E1.10 Fourier Series and Transforms

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Main fact: Complicated time waveforms can be expressed as a sum of sine and cosine waves.

Why bother? Sine/cosine are the only bounded waves that stay the same when differentiated.

Any electronic circuit:
- sine wave in $\Rightarrow$ sine wave out (same frequency).

Hard problem: Complicated waveform $\rightarrow$ electronic circuit $\rightarrow$ output = ?

Easier problem: Complicated waveform $\rightarrow$ sum of sine waves
- $\rightarrow$ linear electronic circuit ($\Rightarrow$ obeys superposition)
- $\rightarrow$ add sine wave outputs $\rightarrow$ output = ?

Syllabus:
- Preliminary maths (1 lecture)
- Fourier series for periodic waveforms (4 lectures)
- Fourier transform for aperiodic waveforms (3 lectures)
A pair of prisms can split light up into its component frequencies (colours). This is called **Fourier Analysis**.

A second pair can re-combine the frequencies. This is called **Fourier Synthesis**.

We want to do the same thing with mathematical signals instead of light.
Organization

- 8 lectures: feel free to ask questions
- Lecture slides (including animations) and problem sheets + answers available via Blackboard or from my website: http://www.ee.ic.ac.uk/hp/staff/dmb/courses/E1Fourier/E1Fourier.htm
- Email me with any errors in slides or problems and if answers are wrong or unclear
1: Sums and Averages
A geometric series is a sum of terms that increase or decrease by a constant factor, $x$:

$$S = a + ax + ax^2 + \ldots + ax^n$$

The sequence of terms themselves is called a geometric progression.

We use a trick to get rid of most of the terms:

$$S = a + ax + ax^2 + \ldots + ax^{n-1} + ax^n$$

$$xS = \quad ax + ax^2 + ax^3 + \ldots \quad + ax^n + ax^{n+1}$$

Now subtract the lines to get: $S - xS = (1 - x) S = a - ax^{n+1}$

Divide by $1 - x$ to get: $a = $ first term, $n + 1 = $ number of terms

$$S = a \times \frac{1-x^{n+1}}{1-x}$$

Example:

$$S = 3 + 6 + 12 + 24$$

$$= 3 \times \frac{1-2^4}{1-2} = 3 \times \frac{-15}{-1} = 45$$

$[a = 3, x = 2, n + 1 = 4]$
A finite geometric series: \( S_n = a + ax + ax^2 + \cdots + ax^n = a \frac{1-x^{n+1}}{1-x} \)

What is the limit as \( n \to \infty \)?

If \( |x| < 1 \) then \( x^{n+1} \to 0 \) which gives

\[
S_\infty = a + ax + ax^2 + \cdots = a \frac{1}{1-x} = \frac{a}{1-x}
\]

Example 1:

\[
0.4 + 0.04 + 0.004 + \ldots = \frac{0.4}{1-0.1} = 0.\dot{4}
\]

\([a = 0.4, x = 0.1]\)

Example 2: (alternating signs)

\[
2 - 1.2 + 0.72 - 0.432 + \cdots = \frac{2}{1-(-0.6)} = 1.25
\]

\([a = 2, x = -0.6]\)

Example 3:

\[
1 + 2 + 4 + \ldots \neq \frac{1}{1-2} = \frac{1}{-1} = -1
\]

\([a = 1, x = 2]\)

The formula \( S = a + ax + ax^2 + \cdots = \frac{a}{1-x} \) is only valid for \( |x| < 1 \)
Dummy Variables

Using a $\sum$ sign, we can write the geometric series more compactly:

$$S_n = a + ax + ax^2 + \ldots + ax^n = \sum_{r=0}^{n} ax^r$$

[Note: $x^0 \triangleq 1$ in this context even when $x = 0$]

Here $r$ is a dummy variable: you can replace it with anything else

$$\sum_{r=0}^{n} ax^r = \sum_{k=0}^{n} ax^k = \sum_{\alpha=0}^{n} ax^\alpha$$

Dummy variables are undefined outside the summation so they sometimes get re-used elsewhere in an expression:

$$\sum_{r=0}^{3} 2^r + \sum_{r=1}^{2} 3^r = \left(1 \times \frac{1-2^4}{1-2}\right) + \left(3 \times \frac{1-3^2}{1-3}\right) = 15 + 12 = 27$$

The two dummy variables are both called $r$ but they have no connection with each other at all (or with any other variable called $r$ anywhere else).
We can derive the formula for the geometric series using $\sum$ notation:

$$S_n = \sum_{r=0}^{n} ax^r \quad \text{and} \quad xS_n = \sum_{r=0}^{n} ax^{r+1}$$

We need to manipulate the second sum to involve $x^r$.

Use the substitution $s = r + 1 \Leftrightarrow r = s - 1$.

Substitute for $r$ everywhere it occurs (including both limits)

$$xS_n = \sum_{s=1}^{n+1} ax^s = \sum_{r=1}^{n+1} ax^r$$

It is essential to sum over exactly the same set of values when substituting for dummy variables.

Subtracting gives

$$(1 - x)S_n = S_n - xS_n = \sum_{r=0}^{n} ax^r - \sum_{r=1}^{n+1} ax^r$$

$r \in [1, n]$ is common to both sums, so extract the remaining terms:

$$(1 - x)S_n = ax^0 - ax^{n+1} + \sum_{r=1}^{n} ax^r - \sum_{r=1}^{n} ax^r$$

$$= ax^0 - ax^{n+1} = a \left(1 - x^{n+1}\right)$$

Hence:

$$S_n = a \frac{1-x^{n+1}}{1-x}$$
If a signal varies with time, we can plot its waveform, $x(t)$.

The **average value** of $x(t)$ in the range $T_1 \leq t \leq T_2$ is

$$\langle x \rangle_{[T_1,T_2]} = \frac{1}{T_2-T_1} \int_{t=T_1}^{T_2} x(t) \, dt$$

The area under the curve $x(t)$ is equal to the area of the rectangle defined by 0 and $\langle x \rangle_{[T_1,T_2]}$.

Angle brackets alone, $\langle x \rangle$, denotes the **average value over all time**

$$\langle x(t) \rangle = \lim_{A,B \to \infty} \langle x(t) \rangle_{[-A,+B]}$$
The properties of averages follow from the properties of integrals:

- **Addition:** \( \langle x(t) + y(t) \rangle = \langle x(t) \rangle + \langle y(t) \rangle \)
- **Add a constant:** \( \langle x(t) + c \rangle = \langle x(t) \rangle + c \)
- **Constant multiple:** \( \langle a \times x(t) \rangle = a \times \langle x(t) \rangle \)

where the constants \( a \) and \( c \) do not depend on time.

For example:

\[
\langle x(t) + y(t) \rangle_{[T_1,T_2]} = \frac{1}{T_2-T_1} \int_{t=T_1}^{T_2} (x(t) + y(t)) \, dt \\
= \frac{1}{T_2-T_1} \int_{t=T_1}^{T_2} x(t) \, dt + \frac{1}{T_2-T_1} \int_{t=T_1}^{T_2} y(t) \, dt \\
= \langle x(t) \rangle_{[T_1,T_2]} + \langle y(t) \rangle_{[T_1,T_2]}
\]

**But beware:** \( \langle x(t) \times y(t) \rangle \neq \langle x(t) \rangle \times \langle y(t) \rangle \).
A **periodic** waveform with period $T$ repeats itself at intervals of $T$:
\[ x(t + T) = x(t) \implies x(t \pm kT) = x(t) \text{ for any integer } k. \]

The **smallest** $T > 0$ for which $x(t + T) = x(t) \forall t$ is the **fundamental period**. The **fundamental frequency** is $F = \frac{1}{T}$.

For a periodic waveform, $\langle x(t) \rangle$ equals the average over one period. It doesn’t make any difference where in a period you start or how many whole periods you take the average over.

**Example:**
\[
x(t) = |\sin t|
\]
\[
\langle x \rangle = \frac{1}{\pi} \int_{t=0}^{\pi} |\sin t| \, dt = \frac{1}{\pi} \int_{t=0}^{\pi} \sin t \, dt
\]
\[
= \frac{1}{\pi} [\cos t]_{0}^{\pi} = \frac{1}{\pi} (1 + 1) = \frac{2}{\pi} \approx 0.637
\]
[proof that $x(t \pm kT) = x(t)$]

**Proof that** $x(t + T) = x(t) \forall t \Rightarrow x(t \pm kT) = x(t) \forall t, \forall k \in \mathbb{Z}$

We use induction. Let $H_k$ be the hypothesis that $x(t + kT) = x(t) \forall t$. Under the assumption that $x(t + T) = x(t) \forall t$, we will show that if $H_k$ is true, then so are $H_{k+1}$ and $H_{k-1}$. Since we know that $H_0$ is definitely true, this implies that $H_k$ is true for all integers $k$, i.e. for all $k \in \mathbb{Z}$.

□ Suppose $H_k$ is true, i.e. $x(\tau + kT) = x(\tau) \forall \tau$. Now set $\tau = t + T$. This gives $x(t + T + kT) = x(t + T) \forall t$. But, we assume that $x(t + T) = x(t)$, so $x(t + (k + 1)T) = x(t + T + kT) = x(t + T) = x(t) \forall t$. Hence $H_{k+1}$ is true.

□ Now suppose $H_k$ is true as before but this time set $\tau = t - T$. Substituting this into $u(\tau + kT) = u(\tau)$ gives $u(t - T + kT) = u(t - T)$. Substituting it also into $u(\tau + T) = u(\tau)$ gives $u(t) = u(t - T)$. Finally, combining these two identities gives $u(t + (k - 1)T) = u(t)$ which is $H_{k-1}$. 
A sine wave, \( x(t) = \sin 2\pi F t \), has a frequency \( F \) and a period \( T = \frac{1}{F} \) so that, \( \sin \left( 2\pi F \left( t + \frac{1}{F} \right) \right) = \sin \left( 2\pi F t + 2\pi \right) = \sin 2\pi F t \).

\[
\langle \sin 2\pi F t \rangle = \frac{1}{T} \int_{t=0}^{T} \sin (2\pi F t) \, dt = 0
\]

Also, \( \langle \cos 2\pi F t \rangle = 0 \) except for the case \( F = 0 \) since \( \cos 2\pi 0 t \equiv 1 \).

Hence: \( \langle \sin 2\pi F t \rangle = 0 \) and \( \langle \cos 2\pi F t \rangle = \begin{cases} 0 & F \neq 0 \\ 1 & F = 0 \end{cases} \)

Also: \( \langle e^{i2\pi F t} \rangle = \langle \cos 2\pi F t + i \sin 2\pi F t \rangle \)
\[
= \langle \cos 2\pi F t \rangle + i \langle \sin 2\pi F t \rangle = \begin{cases} 0 & F \neq 0 \\ 1 & F = 0 \end{cases}
\]
• **Sum of geometric series** (see RHB Chapter 4)
  - Finite series: \( S = a \times \frac{1-x^{n+1}}{1-x} \)
  - Infinite series: \( S = \frac{a}{1-x} \) but only if \(|x| < 1\)

• **Dummy variables**
  - Commonly re-used elsewhere in expressions
  - Substitutions must cover exactly the same set of values

• **Averages**: \( \langle x \rangle_{[T_1,T_2]} = \frac{1}{T_2-T_1} \int_{t=T_1}^{T_2} x(t) dt \)

• **Periodic waveforms**: \( x(t \pm kT) = x(t) \) for any integer \( k \)
  - Fundamental period is the smallest \( T \)
  - Fundamental frequency is \( F = \frac{1}{T} \)
  - For periodic waveforms, \( \langle x \rangle \) is the average over any integer number of periods
  - \( \langle \sin 2\pi F t \rangle = 0 \)
  - \( \langle \cos 2\pi F t \rangle = \langle e^{i2\pi F t} \rangle = \begin{cases} 0 & F \neq 0 \\ 1 & F = 0 \end{cases} \)
2: Fourier Series
A function, \( u(t) \), is periodic with period \( T \) if \( u(t + T) = u(t) \ \forall t \).  
- Periodic with period \( T \) \( \Rightarrow \) Periodic with period \( kT \ \forall k \in \mathbb{Z}^+ \)

The fundamental period is the smallest \( T > 0 \) for which \( u(t + T) = u(t) \).

