = u(t) \otimes v(t) \quad \Leftrightarrow \quad w(t) = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau) d\tau \]

[In the integral, the arguments of \( u^*(\ ) \) and \( v(\ ) \) differ by \( t \)]
\( \otimes \) is not commutative or associative (unlike \( * \))

Cauchy-Schwartz Inequality \( \Rightarrow \) Bound on \( |w(t)| \)
- For all values of \( t \): \( |w(t)|^2 \leq E_u E_v \)
- \( u(t - t_0) \) is an exact multiple of \( v(t) \) \( \Leftrightarrow \) \( |w(t_0)|^2 = E_u E_v \)

Normalized cross-correlation: \( \frac{w(t)}{\sqrt{E_u E_v}} \) has a maximum absolute value of 1

- **Cross-correlation** is used to find the time shift, \( t_0 \), at which two signals match and also how well they match.

- **Auto-correlation** is the cross-correlation of a signal with itself: used to find the period of a signal (i.e. the time shift where it matches itself).