If you add together functions with different periods the fundamental period is the lowest common multiple (LCM) of the individual fundamental periods.

Example:
- \( u(t) = \cos 4\pi t \) \( \Rightarrow \) \( T_u = \frac{2\pi}{4\pi} = 0.5 \)
- \( v(t) = \cos 5\pi t \) \( \Rightarrow \) \( T_v = \frac{2\pi}{5\pi} = 0.4 \)
- \( w(t) = u(t) + 0.1v(t) \) \( \Rightarrow \) \( T_w = \text{lcm}(0.5, 0.4) = 2.0 \)
If \( u(t) \) has fundamental period \( T \) and fundamental frequency \( F = \frac{1}{T} \) then, in most cases, we can express it as a (possibly infinite) sum of sine and cosine waves with frequencies 0, \( F \), 2\( F \), 3\( F \), \( \cdots \).

The Fourier series for \( u(t) \) is

\[
 u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos 2\pi n F t + b_n \sin 2\pi n F t \right)
\]

The Fourier coefficients of \( u(t) \) are \( a_0, a_1, \cdots \) and \( b_1, b_2, \cdots \).

The \( n^{th} \) harmonic of the fundamental is the component at a frequency \( nF \).
Why Sin and Cos Waves?

Why are engineers obsessed with sine waves?

**Answer:** Because ...

1. A sine wave remains a sine wave of the same frequency when you
   (a) multiply by a constant,
   (b) add onto to another sine wave of the same frequency,
   (c) differentiate or integrate or shift in time

2. Almost any function can be expressed as a sum of sine waves
   - Periodic functions $\rightarrow$ Fourier Series
   - Aperiodic functions $\rightarrow$ Fourier Transform

3. Many physical and electronic systems are
   (a) composed entirely of constant-multiply/add/differentiate
   (b) linear: $u(t) \rightarrow x(t)$ and $v(t) \rightarrow y(t)$
     means that $u(t) + v(t) \rightarrow x(t) + y(t)$
     $\Rightarrow$ sum of sine waves $\rightarrow$ sum of sine waves

In these lectures we will use $T$ for the fundamental period and $F = \frac{1}{T}$ for the fundamental frequency.
Dirichlet Conditions

Not all \( u(t) \) can be expressed as a Fourier Series. Peter Dirichlet derived a set of **sufficient** conditions.

**The function** \( u(t) \) **must satisfy:**

- periodic and single-valued
- \( \int_0^T |u(t)| \, dt < \infty \)
- finite number of maxima/minima per period
- finite number of finite discontinuities per period

**Good:**

\[
\begin{align*}
\sin(t) & \\
t^2 & \\
\text{quantized}
\end{align*}
\]

**Bad:**

\[
\begin{align*}
\tan(t) & \\
\sin\left(\frac{1}{t}\right) & \\
\infty \text{ halving steps}
\end{align*}
\]
Suppose that \( u(t) \) satisfies the Dirichlet conditions so that
\[
u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi n Ft + b_n \sin 2\pi n Ft)
\]

**Question:** How do we find \( a_n \) and \( b_n \)?

**Answer:** We use a clever trick that relies on taking averages.

\(<x(t)>\) equals the average of \( x(t) \) over any integer number of periods:
\[
<x(t)> = \frac{1}{T} \int_{t=0}^{T} x(t) dt
\]

Remember, for any integer \( n \),
\[
<\sin 2\pi n Ft> = 0
\]
\[
<\cos 2\pi n Ft> = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}
\]

Finding \( a_n \) and \( b_n \) is called **Fourier analysis**.
\[
\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y
\]
\[
\Rightarrow \sin x \cos y = \frac{1}{2} \sin(x + y) + \frac{1}{2} \sin(x - y)
\]
\[
\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y
\]
\[
\Rightarrow \cos x \cos y = \frac{1}{2} \cos(x + y) + \frac{1}{2} \cos(x - y)
\]
\[
\sin x \sin y = \frac{1}{2} \cos(x - y) - \frac{1}{2} \cos(x + y)
\]

Set \(x = 2\pi mFt, y = 2\pi nFt\) (with \(m + n \neq 0\)) and take time-averages:

- \(\langle \sin (2\pi mFt) \cos (2\pi nFt) \rangle\)
  \[
  = \langle \frac{1}{2} \sin(2\pi (m + n) Ft) \rangle + \langle \frac{1}{2} \sin(2\pi (m - n) Ft) \rangle = 0
  \]

- \(\langle \cos (2\pi mFt) \cos (2\pi nFt) \rangle\)
  \[
  = \langle \frac{1}{2} \cos(2\pi (m + n) Ft) \rangle + \langle \frac{1}{2} \cos(2\pi (m - n) Ft) \rangle = \begin{cases} 
    0 & m \neq n \\
    \frac{1}{2} & m = n 
  \end{cases}
  \]

- \(\langle \sin (2\pi mFt) \sin (2\pi nFt) \rangle\)
  \[
  = \langle \frac{1}{2} \cos(2\pi (m - n) Ft) \rangle - \langle \frac{1}{2} \cos(2\pi (m + n) Ft) \rangle = \begin{cases} 
    0 & m \neq n \\
    \frac{1}{2} & m = n 
  \end{cases}
  \]

Summary:
\[
\langle \sin \cos \rangle = 0 \quad [\text{provided that } m + n \neq 0]
\]
\[
\langle \sin \sin \rangle = \langle \cos \cos \rangle = \frac{1}{2} \text{ if } m = n \text{ or otherwise } = 0.
\]
Proof that $\cos x \cos y = \frac{1}{2} \cos(x + y) + \frac{1}{2} \cos(x - y)$

We know that

\[
\begin{align*}
\cos(x + y) &= \cos x \cos y - \sin x \sin y \\
\cos(x - y) &= \cos x \cos y + \sin x \sin y
\end{align*}
\]

Adding these two gives

\[
\cos(x + y) + \cos(x - y) = 2 \cos x \cos y
\]

From which: $\cos x \cos y = \frac{1}{2} \cos(x + y) + \frac{1}{2} \cos(x - y)$

Subtracting instead of adding gives: $\sin x \sin y = \frac{1}{2} \cos(x - y) - \frac{1}{2} \cos(x + y)$

Proof that $\langle \frac{1}{2} \cos(2\pi (m + n) F t) \rangle + \langle \frac{1}{2} \cos(2\pi (m - n) F t) \rangle = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \end{cases}$

We are assuming that $m$ and $n$ are integers with $m + n \neq 0$ and we use the result that $\langle \cos 2\pi ft \rangle$ is zero unless $f = 0$ in which case $\langle \cos 2\pi 0 t \rangle = 1$. The frequency of the first term, $\cos(2\pi (m + n) F t)$, is $(m + n) F$ which is definitely non-zero (because of our assumption that $m + n \neq 0$) and so the average of this cosine wave is zero. The frequency of the second term is $(m - n) F$ and this is zero only if $m = n$. So it follows that the entire expression is zero unless $m = n$ in which case the second term gives a value of $\frac{1}{2}$. Since $m$ and $n$ are integers, we can take the averages over a time interval $T$ and be sure that this includes an integer number of periods for both terms.
Find $a_n$ and $b_n$ in $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi n F t + b_n \sin 2\pi n F t)$

**Answer:**

\[ a_n = 2 \langle u(t) \cos (2\pi n Ft) \rangle = \frac{2}{T} \int_0^T u(t) \cos (2\pi n Ft) \, dt \]
\[ b_n = 2 \langle u(t) \sin (2\pi n Ft) \rangle = \frac{2}{T} \int_0^T u(t) \sin (2\pi n Ft) \, dt \]

**Proof $[a_0]$:**

\[ 2 \langle u(t) \cos (2\pi 0 Ft) \rangle = 2 \langle u(t) \rangle = 2 \times \frac{a_0}{2} = a_0 \]

**Proof $[a_n, n > 0]$:**

\[ 2 \langle u(t) \cos (2\pi n Ft) \rangle \]
\[ = 2 \langle \frac{a_0}{2} \cos (2\pi n Ft) \rangle + \sum_{r=1}^{\infty} 2 \langle a_r \cos (2\pi r F t) \cos (2\pi n F t) \rangle + \sum_{r=1}^{\infty} 2 \langle b_r \sin (2\pi r F t) \cos (2\pi n F t) \rangle \]

**Term 1:**

\[ 2 \langle \frac{a_0}{2} \cos (2\pi n F t) \rangle = 0 \]

**Term 2:**

\[ 2 \langle a_r \cos (2\pi r F t) \cos (2\pi n F t) \rangle = \begin{cases} a_n & r = n \\ 0 & r \neq n \end{cases} \]
\[ \Rightarrow \sum_{r=1}^{\infty} 2 \langle a_r \cos (2\pi r F t) \cos (2\pi n F t) \rangle = a_n \]

**Term 3:**

\[ 2 \langle b_r \sin 2\pi r F t \cos (2\pi n F t) \rangle = 0 \]

**Proof $[b_n, n > 0]$:** same method as for $a_n$
Truncated Series:

\[ u_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N} \left( a_n \cos \frac{2\pi n F t}{T} + b_n \sin \frac{2\pi n F t}{T} \right) \]

Pulse: \( T = 20 \), width \( W = \frac{T}{4} \), height \( A = 8 \)

\[ a_n = \frac{2}{T} \int_{0}^{T} u(t) \cos \frac{2\pi n t}{T} \, dt \]
\[ = \frac{2}{T} \int_{0}^{W} A \cos \frac{2\pi n t}{T} \, dt \]
\[ = \frac{2AT}{2n\pi T} \sin \left( \frac{2\pi n W}{T} \right) \]
\[ = \frac{A}{n\pi} \sin \frac{2\pi n W}{T} = \frac{A}{n\pi} \sin \frac{n\pi}{2} \]

\[ b_n = \frac{2}{T} \int_{0}^{T} u(t) \sin \frac{2\pi n t}{T} \, dt \]
\[ = \frac{2AT}{2\pi n T} \sin \left( \frac{2\pi n W}{T} \right) \]
\[ = \frac{A}{n\pi} \left( 1 - \cos \frac{n\pi}{2} \right) \]

<table>
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<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td>( -\frac{8}{3\pi} )</td>
<td>0</td>
<td>( \frac{8}{5\pi} )</td>
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<tr>
<td>( b_n )</td>
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In the previous example, we can obtain $a_0$ by noting that $\frac{a_0}{2} = \langle u(t) \rangle$, the average value of the waveform which must be $\frac{AW}{2} = 2$. From this, $a_0 = 4$. We can, however, also derive this value from the general expression.

The expression for $a_m$ is $a_m = \frac{A}{n\pi} \sin \frac{n\pi}{2}$. For the case, $n = 0$, this is difficult to evaluate because both the numerator and denominator are zero. The general way of dealing with this situation is L'Hôpital's rule (see section 4.7 of RHB) but here we can use a simpler and very useful technique that is often referred to as the “small angle approximation”. For small values of $\theta$ we can approximate the standard trigonometrical functions as: $\sin \theta \approx \theta$, $\cos \theta \approx 1 - 0.5\theta^2$ and $\tan \theta \approx \theta$. These approximations are obtained by taking the first three terms of the Taylor series; they are accurate to $O(\theta^3)$ and are exactly correct when $\theta = 0$. When $m = 0$ we can therefore make an exact approximation to $a_n$ by writing $a_n = \frac{A}{n\pi} \sin \frac{n\pi}{2} \approx \frac{A}{n\pi} \times \frac{n\pi}{2} = \frac{A}{2} = 4$. What we have implicitly done here is to assume that $n$ is a real number (instead of an integer) and then taken the limit of $a_n$ as $n \to 0$. 
Fourier analysis maps a function of time onto a set of Fourier coefficients:

\[ u(t) \rightarrow \{a_n, b_n\} \]

This mapping is **linear** which means:

1. For any \( \gamma \), if \( u(t) \rightarrow \{a_n, b_n\} \) then \( \gamma u(t) \rightarrow \{\gamma a_n, \gamma b_n\} \)
2. If \( u(t) \rightarrow \{a_n, b_n\} \) and \( u'(t) \rightarrow \{a'_n, b'_n\} \) then
   \[ (u(t) + u'(t)) \rightarrow \{a_n + a'_n, b_n + b'_n\} \]

**Proof for \( a_n \):** (proof for \( b_n \) is similar)

1. If \( \frac{2}{T} \int_0^T u(t) \cos (2\pi n Ft) \, dt = a_n \), then
   \[
   \frac{2}{T} \int_0^T (\gamma u(t)) \cos (2\pi n Ft) \, dt \\
   = \gamma \frac{2}{T} \int_0^T u(t) \cos (2\pi n Ft) \, dt = \gamma a_n
   \]

2. If \( \frac{2}{T} \int_0^T u(t) \cos (2\pi n Ft) \, dt = a_n \) and
   \[
   \frac{2}{T} \int_0^T u'(t) \cos (2\pi n Ft) \, dt = a'_n \) then
   \[
   \frac{2}{T} \int_0^T (u(t) + u'(t)) \cos (2\pi n Ft) \, dt \\
   = \frac{2}{T} \int_0^T u(t) \cos (2\pi n Ft) \, dt + \frac{2}{T} \int_0^T u'(t) \cos (2\pi n Ft) \, dt \\
   = a_n + a'_n
   \]
Fourier Series:

\[ u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos 2\pi n Ft + b_n \sin 2\pi n Ft \right) \]

Dirichlet Conditions: sufficient conditions for FS to exist
- Periodic, Single-valued, Bounded absolute integral
- Finite number of (a) max/min and (b) finite discontinuities

Fourier Analysis = “finding \( a_n \) and \( b_n \)”
- \( a_n = 2 \langle u(t) \cos (2\pi n F t) \rangle \)
  \[ \triangleq \frac{2}{T} \int_{0}^{T} u(t) \cos (2\pi n F t) \, dt \]
- \( b_n = 2 \langle u(t) \sin (2\pi n F t) \rangle \)
  \[ \triangleq \frac{2}{T} \int_{0}^{T} u(t) \sin (2\pi n F t) \, dt \]

The mapping \( u(t) \rightarrow \{a_n, b_n\} \) is linear

For further details see RHB 12.1 and 12.2.
3: Complex Fourier Series

Euler’s Equation
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Summary
Euler’s Equation

Euler’s Equation: \( e^{i\theta} = \cos\theta + i\sin\theta \)

Hence:
\[
\begin{align*}
\cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2} e^{i\theta} + \frac{1}{2} e^{-i\theta} \\
\sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} = -\frac{1}{2} i e^{i\theta} + \frac{1}{2} i e^{-i\theta}
\end{align*}
\]

Most maths becomes simpler if you use \( e^{i\theta} \) instead of \( \cos\theta \) and \( \sin\theta \)

The Complex Fourier Series is the Fourier Series but written using \( e^{i\theta} \)

Examples where using \( e^{i\theta} \) makes things simpler:

<table>
<thead>
<tr>
<th>Using ( e^{i\theta} )</th>
<th>Using ( \cos\theta ) and ( \sin\theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^{i(\theta+\phi)} = e^{i\theta} e^{i\phi} )</td>
<td>( \cos (\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi )</td>
</tr>
<tr>
<td>( e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)} )</td>
<td>( \cos\theta \cos\phi = \frac{1}{2} \cos (\theta + \phi) + \frac{1}{2} \cos (\theta - \phi) )</td>
</tr>
<tr>
<td>( \frac{d}{d\theta} e^{i\theta} = i e^{i\theta} )</td>
<td>( \frac{d}{d\theta} \cos\theta = -\sin\theta )</td>
</tr>
</tbody>
</table>
Fourier Series: $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi n Ft + b_n \sin 2\pi n Ft)$

Substitute: $\cos \theta = \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}$ and $\sin \theta = -\frac{1}{2}ie^{i\theta} + \frac{1}{2}ie^{-i\theta}$

$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \left(\frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}\right) + b_n \left(-\frac{1}{2}ie^{i\theta} + \frac{1}{2}ie^{-i\theta}\right))$

$\quad = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left((\frac{1}{2}a_n - \frac{1}{2}ib_n) e^{i2\pi n Ft}\right)$
$\quad \quad + \sum_{n=1}^{\infty} \left((\frac{1}{2}a_n + \frac{1}{2}ib_n) e^{-i2\pi n Ft}\right)$

$\quad = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n Ft}$

where

$U_n = \begin{cases} 
\frac{1}{2}a_n - \frac{1}{2}ib_n & n \geq 1 \\
\frac{1}{2}a_0 & n = 0 \\
\frac{1}{2}|a_n| + \frac{1}{2}ib_n & n \leq -1 
\end{cases}$

$U_{\pm n} = \frac{1}{2} (a_{|n|} \mp ib_{|n|})$

The $U_n$ are normally complex except for $U_0$ and satisfy $U_n = U_{-n}^*$

Complex Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n Ft}$ [simpler]
If $x(t)$ has period $\frac{T}{n}$ for some integer $n$ (i.e. frequency $\frac{n}{T} = nF$):

$$\langle x(t) \rangle \triangleq \frac{1}{T} \int_{t=0}^{T} x(t) dt$$

This is the average over an integer number of cycles.

For a complex exponential:

$$\langle e^{i2\pi nFt} \rangle = \langle \cos (2\pi nFt) + i \sin (2\pi nFt) \rangle$$

$$= \langle \cos (2\pi nFt) \rangle + i \langle \sin (2\pi nFt) \rangle$$

$$= \begin{cases} 
1 + 0i & n = 0 \\
0 + 0i & n \neq 0 
\end{cases}$$

Hence:

$$\langle e^{i2\pi nFt} \rangle = \begin{cases} 
1 & n = 0 \\
0 & n \neq 0 
\end{cases}$$
Complex Fourier Series: \( u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n Ft} \)

To find the coefficient, \( U_n \), we multiply by something that makes all the terms involving the other coefficients average to zero.

\[
\langle u(t) e^{-i2\pi n F t} \rangle = \langle \sum_{r=-\infty}^{\infty} U_r e^{i2\pi F r t} e^{-i2\pi n F t} \rangle \\
= \langle \sum_{r=-\infty}^{\infty} U_r e^{i2\pi (r-n) F t} \rangle \\
= \sum_{r=-\infty}^{\infty} U_r \langle e^{i2\pi (r-n) F t} \rangle
\]

All terms in the sum are zero, except for the one when \( n = r \) which equals \( U_n \):

\[
U_n = \langle u(t) e^{-i2\pi n F t} \rangle \tag{\S}
\]

This shows that the Fourier series coefficients are unique: you cannot have two different sets of coefficients that result in the same function \( u(t) \).

Note the sign of the exponent: “+” in the Fourier Series but “−” for Fourier Analysis (in order to cancel out the “+”).
\[
\begin{align*}
    u(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi n F t + b_n \sin 2\pi n F t) \\
    &= \sum_{n=-\infty}^{\infty} U_n e^{i 2\pi n F t}
\end{align*}
\]

We can easily convert between the two forms.

**Fourier Coefficients \(\rightarrow\) Complex Fourier Coefficients:**

\[
U_{\pm n} = \frac{1}{2} \left( a_{|n|} \mp i b_{|n|} \right)
\]

\[U_n = U_{-n}^*\]

**Complex Fourier Coefficients \(\rightarrow\) Fourier Coefficients:**

\[
\begin{align*}
    a_n &= U_n + U_{-n} = 2 \Re (U_n) \\
    b_n &= i (U_n - U_{-n}) = -2 \Im (U_n)
\end{align*}
\]

\[\Re = \text{“real part”}\]

\[\Im = \text{“imaginary part”}\]

The formula for \(a_n\) works even for \(n = 0\).
In these lectures, we are assuming that $u(t)$ is a periodic real-valued function of time. In this case we can represent $u(t)$ using either the Fourier Series or the Complex Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos 2\pi n Ft + b_n \sin 2\pi n Ft \right) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n Ft}$$

We have seen that the $U_n$ coefficients are complex-valued and that $U_n$ and $U_{-n}$ are complex conjugates so that we can write $U_{-n} = U_n^*$. In fact, the complex Fourier series can also be used when $u(t)$ is a complex-valued function of time (this is sometimes useful in the fields of communications and signal processing). In this case, it is still true that $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n Ft}$, but now $U_n$ and $U_{-n}$ are completely independent and normally $U_{-n} \neq U_n^*$. 
**Complex Fourier Analysis Example**

**Method 1:**
\[ U_{\pm n} = \frac{1}{2}a_n \mp i\frac{1}{2}b_n \]

**Method 2:**
\[
U_n = \langle u(t)e^{-i2\pi nFt} \rangle \\
= \frac{1}{T} \int_0^T u(t)e^{-i2\pi nFt} dt \\
= \frac{1}{T} \int_0^W Ae^{-i2\pi nFt} dt \\
= \frac{A}{-i2\pi nFT} \left[ e^{-i2\pi nFt} \right]_0^W \\
= \frac{A}{i2\pi n} \left( 1 - e^{-i2\pi nFW} \right) \\
= \frac{Ae^{-i\pi nFW}}{i2\pi n} \left( e^{i\pi nFW} - e^{-i\pi nFW} \right) \\
= \frac{Ae^{-i\pi nFW}}{n\pi} \sin (n\pi FW) \\
= \frac{8}{n\pi} \sin \left( \frac{n\pi}{4} \right) e^{-i\frac{n\pi}{4}}
\]

**E1.10 Fourier Series and Transforms (2014-5543)**

- Complex Fourier Series
  - Euler’s Equation
  - Complex Fourier Series
  - Averaging Complex Exponentials
  - Complex Fourier Analysis
  - Fourier Series ↔ Complex Fourier Series
  - Time Shifting
  - Even/Odd Symmetry
  - Antiperiodic ⇒ Odd Harmonics Only
  - Symmetry Examples
  - Summary

<table>
<thead>
<tr>
<th>n</th>
<th>(a_n)</th>
<th>(b_n)</th>
<th>(U_n)</th>
</tr>
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<tbody>
<tr>
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<td></td>
<td></td>
<td>(i\frac{8}{6\pi})</td>
</tr>
<tr>
<td>-5</td>
<td></td>
<td></td>
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</tr>
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<td></td>
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</tr>
<tr>
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<td>(\frac{8}{2\pi})</td>
<td></td>
<td></td>
</tr>
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<td>(\frac{16}{6\pi})</td>
<td>(i\frac{-8}{6\pi})</td>
</tr>
</tbody>
</table>
Time Shifting

Complex Fourier Series: \( u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt} \)

If \( v(t) \) is the same as \( u(t) \) but delayed by a time \( \tau \): \( v(t) = u(t - \tau) \)

\[
v(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nF(t-\tau)} = \sum_{n=-\infty}^{\infty} \left( U_n e^{-i2\pi nF\tau} \right) e^{i2\pi nFt}
\]

\[
= \sum_{n=-\infty}^{\infty} V_n e^{i2\pi nFt}
\]

where \( V_n = U_n e^{-i2\pi nF\tau} \)

Example:

\( u(t) = 6 \cos (2\pi F t) \)

\[
\begin{align*}
\text{Fourier: } & a_1 = 6, \quad b_1 = 0 \\
\text{Complex: } & U_{\pm 1} = \frac{1}{2} a_1 \mp \frac{1}{2} i b_1 = 3
\end{align*}
\]

\( v(t) = 6 \sin (2\pi F t) = u(t - \tau) \)

\[
\begin{align*}
\text{Time delay: } & \tau = \frac{T}{4} \Rightarrow F \tau = \frac{1}{4} \\
\text{Complex: } & V_1 = U_1 e^{-i\frac{\pi}{2}} = -3i \\
& V_{-1} = U_{-1} e^{i\frac{\pi}{2}} = +3i
\end{align*}
\]

Note: If \( u(t) \) is a sine wave, \( U_1 \) equals half the corresponding phasor.
Even/Odd Symmetry

(1) \( u(t) \) real-valued \( \iff \ U_n \) conjugate symmetric \( [U_n = U^*_{-n}] \)

(2) \( u(t) \) even \( [u(t) = u(-t)] \) \( \iff \ U_n \) even \( [U_n = U_{-n}] \)

(3) \( u(t) \) odd \( [u(t) = -u(-t)] \) \( \iff \ U_n \) odd \( [U_n = -U_{-n}] \)

(1)+(2) \( u(t) \) real & even \( \iff \ U_n \) real & even \( [U_n = U^*_{-n} = U_{-n}] \)

(1)+(3) \( u(t) \) real & odd \( \iff \ U_n \) imaginary & odd \( [U_n = U^*_{-n} = -U_{-n}] \)

Proof of (2): \( u(t) \) even \( \Rightarrow \) \( U_n \) even

\[
U_{-n} = \frac{1}{T} \int_{0}^{T} u(t)e^{-i2\pi (-n)Ft} dt
= \frac{1}{T} \int_{-T}^{0} u(-x)e^{-i2\pi nFx} (-dx) \quad \text{[substitute } x = -t]\]
\[
= \frac{1}{T} \int_{x=0}^{x=-T} u(-x)e^{-i2\pi nFx} dx \quad \text{[reverse the limits]}
= \frac{1}{T} \int_{x=-T}^{0} u(x)e^{-i2\pi nFx} dx = U_n \quad \text{[even: } u(-x) = u(x)]
\]

Proof of (3): \( u(t) \) odd \( \Rightarrow \) \( U_n \) odd

Same as before, except for the last line:
\[
= \frac{1}{T} \int_{x=-T}^{0} -u(x)e^{-i2\pi nFx} dx = -U_n \quad \text{[odd: } u(-x) = -u(x)]
\]
A waveform, \( u(t) \), is **anti-periodic** if \( u(t + \frac{1}{2}T) = -u(t) \).

If \( u(t) \) is anti-periodic then \( U_n = 0 \) for \( n \) even.

**Proof:**

Define \( v(t) = u(t + \frac{T}{2}) \), then

1. \( v(t) = -u(t) \) \( \Rightarrow \) \( V_n = -U_n \)
2. \( v(t) \) equals \( u(t) \) but delayed by \( -\frac{T}{2} \)

\[ V_n = U_n e^{i2\pi n F \frac{T}{2}} = U_n e^{in\pi} = \begin{cases} U_n & n \text{ even} \\ -U_n & n \text{ odd} \end{cases} \]

Hence for \( n \) even: \( V_n = -U_n = U_n \) \( \Rightarrow \) \( U_n = 0 \)

**Example:**

\( U_{0:5} = [0, 3 + 2i, 0, i, 0, 1] \)

Odd harmonics only \( \Leftrightarrow \)

Second half of each period is the negative of the first half.
All these examples assume that \( u(t) \) is real-valued \( \iff U_{-n} = U_{+n}^* \).

1. **Even** \( u(t) \) \( \iff \) real \( U_n \)
   \[
   U_{0:2} = [0, 2, -1]
   \]

2. **Odd** \( u(t) \) \( \iff \) imaginary \( U_n \)
   \[
   U_{0:3} = [0, -2i, i, i]
   \]

3. **Anti-periodic** \( u(t) \)
   \( \iff \) odd harmonics only
   \[
   U_{0:1} = [0, -i]
   \]

4. **Even harmonics only**
   \( \iff \) period is really \( \frac{1}{2}T \)
   \[
   U_{0:4} = [2, 0, 2, 0, 1]
   \]
- **Fourier Series:**
  
  \[ u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos 2\pi n F t + b_n \sin 2\pi n F t \right) \]

- **Complex Fourier Series:**
  
  \[ u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t} \]
  
  - \( U_n = \langle u(t) e^{-i2\pi n F t} \rangle \triangleq \frac{1}{T} \int_0^T u(t) e^{-i2\pi n F t} dt \)
  
  - Since \( u(t) \) is real-valued, \( U_n = U_n^* \)
  
  - \( \text{FS} \rightarrow \text{CFS} \): \( U_{\pm n} = \frac{1}{2} a_{|n|} \mp i \frac{1}{2} b_{|n|} \)
  
  - \( \text{CFS} \rightarrow \text{FS} \): \( a_n = U_n + U_{-n} \)
    
    \[ b_n = i (U_n - U_{-n}) \]

- **\( u(t) \) real and even** \( \Leftrightarrow \) \( u(-t) = u(t) \)
  
  \( \Leftrightarrow \) \( U_n \) is real-valued and even \( \Leftrightarrow \) \( b_n = 0 \ \forall n \)

- **\( u(t) \) real and odd** \( \Leftrightarrow \) \( u(-t) = -u(t) \)
  
  \( \Leftrightarrow \) \( U_n \) is purely imaginary and odd \( \Leftrightarrow \) \( a_n = 0 \ \forall n \)

- **\( u(t) \) anti-periodic** \( \Leftrightarrow \) \( u(t + \frac{T}{2}) = -u(t) \)
  
  \( \Leftrightarrow \) odd harmonics only \( \Leftrightarrow \) \( a_{2n} = b_{2n} = U_{2n} = 0 \ \forall n \)

For further details see RHB 12.3 and 12.7.
4: Parseval’s Theorem and Convolution
Suppose we have two signals with the same period, \( T = \frac{1}{F} \),

\[
\begin{align*}
  u(t) &= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n Ft} \\
  \Rightarrow u^*(t) &= \sum_{n=-\infty}^{\infty} U_n^* e^{-i2\pi n Ft} \\
  v(t) &= \sum_{n=-\infty}^{\infty} V_n e^{i2\pi n Ft}
\end{align*}
\]

\[
[u(t) = u^*(t) \text{ if real}]
\]

Now multiply \( u^*(t) \) and \( v(t) \) together and take the average over \([0, T]\).

[Use different “dummy variables”, \( n \) and \( m \), so they don’t get mixed up]

\[
\langle u^*(t)v(t) \rangle = \langle \sum_{n=-\infty}^{\infty} U_n^* e^{-i2\pi n Ft} \sum_{m=-\infty}^{\infty} V_m e^{i2\pi m Ft} \rangle \\
= \sum_{n=-\infty}^{\infty} U_n^* \sum_{m=-\infty}^{\infty} V_m \langle e^{-i2\pi n Ft} e^{i2\pi m Ft} \rangle \\
= \sum_{n=-\infty}^{\infty} U_n^* \sum_{m=-\infty}^{\infty} V_m \langle e^{i2\pi (m-n) Ft} \rangle
\]

The quantity \( \langle \cdot \cdot \cdot \rangle \) equals 1 if \( m = n \) and 0 otherwise, so the only non-zero element in the second sum is when \( m = n \), so the second sum equals \( V_n \).

Hence Parseval’s theorem:

\[
\langle u^*(t)v(t) \rangle = \sum_{n=-\infty}^{\infty} U_n^* V_n
\]

If \( v(t) = u(t) \) we get:

\[
\langle |u(t)|^2 \rangle = \sum_{n=-\infty}^{\infty} U_n^* U_n = \sum_{n=-\infty}^{\infty} |U_n|^2
\]
If you have a multiplicative expression involving two or more sums, then you **must** use different dummy variables for each of the sums:

\[ \sum_n af(n) \sum_m bg(m) \]

1. You can always move any quantities to the right

\[ \sum_n af(n) \sum_m bg(m) = \sum_n a \sum_m bf(n)g(m) = \sum_n \sum_m abf(n)g(m) \]

2. You can move quantities to the left past a summation provided that they do not involve the dummy variable of the summation:

\[ \sum_n \sum_m abf(n)g(m) = \sum_n af(n) \sum_m bg(m) \neq \sum_n af(n)g(m) \sum_m b \]

The last expression doesn’t make sense in any case since \( m \) is undefined outside \( \sum_m \)

3. You can swap the summation order if the sum converges absolutely

\[ \sum_n \sum_m h(n, m) = \sum_m \sum_n h(n, m) \text{ provided that } \sum_n \sum_m |h(n, m)| < \infty \]

The equality on the left is not necessarily true if the sum does not converge absolutely. Of course, if the sum has only a finite number of terms, it is bound to converge absolutely.
The **average power** of a periodic signal is given by \( P_u \triangleq \langle |u(t)|^2 \rangle. \)

This is the average electrical power that would be dissipated if the signal represents the voltage across a 1Ω resistor.

**Parseval’s Theorem:**

\[
P_u = \langle |u(t)|^2 \rangle = \sum_{n=-\infty}^{\infty} |U_n|^2
\]

\[
= |U_0|^2 + 2 \sum_{n=1}^{\infty} |U_n|^2 \quad \text{[assume } u(t) \text{ real]}
\]

\[
= \frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \quad \text{[} U_n = \frac{a_n - ib_n}{2} \text{]}
\]

Parseval’s theorem ⇒ the average power in \( u(t) \) is equal to the sum of the average powers in each of its Fourier components.

**Example:**

\[
u(t) = 2 + 2 \cos 2\pi F t + 4 \sin 2\pi F t - 2 \sin 6\pi F t
\]

\[
\langle |u(t)|^2 \rangle = 4 + \frac{1}{2} \left( 2^2 + 4^2 + (-2)^2 \right) = 16
\]

\[U_{0:3} = [2, 1-2i, 0, i] \quad \Rightarrow \quad |U_0|^2 + 2 \sum_{n=1}^{\infty} |U_n|^2 = 16\]
The spectrum of a periodic signal is the values of \( \{U_n\} \) versus \( nF \).

The magnitude spectrum is the values of \( \{ |U_n| \} = \left\{ \frac{1}{2} \sqrt{a^2_{|n|} + b^2_{|n|}} \right\} \).

The power spectrum is the values of \( \{ |U_n|^2 \} = \left\{ \frac{1}{4} \left( a^2_{|n|} + b^2_{|n|} \right) \right\} \).

Example:

\[ u(t) = 2 + 2 \cos 2\pi F t + 4 \sin 2\pi F t - 2 \sin 6\pi F t \]

Fourier Coefficients: \( a_{0:3} = [4, 2, 0, 0] \quad b_{1:3} = [4, 0, -2] \)

Spectrum: \( U_{-3:3} = [-i, 0, 1 + 2i, 2, 1 - 2i, 0, i] \)

Magnitude Spectrum: \( |U_{-3:3}| = [1, 0, \sqrt{5}, 2, \sqrt{5}, 0, 1] \)

Power Spectrum: \( |U_{-3:3}^2| = [1, 0, 5, 4, 5, 0, 1] \quad [\Sigma = \langle u^2(t) \rangle] \)

The magnitude and power spectra of a real periodic signal are symmetrical.

A one-sided power power spectrum shows \( U_0 \) and then \( 2 |U_n|^2 \) for \( n \geq 1 \).
Suppose we have two signals with the same period, $T = \frac{1}{F}$,

\[ u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t} \]
\[ v(t) = \sum_{m=-\infty}^{\infty} V_n e^{i2\pi m F t} \]

If $w(t) = u(t)v(t)$ then $W_r = \sum_{m=-\infty}^{\infty} U_{r-m} V_m \triangleq U_r \ast V_r$

Proof:

\[ w(t) = u(t)v(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t} \sum_{m=-\infty}^{\infty} V_m e^{i2\pi m F t} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U_n V_m e^{i2\pi (m+n) F t} \]

Now we change the summation variable to use $r$ instead of $n$:

\[ r = m + n \Rightarrow n = r - m \]

This is a one-to-one mapping: every pair $(m, n)$ in the range $\pm \infty$ corresponds to exactly one pair $(m, r)$ in the same range.

\[ w(t) = \sum_{r=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U_{r-m} V_m e^{i2\pi r F t} = \sum_{r=-\infty}^{\infty} W_r e^{i2\pi r F t} \]

where $W_r = \sum_{m=-\infty}^{\infty} U_{r-m} V_m \triangleq U_r \ast V_r$.

$W_r$ is the sum of all products $U_n V_m$ for which $m + n = r$.

The spectrum $W_r = U_r \ast V_r$ is called the convolution of $U_r$ and $V_r$. 
Convolution Properties

Convolution behaves algebraically like multiplication:

1) **Commutative**: \( U_r \ast V_r = V_r \ast U_r \)
2) **Associative**: \( U_r \ast V_r \ast W_r = (U_r \ast V_r) \ast W_r = U_r \ast (V_r \ast W_r) \)
3) **Distributive over addition**: \( W_r \ast (U_r + V_r) = W_r \ast U_r + W_r \ast V_r \)
4) **Identity Element or “1”**: If \( I_r = \begin{cases} 1 & r = 0 \\ 0 & r \neq 0 \end{cases} \), then \( I_r \ast U_r = U_r \)

Proofs: (all sums are over \( \pm \infty \))

1) Substitute for \( m \): \( n = r - m \Leftrightarrow m = r - n \) \([1 \leftrightarrow 1 \text{ for any } r]\)
\[
\sum_m U_{r-m}V_m = \sum_n U_nV_{r-n}
\]

2) Substitute for \( n \): \( k = r + m - n \Leftrightarrow n = r + m - k \) \([1 \leftrightarrow 1]\)
\[
\sum_n \left( \sum_m U_{n-m}V_m \right)W_{r-n} = \sum_k \left( \sum_m U_{r-k}V_m \right)W_{k-m}
\]
\[
= \sum_k \sum_m U_{r-k}V_mW_{k-m} = \sum_k \left( U_{r-k} \left( \sum_m V_mW_{k-m} \right) \right)
\]

3) \( \sum_m W_{r-m} (U_m + V_m) = \sum_m W_{r-m}U_m + \sum_m W_{r-m}V_m \)

4) \( I_{r-m}U_m = 0 \) unless \( m = r \). Hence \( \sum_m I_{r-m}U_m = U_r \).
Convolution Example

\[ u(t) = 10 + 8 \sin 2\pi t \quad v(t) = 4 \cos 6\pi t \]

\[ U_{-1:1} = [4i, 10, -4i] \quad V_{-3:3} = [2, 0, 0, 0, 0, 0, 2] \]

\[ w(t) = u(t)v(t) = (10 + 8 \sin 2\pi t) \cdot 4 \cos 6\pi t \]

\[ = 40 \cos 6\pi t + 32 \sin 2\pi t \cos 6\pi t \]

\[ = 40 \cos 6\pi t + 16 \sin 8\pi t - 16 \sin 4\pi t \]

\[ W_{-4:4} = [8i, 20, -8i, 0, 0, 0, 8i, 20, -8i] \]

To convolve \( U_n \) and \( V_n \):
Replace each harmonic in \( V_n \) by a scaled copy of the entire \( \{U_n\} \) (or vice versa) and sum the complex-valued coefficients of any overlapping harmonics.
Two polynomials: \( u(x) = U_3 x^3 + U_2 x^2 + U_1 x + U_0 \)
\[ v(x) = V_2 x^2 + V_1 x + V_0 \]

Now multiply the two polynomials together:
\[ w(x) = u(x)v(x) \]
\[ = U_3 V_2 x^5 + (U_3 V_1 + U_2 V_2) x^4 + (U_3 V_0 + U_2 V_1 + U_1 V_2) x^3 \]
\[ + (U_2 V_0 + U_1 V_1 + U_0 V_2) x^2 + (U_1 V_0 + U_0 V_1) x + U_0 V_0 \]

The coefficient of \( x^r \) consists of all the coefficient pair from \( U \) and \( V \) where the subscripts add up to \( r \). For example, for \( r = 3 \):
\[ W_3 = U_3 V_0 + U_2 V_1 + U_1 V_2 = \sum_{m=0}^{2} U_{3-m} V_m \]

If all the missing coefficients are assumed to be zero, we can write
\[ W_r = \sum_{m=-\infty}^{\infty} U_{r-m} V_m \triangleq U_r * V_r \]

So, to multiply two polynomials, you convolve their coefficient sequences.

Actually, the complex Fourier Series is just a polynomial:
\[ u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i 2 \pi n F_t} = \sum_{n=-\infty}^{\infty} U_n (e^{i 2 \pi F_t})^n \]
Summary

- **Parseval’s Theorem:** \[ \langle u^*(t)v(t) \rangle = \sum_{n=-\infty}^{\infty} U_n^* V_n \]
  - Power Conservation: \[ \langle |u(t)|^2 \rangle = \sum_{n=-\infty}^{\infty} |U_n|^2 \]
  - Or in terms of \( a_n \) and \( b_n \):
    \[ \langle |u(t)|^2 \rangle = \frac{1}{4}a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \]

- **Linearity:** \( w(t) = au(t) + bv(t) \iff W_n = aU_n + bV_n \)

- **Product of signals \( \iff \) Convolution of complex Fourier coefficients:**
  \[ w(t) = u(t)v(t) \iff W_n = U_n * V_n \triangleq \sum_{m=-\infty}^{\infty} U_{n-m}V_m \]

- **Convolution acts like multiplication:**
  - Commutative: \( U * V = V * U \)
  - Associative: \( U * V * W \) is unambiguous
  - Distributes over addition: \( U * (V + W) = U * V + U * W \)
  - Has an identity: \( I_r = 1 \) if \( r = 0 \) and \( = 0 \) otherwise

- **Polynomial multiplication \( \iff \) convolution of coefficients**

For further details see RHB Chapter 12.8.
5: Gibbs Phenomenon

Discontinuities
Discontinuous Waveform
Gibbs Phenomenon
Integration
Rate at which coefficients decrease with $m$
Differentiation
Periodic Extension
$t^2$ Periodic Extension: Method (a)
$t^2$ Periodic Extension: Method (b)
Summary

5: Gibbs Phenomenon
Discontinuities

A function, $v(t)$, has a **discontinuity** of amplitude $b$ at $t = a$ if

$$\lim_{e \to 0} (v(a + e) - v(a - e)) = b \neq 0$$

Conversely, $v(t)$, is **continuous** at $t = a$ if the limit, $b$, equals zero.

**Examples:**

We will see that if a periodic function, $v(t)$, is discontinuous, then its Fourier series behaves in a strange way.
Discontinuous Waveform

Pulse: \( T = \frac{1}{F} = 20 \), width = \( \frac{1}{2} T \), height \( A = 1 \)

\[
U_m = \frac{1}{T} \int_0^{0.5T} A e^{-i2\pi m Ft} dt = \frac{i}{2\pi m FT} \left[ e^{-i2\pi m Ft} \right]^{0.5T}_0 = \frac{i}{2\pi m} \left( e^{-i\pi m} - 1 \right) = \frac{((-1)^m - 1)i}{2\pi m}
\]

\[
= \begin{cases} 
0 & m \neq 0, \text{ even} \\
0.5 & m = 0 \\
-\frac{i}{m\pi} & m \text{ odd}
\end{cases}
\]

So, \( u(t) = \frac{1}{2} + \frac{2}{\pi} \left( \sin 2\pi Ft + \frac{1}{3} \sin 6\pi Ft + \frac{1}{5} \sin 10\pi Ft + \ldots \right) \)

Define: \( u_N(t) = \sum_{m=-N}^{N} U_m e^{i2\pi m Ft} \)

\( u_N(0) = 0.5 \ \forall N \)

\[
\max_t u_N(t) \xrightarrow{N \to \infty} \frac{1}{2} + \frac{1}{\pi} \int_0^\pi \frac{\sin t}{t} dt \approx 1.0895
\]
Expressions involving \((-1)^m\) or, less commonly, \(i^m\) arise quite frequently and it is worth becoming familiar with them. They can arise in several guises:

\[
e^{-i\pi m} = e^{i\pi m} = (e^{i\pi})^m = \cos(\pi m) = (-1)^m
\]
\[
e^{i\frac{1}{2}\pi m} = (e^{i\frac{1}{2}\pi})^m = i^m
\]
\[
e^{-i\frac{1}{2}\pi m} = (e^{-i\frac{1}{2}\pi})^m = (-i)^m
\]

As \(m\) increases these expressions repeat with periods of 2 or 4. Simple expressions involving these quantities make useful sequences.

<table>
<thead>
<tr>
<th>(m)</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-1)^m = \cos \pi m = e^{i\pi m})</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>(i^m = e^{i0.5\pi m})</td>
<td>1</td>
<td>(i)</td>
<td>-1</td>
<td>-i</td>
<td>1</td>
<td>(i)</td>
<td>-1</td>
<td>-i</td>
<td>1</td>
</tr>
<tr>
<td>((-i)^m = e^{-i0.5\pi m})</td>
<td>1</td>
<td>-i</td>
<td>-1</td>
<td>i</td>
<td>1</td>
<td>-i</td>
<td>-1</td>
<td>i</td>
<td>1</td>
</tr>
</tbody>
</table>
| \(
\frac{1}{2} (1 + (-1)^m) \n\) | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| \(
\frac{1}{2} (1 - (-1)^m) \n\) | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| \(
\frac{1}{2} (i^m + (-i)^m) = \cos 0.5\pi m \n\) | 1 | 0 | -1 | 0 | 1 | 0 | -1 | 0 | 1 |
| \(
\frac{1}{4} (1 + (-1)^m + i^m + (-i)^m) \n\) | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
Gibbs Phenomenon

Truncated Fourier Series: 
\[ u_N(t) = \sum_{m=-N}^{N} U_m e^{i2\pi mt} \]

If \( u(t) \) has a discontinuity of height \( b \) at \( t = a \) then:

1. \( u_N(a) \to \lim_{N \to \infty} \frac{u(a-e)+u(a+e)}{2} \)

2. \( u_N(t) \) has an overshoot of about 9% of \( b \) at the discontinuity. For large \( N \) the overshoot moves closer to the discontinuity but does not get smaller (Gibbs phenomenon). In the limit the overshoot equals \( \left( -\frac{1}{2} + \frac{1}{\pi} \int_0^{\pi} \frac{\sin t}{t} dt \right) b \approx 0.0895b \).

3. For large \( m \), the coefficients, \( U_m \) decrease no faster than \( |m|^{-1} \).

Example:

\[ u_N(0) \to 0.5 \]
\[ \max_t u_N(t) \to 1.0895 \ldots \]

\[ U_m = \begin{cases} 0 & m \neq 0, \text{ even} \\ 0.5 & m = 0 \\ -\frac{i}{m\pi} & m \text{ odd} \end{cases} \]
This topic is included for interest but is not examinable.

Our first goal is to express the partial Fourier series, $u_N(t)$, in terms of the original signal, $u(t)$. We begin by substituting the integral expression for $U_n$ in the partial Fourier series

$$u_N(t) = \sum_{n=-N}^{+N} U_n e^{i2\pi n F t} = \sum_{n=-N}^{+N} \left( \frac{1}{T} \int_0^T u(\tau) e^{-i2\pi n F \tau} d\tau \right) e^{i2\pi n F t}$$

Now we swap the order of the integration and the finite summation (OK if the integral converges $\forall n$)

$$u_N(t) = \frac{1}{T} \int_0^T u(\tau) \left( \sum_{n=-N}^{+N} e^{i2\pi n F (t-\tau)} \right) d\tau$$

Now apply the formula for the sum of a geometric progression with $z = e^{i2\pi F(t-\tau)}$:

$$\sum_{n=-N}^{+N} z^n = \frac{z^{-N} - z^{N+1}}{1-z} = \frac{z^{-(N+0.5)} - z^{N+0.5}}{z^{0.5} - z^{0.5}}$$

$$u_N(t) = \frac{1}{T} \int_0^T u(\tau) \frac{e^{i2\pi (N+0.5) F (\tau-t)} - e^{-i2\pi (N+0.5) F (\tau-t)}}{e^{i2\pi 0.5 F (\tau-t)} - e^{-i2\pi 0.5 F (\tau-t)}} d\tau$$

$$= \frac{1}{T} \int_0^T u(\tau) \frac{\sin \pi (2N+1) F (\tau-t)}{\sin \pi F (\tau-t)} d\tau$$

So if we define the Dirichlet Kernel to be $D_N(x) = \frac{\sin((N+0.5)x)}{\sin 0.5x}$, and set $x = 2\pi F(\tau-t)$, we obtain

$$u_N(t) = \frac{1}{T} \int_0^T u(\tau) D_N(2\pi F(\tau-t)) d\tau$$

So what we have shown is that $u_N(t)$ can be obtained by multiplying $u(\tau)$ by a time-shifted Dirichlet Kernel and then integrating over one period. Next we will look at the properties of the Dirichlet Kernel.
This topic is included for interest but is not examinable.

**Dirichlet Kernel** definition: 

\[ D_N(x) = \sum_{n=-N}^{N} e^{inx} = 1 + 2 \sum_{n=1}^{N} \cos nx = \frac{\sin((N+0.5)x)}{\sin 0.5x} \]

\( D_N(x) \) is plotted below for \( N = \{2, 5, 10, 21\} \). The vertical red lines at \( \pm \pi \) mark one period.

- **Periodic:** with period \( 2\pi \)
- **Average value:** \( \langle D_N(x) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x)dx = 1 \)
- **First Zeros:** \( D_N(x) = 0 \) at \( x = \pm \frac{\pi}{N+0.5} \) define the main lobe as \( -\frac{\pi}{N+0.5} < x < \frac{\pi}{N+0.5} \)
- **Peak value:** \( 2N + 1 \) at \( x = 0 \). The main lobe gets narrower but higher as \( N \) increases.
- **Main Lobe semi-integral:**

\[
\int_{x=0}^{\pi} D_N(x)dx = \int_{x=0}^{\pi} \frac{\sin((N+0.5)x)}{\sin 0.5x} \, dx = \frac{1}{N+0.5} \int_{y=0}^{\pi} \frac{\sin y}{\sin \frac{y}{2N+1}} \, dy \quad [y = (N + 0.5)x]
\]

where we substituted \( y = (N + 0.5)x \). Now, for large \( N \), we can approximate \( \sin \frac{y}{2N+1} \approx \frac{y}{2N+1} \):

\[
\int_{x=0}^{\pi} D_N(x)dx \approx \frac{1}{N+0.5} \int_{y=0}^{\pi} \frac{\sin y}{2N+1} \, dy = 2 \int_{y=0}^{\pi} \frac{\sin y}{y} \, dy \approx 3.7038741 \approx 2\pi \times 0.58949
\]

We see that, for large enough \( N \), the main lobe semi-integral is independent of \( N \).

\[ [\text{In MATLAB } D_N(x) = (2N + 1) \times \text{diric}(x, 2N + 1)] \]
This topic is included for interest but is not examinable.

The partial Fourier Series, \( u_N(t) \), can be obtained by multiplying \( u(t) \) by a shifted Dirichlet Kernel and integrating over one period:

\[
u_N(t) = \frac{1}{T} \int_{0}^{T} u(\tau) D_N \left(2\pi F (\tau - t)\right) d\tau
\]

For the special case when \( u(t) \) is a pulse of height 1 and width \( 0.5T \):

\[
u_N(t) = \frac{1}{T} \int_{0}^{0.5T} D_N \left(2\pi F (\tau - t)\right) d\tau
\]

Substitute \( x = 2\pi F (\tau - t) \)

\[
u_N(t) = \frac{1}{2\pi F T} \int_{-2\pi F t}^{2\pi F t} D_N \left(x \right) dx = \frac{1}{2\pi} \int_{-2\pi F t}^{2\pi F t} D_N \left(x \right) dx
\]

- For \( t = 0 \) (the blue integral and the blue circle on the upper graph):
  \( u_N(0) = \frac{1}{2\pi} \int_{0}^{\pi} D_N \left(x \right) dx = 0.5 \)

- For \( t = \frac{T}{2N + 1} \) (the red integral and the red circle on the upper graph):

\[
u_N \left(\frac{T}{2N + 1}\right) = \frac{1}{2\pi} \int_{\frac{-N + 0.5}{N + 0.5}}^{\frac{-N + 0.5}{N + 0.5}} D_N \left(x \right) dx = \frac{1}{2\pi} \int_{0}^{\frac{-N + 0.5}{N + 0.5}} D_N \left(x \right) dx + \frac{1}{2\pi} \int_{0}^{\frac{-N + 0.5}{N + 0.5}} D_N \left(x \right) dx
\]

For large \( N \), we replace the first term by a constant (since it is the semi-integral of the main lobe) and replace the upper limit of the second term by \( \pi \):

\[
\approx 0.58949 + \frac{1}{2\pi} \int_{0}^{\pi} D_N \left(x \right) dx = 1.08949
\]

- For \( 0 < t < 0.5T \), \( u_N(t) \approx 1 \) (the green integral and the green circle on the upper graph).
Suppose \( u(t) = \sum_{m=-\infty}^{\infty} U_m e^{i2\pi m Ft} \)

Define \( v(t) \) to be the integral of \( u(t) \) \[[\text{boundedness requires } U_0 = 0]\]

\[
v(t) = \int^t u(\tau) d\tau = \int^t \sum_{m=-\infty}^{\infty} U_m e^{i2\pi m F \tau} d\tau
= \sum_{m=-\infty}^{\infty} U_m \int^t e^{i2\pi m F \tau} d\tau
= c + \sum_{m=-\infty}^{\infty} U_m \frac{1}{i2\pi m F} e^{i2\pi m F t}
= c + \sum_{m=-\infty}^{\infty} V_m e^{i2\pi m F t}
\]

where \( c \) is an integration constant

Hence \( V_m = \frac{-i}{2\pi m F} U_m \) except for \( V_0 = c \) (arbitrary constant)

Example:

Square wave: \( U_m = \frac{-2i}{m\pi} \) for odd \( m \) \( (0 \text{ for even } m) \)

Triangle wave: \( V_m = \frac{-i}{2\pi m F} \times \frac{-2i}{m\pi} = \frac{-1}{\pi^2 m^2 F} \) for odd \( m \) \( (0 \text{ for even } m) \)

Convergence: \( v(t) \) always converges if \( u(t) \) does since \( V_m \propto \frac{1}{m} U_m \),

\( v_N(t) \) is a good approximation even for small \( N \)
Rate at which coefficients decrease with $m$

Square wave: $U_m = \frac{-2i}{\pi} m^{-1}$ for odd $m$ (0 for even $m$)

Triangle wave: $V_m = \frac{-1}{\pi^2 F} m^{-2}$ for odd $m$ (0 for even $m$)

Integrating $u(t)$ multiplies the $U_m$ by $\frac{-i}{2\pi F} \times m^{-1}$ ⇒ they decrease faster.

The rate at which the coefficients, $U_m$, decrease with $m$ depends on the lowest derivative that has a discontinuity:

- **Discontinuity in $u(t)$ itself** (e.g. square wave)
  For large $|m|$, $U_m$ decreases as $|m|^{-1}$

- **Discontinuity in $u'(t)$** (e.g. triangle wave)
  For large $|m|$, $U_m$ decreases as $|m|^{-2}$

- **Discontinuity in $u^{(n)}(t)$**
  For large $|m|$, $U_m$ decreases as $|m|^{-(n+1)}$

- **No discontinuous derivatives**
  For large $|m|$, $U_m$ decreases faster than any power (e.g. $e^{-|m|}$)
Integration multiplies $U_m$ by $\frac{-i}{2\pi mF}$.

Hence differentiation multiplies $U_m$ by $\frac{2\pi mF}{-i} = i2\pi mF$

If $u(t)$ is a continuous differentiable function and $w(t) = \frac{du(t)}{dt}$ then, provided that $w(t)$ satisfies the Dirichlet conditions, its Fourier coefficients are:

$$W_m = \begin{cases} 0 & m = 0 \\ i2\pi mFU_m & m \neq 0 \end{cases}$$

Since we are multiplying $U_m$ by $m$ the coefficients $W_m$ decrease more slowly with $m$ and so the Fourier series for $w(t)$ may not converge (i.e. $w(t)$ may not satisfy the Dirichlet conditions).

Differentiation makes waveforms spikier and less smooth.
Suppose \( y(t) \) is only defined over a finite interval \((a, b)\).

You have two reasonable choices to make a periodic version:

(a) \( T = b - a \), \( u(t) = y(t) \) for \( a \leq t < b \)

(b) \( T = 2(b - a) \), \( u(t) = \begin{cases} 
    y(t) & a \leq t \leq b \\
    y(2b - t) & b \leq t \leq 2b - a 
\end{cases} \)

Example:
\( y(t) = t^2 \) for \( 0 \leq t < 2 \)

Option (b) has twice the period, no discontinuities, no Gibbs phenomenon overshoots and if \( y(t) \) is continuous the coefficients decrease at least as fast as \( |m|^{-2} \).
$t^2$  Periodic Extension: Method (a)

$y(t) = t^2$ for $0 \leq t < 2$

Method (a): $T = \frac{1}{F} = 2$

$$U_m = \frac{1}{T} \int_0^T t^2 e^{-i2\pi m F t} dt$$

$$= \frac{1}{T} \left[ \frac{t^2 e^{-i2\pi m F t}}{-i2\pi m F} - \frac{2te^{-i2\pi m F t}}{(-i2\pi m F)^2} + \frac{2e^{-i2\pi m F t}}{(-i2\pi m F)^3} \right]_0^T$$

Substitute $e^{-i2\pi m F 0} = e^{-i2\pi m F T} = 1$ [for integer $m$]

$$= \frac{1}{T} \left[ \frac{T^2}{-i2\pi m F} - \frac{2T}{(-i2\pi m F)^2} \right]$$

$$= \frac{2i}{\pi m} + \frac{2}{\pi^2 m^2}$$

$U_{0:3} = [1.333, 0.203 + 0.637i, 0.051 + 0.318i, 0.023 + 0.212i]$
Periodic Extension: Method (b)

\[ y(t) = t^2 \text{ for } 0 \leq t < 2 \]

Method (b): \[ T = \frac{1}{F} = 4 \]

\[
U_m = \frac{1}{T} \int_{-0.5T}^{0.5T} t^2 e^{-i2\pi m F t} \, dt
\]
\[
= \frac{1}{T} \left[ \frac{t^2 e^{-i2\pi m F t}}{-i2\pi m F} - 2t e^{-i2\pi m F t} \left( -i2\pi m F \right)^2 + 2e^{-i2\pi m F t} \left( -i2\pi m F \right)^3 \right] \bigg|_{-0.5T}^{0.5T}
\]

Substitute \( e^{\pm i\pi m F T} = e^{\pm i\pi m} = (-1)^m \) [for integer m]

\[
= \frac{(-1)^m}{T} \left[ \frac{-2T}{(-i2\pi m F)^2} \right]
\]

[all even powers of t cancel out]

\[
= \frac{(-1)^m T^2}{2\pi^2 m^2} = \frac{(-1)^m 8}{\pi^2 m^2}
\]

\[ U_{0:3} = [1.333, -0.811, 0.203, -0.090] \]

[\( u(t) \) real+even \( \Rightarrow U_m \) real]

Convergence is noticeably faster than for method (a)
Summary

- **Discontinuity** at $t = a$
  - Gibbs phenomenon: $u_N(t)$ overshoots by 9% of iump
  - $u_N(a) \to$ mid point of iump

- **Integration**: If $v(t) = \int^t u(\tau)d\tau$, then $V_m = \frac{-i}{2\pi m F} U_m$ and $V_0 = c$, an arbitrary constant. $U_0$ must be zero.

- **Differentiation**: If $w(t) = \frac{du(t)}{dt}$, then $W_m = i2\pi m F U_m$ provided $w(t)$ satisfies Dirichlet conditions (it might not)

- **Rate of decay**:
  - For large $n$, $U_n$ decreases at a rate $|n|^{-(k+1)}$ where $\frac{d^k u(t)}{dt^k}$ is the first discontinuous derivative
  - Error power: $\left\langle (u(t) - u_N(t))^2 \right\rangle = \sum_{|n|>N} |U_n|^2$

- **Periodic Extension** of finite domain signal of length $L$
  - (a) Repeat indefinitely with period $T = L$
  - (b) Reflect alternate repetitions for period $T = 2L$ no discontinuities or Gibbs phenomenon

For further details see RHB Chapter 12.4, 12.5, 12.6
6: Fourier Transform

Fourier Series as $T \to \infty$

Fourier Transform

Examples

Dirac Delta Function

Dirac Delta Function: Scaling and Translation

Dirac Delta Function: Products and Integrals

Periodic Signals

Duality

Time Shifting and Scaling

Gaussian Pulse

Summary
Fourier Series as $T \to \infty$

**Fourier Series:**

$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$

The harmonic frequencies are $nF \ \forall n$ and are spaced $F = \frac{1}{T}$ apart.

As $T$ gets larger, the harmonic spacing becomes smaller.

- e.g. $T = 1 \text{ s} \Rightarrow F = 1 \text{ Hz}$
- $T = 1 \text{ day} \Rightarrow F = 11.57 \mu\text{Hz}$

If $T \to \infty$ then the harmonic spacing becomes zero, the sum becomes an integral and we get the Fourier Transform:

$$u(t) = \int_{f=-\infty}^{+\infty} U(f)e^{i2\pi ft} df$$

Here, $U(f)$, is the *spectral density* of $u(t)$.

- $U(f)$ is a continuous function of $f$.
- $U(f)$ is complex-valued.
- $u(t)$ real $\Rightarrow$ $U(f)$ is conjugate symmetric $\Leftrightarrow U(-f) = U(f)^*.$
- **Units:** if $u(t)$ is in volts, then $U(f)df$ must also be in volts $\Rightarrow U(f)$ is in volts/Hz (hence “spectral density”).
Fourier Transform

Fourier Series: \[ u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t} \]

The summation is over a set of equally spaced frequencies \( f_n = nF \) where the spacing between them is \( \Delta f = F = \frac{1}{T} \).

\[ U_n = \langle u(t) e^{-i2\pi n F t} \rangle = \Delta f \int_{t=-0.5T}^{0.5T} u(t) e^{-i2\pi n F t} dt \]

Spectral Density: If \( u(t) \) has finite energy, \( U_n \to 0 \) as \( \Delta f \to 0 \). So we define a spectral density, \( U(f_n) = \frac{U_n}{\Delta f} \), on the set of frequencies \( \{f_n\} \):

\[ U(f_n) = \frac{U_n}{\Delta f} = \int_{t=-0.5T}^{0.5T} u(t) e^{-i2\pi f_n t} dt \]

we can write \[ u(t) = \sum_{n=-\infty}^{\infty} U(f_n) e^{i2\pi f_n t} \Delta f \] [Substitute \( U_n = U(f_n) \Delta f \)]

Fourier Transform: Now if we take the limit as \( \Delta f \to 0 \), we get

\[ u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi f t} df \] [Fourier Synthesis]

\[ U(f) = \int_{t=-\infty}^{\infty} u(t) e^{-i2\pi f t} dt \] [Fourier Analysis]

For non-periodic signals \( U_n \to 0 \) as \( \Delta f \to 0 \) and \( U(f_n) = \frac{U_n}{\Delta f} \) remains finite. However, if \( u(t) \) contains an exactly periodic component, then the corresponding \( U(f_n) \) will become infinite as \( \Delta f \to 0 \). We will deal with it.
**Fourier Transform Examples**

Example 1:

\[ u(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases} \]

\[ U(f) = \int_{-\infty}^{\infty} u(t) e^{-i2\pi ft} dt \]

\[ = \int_{0}^{\infty} e^{-at} e^{-i2\pi ft} dt \]

\[ = \int_{0}^{\infty} e^{(-a-i2\pi f)t} dt \]

\[ = \frac{-1}{a+i2\pi f} \left[ e^{(-a-i2\pi f)t} \right]_{0}^{\infty} = \frac{1}{a+i2\pi f} \]

Example 2:

\[ v(t) = e^{-a|t|} \]

\[ V(f) = \int_{-\infty}^{\infty} v(t) e^{-i2\pi ft} dt \]

\[ = \int_{-\infty}^{0} e^{at} e^{-i2\pi ft} dt + \int_{0}^{\infty} e^{-at} e^{-i2\pi ft} dt \]

\[ = \frac{1}{a-i2\pi f} \left[ e^{(a-i2\pi f)t} \right]_{-\infty}^{0} + \frac{-1}{a+i2\pi f} \left[ e^{(-a-i2\pi f)t} \right]_{0}^{\infty} \]

\[ = \frac{1}{a-i2\pi f} + \frac{1}{a+i2\pi f} = \frac{2a}{a^2+4\pi^2f^2} \]

\[ [v(t) \text{ real+symmetric}] \quad \Rightarrow \quad [V(f) \text{ real+symmetric}] \]
We define a unit area pulse of width \( w \) as

\[
d_w(x) = \begin{cases} 
\frac{1}{w} & -0.5w \leq x \leq 0.5w \\
0 & \text{otherwise}
\end{cases}
\]

This pulse has the property that its integral equals 1 for all values of \( w \):

\[
\int_{x=-\infty}^{\infty} d_w(x) \, dx = 1
\]

If we make \( w \) smaller, the pulse height increases to preserve unit area.

We define the Dirac delta function as \( \delta(x) = \lim_{w \to 0} d_w(x) \)

- \( \delta(x) \) equals zero everywhere except at \( x = 0 \) where it is infinite.
- However its area still equals 1 \( \Rightarrow \int_{-\infty}^{\infty} \delta(x) \, dx = 1 \)
- We plot the height of \( \delta(x) \) as its area rather than its true height of \( \infty \).

\( \delta(x) \) is not quite a proper function: it is called a generalized function.
**Translation:** \(\delta(x - a)\)

\(\delta(x)\) is a pulse at \(x = 0\)

\(\delta(x - a)\) is a pulse at \(x = a\)

**Amplitude Scaling:** \(b\delta(x)\)

\(\delta(x)\) has an area of 1 \(\Leftrightarrow \int_{-\infty}^{\infty} \delta(x)dx = 1\)

\(b\delta(x)\) has an area of \(b\) since

\[\int_{-\infty}^{\infty} (b\delta(x)) \, dx = b \int_{-\infty}^{\infty} \delta(x) \, dx = b\]

\(b\) can be a complex number (on a graph, we then plot only its magnitude)

**Time Scaling:** \(\delta(cx)\)

\(c > 0:\) \[\int_{x=-\infty}^{\infty} \delta(cx) \, dx = \frac{1}{c} \int_{y=-\infty}^{\infty} \delta(y) \, dy = \frac{1}{c} = \frac{1}{|c|}\] [sub \(y = cx\)]

\(c < 0:\) \[\int_{x=-\infty}^{\infty} \delta(cx) \, dx = \frac{-1}{c} \int_{y=+\infty}^{-\infty} \delta(y) \, dy = \frac{-1}{c} = \frac{1}{|c|}\] [sub \(y = cx\)]

In general, \(\delta(cx) = \frac{1}{|c|} \delta(x)\) for \(c \neq 0\)
If we multiply $\delta(x - a)$ by a function of $x$:

$$y = x^2 \times \delta(x - 2)$$

The product is 0 everywhere except at $x = 2$.

So $\delta(x - 2)$ is multiplied by the value taken by $x^2$ at $x = 2$:

$$x^2 \times \delta(x - 2) = \left[x^2\right]_{x=2} \times \delta(x - 2) = 4 \times \delta(x - 2)$$

In general for any function, $f(x)$, that is continuous at $x = a$,

$$f(x)\delta(x - a) = f(a)\delta(x - a)$$

Integrals:

$$\int_{-\infty}^{\infty} f(x)\delta(x - a)\,dx = \int_{-\infty}^{\infty} f(a)\delta(x - a)\,dx$$

$$= f(a) \int_{-\infty}^{\infty} \delta(x - a)\,dx$$

$$= f(a) \quad [\text{if } f(x) \text{ continuous at } a]$$

Example: $\int_{-\infty}^{\infty} (3x^2 - 2x) \delta(x - 2)\,dx = \left[3x^2 - 2x\right]_{x=2} = 8$
Fourier Transform: \( u(t) = \int_{-\infty}^{\infty} U(f)e^{i2\pi ft} df \)
\( U(f) = \int_{t=-\infty}^{\infty} u(t)e^{-i2\pi ft} dt \)

Example: \( U(f) = 1.5\delta(f+2) + 1.5\delta(f-2) \)
\( u(t) = \int_{-\infty}^{\infty} U(f)e^{i2\pi ft} df \)
\[ = \int_{-\infty}^{\infty} 1.5\delta(f+2)e^{i2\pi ft} df \]
\[ + \int_{-\infty}^{\infty} 1.5\delta(f-2)e^{i2\pi ft} df \]
\[ = 1.5 \left[ e^{i2\pi ft} \right]_f=-2 + 1.5 \left[ e^{i2\pi ft} \right]_f=+2 \]
\[ = 1.5 \left( e^{i4\pi t} + e^{-i4\pi t} \right) = 3\cos 4\pi t \]

If \( u(t) \) is periodic then \( U(f) \) is a sum of Dirac delta functions:
\[ u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt} \Rightarrow U(f) = \sum_{n=-\infty}^{\infty} U_n \delta(f-nF) \]

Proof: \( u(t) = \int_{-\infty}^{\infty} U(f)e^{i2\pi ft} df \)
\[ = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} U_n \delta(f-nF) e^{i2\pi ft} df \]
\[ = \sum_{n=-\infty}^{\infty} U_n \int_{-\infty}^{\infty} \delta(f-nF) e^{i2\pi ft} df \]
\[ = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt} \]
Duality

**Fourier Transform:**
\[ u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df \]
\[ U(f) = \int_{-\infty}^{\infty} u(t) e^{-i2\pi ft} dt \]  

[Fourier Synthesis]  
[Fourier Analysis]

**Dual transform:**

Suppose \( v(t) = U(t) \), then
\[ V(f) = \int_{-\infty}^{\infty} v(t) e^{-i2\pi ft} d\tau \]
\[ V(g) = \int_{-\infty}^{\infty} U(t) e^{-i2\pi gt} dt \]
\[ = \int_{f=-\infty}^{\infty} U(f) e^{-i2\pi gf} df \]
\[ = u(-g) \]

So:
\[ v(t) = U(t) \implies V(f) = u(-f) \]

**Example:**
\[ u(t) = e^{-|t|} \implies U(f) = \frac{2}{1+4\pi^2 f^2} \]  
[from earlier]
\[ v(t) = \frac{2}{1+4\pi^2 t^2} \implies V(f) = e^{-|f|} = e^{-|f|} \]
Time Shifting and Scaling

Fourier Transform: 
\[ u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df \]
\[ U(f) = \int_{-\infty}^{\infty} u(t) e^{-i2\pi ft} dt \]

[Fourier Synthesis] [Fourier Analysis]

Time Shifting and Scaling:

Suppose \( v(t) = u(at + b) \), then
\[
V(f) = \int_{t=-\infty}^{\infty} u(at + b)e^{-i2\pi ft} dt
\]
\[
= \text{sgn}(a) \int_{\tau=-\infty}^{\infty} u(\tau)e^{-i2\pi f\left(\frac{\tau-b}{a}\right)} \frac{1}{|a|} d\tau
\]

note that \( \pm \infty \) limits swap if \( a < 0 \) hence \( \text{sgn}(a) = \begin{cases} 1 & a > 0 \\ -1 & a < 0 \end{cases} \)

\[
= \frac{1}{|a|} e^{i2\pi \frac{fb}{a}} \int_{\tau=-\infty}^{\infty} u(\tau)e^{-i2\pi \frac{f}{a} \tau} d\tau
\]
\[
= \frac{1}{|a|} e^{i2\pi \frac{fb}{a}} U \left( \frac{f}{a} \right)
\]

So: \( v(t) = u(at + b) \quad \Rightarrow \quad V(f) = \frac{1}{|a|} e^{i2\pi \frac{fb}{a}} U \left( \frac{f}{a} \right) \)
Gaussian Pulse: \( u(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} \)

This is a Normal (or Gaussian) probability distribution, so \( \int_{-\infty}^{\infty} u(t) dt = 1 \).

\[
U(f) = \int_{-\infty}^{\infty} u(t) e^{-i2\pi ft} dt = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} e^{-i2\pi ft} dt
\]

\[
= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} (t^2 + i4\pi\sigma^2 ft)} dt
\]

\[
= e^{\frac{1}{2\sigma^2} (i2\pi\sigma^2 f)^2} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} (t+i2\pi\sigma^2 f)^2} dt
\]

\[
= e^{\frac{1}{2\sigma^2} (i2\pi\sigma^2 f)^2}
\]

\[
[\text{(i) uses a result from complex analysis theory that:}]
\]

\[
\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} (t+i2\pi\sigma^2 f)^2} dt = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} t^2} dt = 1
\]

Uniquely, the Fourier Transform of a Gaussian pulse is a Gaussian pulse.
Summary

- **Fourier Transform:**
  - Inverse transform (synthesis): \( u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df \)
  - Forward transform (analysis): \( U(f) = \int_{t=-\infty}^{\infty} u(t) e^{-i2\pi ft} dt \)
    - \( U(f) \) is the spectral density function (e.g. Volts/Hz)
- **Dirac Delta Function:**
  - \( \delta(t) \) is a zero-width infinite-height pulse with \( \int_{-\infty}^{\infty} \delta(t) dt = 1 \)
  - Integral: \( \int_{-\infty}^{\infty} f(t) \delta(t - a) = f(a) \)
  - Scaling: \( \delta(ct) = \frac{1}{|c|} \delta(t) \)
- **Periodic Signals:** \( u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n Ft} \)
  \( \Rightarrow U(f) = \sum_{n=-\infty}^{\infty} U_n \delta(f - nF) \)
- **Fourier Transform Properties:**
  - \( v(t) = U(t) \quad \Rightarrow \quad V(f) = u(-f) \)
  - \( v(t) = u(at + b) \quad \Rightarrow \quad V(f) = \frac{1}{|a|} e^{i\frac{2\pi fb}{a}} U \left( \frac{f}{a} \right) \)
  - \( v(t) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2} \left( \frac{t}{\sigma} \right)^2} \quad \Rightarrow \quad V(f) = e^{-\frac{1}{2} \left( 2\pi \sigma f \right)^2} \)

For further details see RHB Chapter 13.1 (uses \( \omega \) instead of \( f \))
7: Fourier Transforms: Convolution and Parseval’s Theorem
Multiplication of Signals

Question: What is the Fourier transform of \( w(t) = u(t)v(t) \)?

Let \( u(t) = \int_{h=-\infty}^{+\infty} U(h)e^{i2\pi ht} \, dh \) and \( v(t) = \int_{g=-\infty}^{+\infty} V(g)e^{i2\pi gt} \, dg \)

[Note use of different dummy variables]

\[
\begin{align*}
  w(t) &= u(t)v(t) \\
  &= \int_{h=-\infty}^{+\infty} U(h)e^{i2\pi ht} \, dh \int_{g=-\infty}^{+\infty} V(g)e^{i2\pi gt} \, dg \\
  &= \int_{h=-\infty}^{+\infty} U(h) \int_{g=-\infty}^{+\infty} V(g)e^{i2\pi (h+g)t} \, dg \, dh \\
  &\quad \text{[merge } e^{i2\pi \cdot \cdot \cdot}] \\
\end{align*}
\]

Now we make a change of variable in the second integral: \( g = f - h \)

\[
\begin{align*}
  &= \int_{h=-\infty}^{+\infty} U(h) \int_{f=-\infty}^{+\infty} V(f-h)e^{i2\pi ft} \, df \, dh \\
  &= \int_{f=-\infty}^{+\infty} \int_{h=-\infty}^{+\infty} U(h)V(f-h)e^{i2\pi ft} \, dh \, df \\
  &\quad \text{[swap } \int] \\
  &= \int_{f=-\infty}^{+\infty} W(f)e^{i2\pi ft} \, df \\
\end{align*}
\]

where \( W(f) = \int_{h=-\infty}^{+\infty} U(h)V(f-h) \, dh \int_{h=-\infty}^{+\infty} U(h)V(f-h) \, dh \triangleq U(f) * V(f) \)

This is the convolution of the two spectra \( U(f) \) and \( V(f) \).

\[
\begin{align*}
  w(t) = u(t)v(t) &\quad \iff \quad W(f) = U(f) * V(f)
\end{align*}
\]
Multiplication Example

\[ u(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases} \]

\[ U(f) = \frac{1}{a+i2\pi f} \quad \text{[from before]} \]

\[ v(t) = \cos 2\pi F t \]

\[ V(f) = 0.5(\delta(f+F) + \delta(f-F)) \]

\[ w(t) = u(t)v(t) \]

\[ W(f) = U(f) \ast V(f) = \frac{0.5}{a+i2\pi(f+F)} + \frac{0.5}{a+i2\pi(f-F)} \]

If \( V(f) \) consists entirely of Dirac impulses then \( U(f) \ast V(f) \) just replaces each impulse with a complete copy of \( U(f) \) scaled by the area of the impulse and shifted so that 0 Hz lies on the impulse. Then add the overlapping complex spectra.
Convolution Theorem:
\[ w(t) = u(t)v(t) \iff W(f) = U(f) * V(f) \]
\[ w(t) = u(t) * v(t) \iff W(f) = U(f)V(f) \]

Convolution in the time domain is equivalent to multiplication in the frequency domain and vice versa.

Proof of second line:

Given \( u(t) \), \( v(t) \) and \( w(t) \) satisfying
\[ w(t) = u(t)v(t) \iff W(f) = U(f) * V(f) \]
define dual waveforms \( x(t) \), \( y(t) \) and \( z(t) \) as follows:

\[ x(t) = U(t) \iff X(f) = u(-f) \]
\[ y(t) = V(t) \iff Y(f) = v(-f) \]
\[ z(t) = W(t) \iff Z(f) = w(-f) \]

Now the convolution property becomes:
\[ w(-f) = u(-f)v(-f) \iff W(t) = U(t) * V(t) \quad \text{[sub } t \leftrightarrow \pm f\text{]} \]
\[ Z(f) = X(f)Y(f) \iff z(t) = x(t) * y(t) \quad \text{[duality]} \]
Convolution Example

\[ u(t) = \begin{cases} 
1 - t & 0 \leq t < 1 \\
0 & \text{otherwise}
\end{cases} \]

\[ v(t) = \begin{cases} 
e^{-t} & t \geq 0 \\
0 & t < 0
\end{cases} \]

\[ w(t) = u(t) \ast v(t) = \int_{-\infty}^{\infty} u(\tau)v(t - \tau) d\tau = \int_{0}^{\min(t,1)} (1 - \tau)e^{\tau} - t d\tau = [2 - \tau]^{\min(t,1)}_{\tau=0} = \begin{cases} 
0 & t < 0 \\
2 - t - 2e^{-t} & 0 \leq t < 1 \\
(e - 2)e^{-t} & t \geq 1
\end{cases} \]

Note how \( v(t - \tau) \) is time-reversed (because of the \(-\tau\)) and time-shifted to put the time origin at \( \tau = t \).
Convolution Properties

**Convolution:** \( w(t) = u(t) \ast v(t) \triangleq \int_{-\infty}^{\infty} u(\tau)v(t - \tau)d\tau \)

Convolution behaves algebraically like multiplication:

1) **Commutative:** \( u(t) \ast v(t) = v(t) \ast u(t) \)

2) **Associative:**
   \[
   u(t) \ast v(t) \ast w(t) = (u(t) \ast v(t)) \ast w(t) = u(t) \ast (v(t) \ast w(t))
   \]

3) **Distributive over addition:**
   \[
   w(t) \ast (u(t) + v(t)) = w(t) \ast u(t) + w(t) \ast v(t)
   \]

4) **Identity Element or “1”:** \( u(t) \ast \delta(t) = \delta(t) \ast u(t) = u(t) \)

5) **Bilinear:** \( (au(t)) \ast (bv(t)) = ab (u(t) \ast v(t)) \)

Proof: In the frequency domain, convolution is multiplication.

Also, if \( u(t) \ast v(t) = w(t) \), then

6) **Time Shifting:** \( u(t+a) \ast v(t+b) = w(t+a+b) \)

7) **Time Scaling:** \( u(at) \ast v(at) = \frac{1}{|a|}w(at) \)

How to recognise a convolution integral:

the arguments of \( u(\cdot\cdot\cdot) \) and \( v(\cdot\cdot\cdot) \) sum to a constant.
Lemma:

\[ X(f) = \delta(f-g) \Rightarrow x(t) = \int \delta(f-g) e^{i2\pi ft} df = e^{i2\pi g t} \]

\[ \Rightarrow X(f) = \int e^{i2\pi g t} e^{-i2\pi ft} dt = \int e^{i2\pi(g-f)t} dt = \delta(g-f) \]

Parseval's Theorem:

\[ \int_{t=-\infty}^{\infty} u^*(t)v(t) dt = \int_{f=-\infty}^{\infty} U^*(f)V(f) df \]

Proof:

Let \( u(t) = \int_{f=-\infty}^{\infty} U(f)e^{i2\pi ft} df \) and \( v(t) = \int_{g=-\infty}^{\infty} V(g)e^{i2\pi gt} dg \)

[Note use of different dummy variables]

Now multiply \( u^*(t) = u(t) \) and \( v(t) \) together and integrate over time:

\[ \int_{t=-\infty}^{\infty} u^*(t)v(t) dt \]

\[ = \int_{t=-\infty}^{\infty} \int_{f=-\infty}^{\infty} U^*(f)e^{-i2\pi ft} df \int_{g=-\infty}^{\infty} V(g)e^{i2\pi gt} dg dt \]

\[ = \int_{f=-\infty}^{\infty} U^*(f) \int_{g=-\infty}^{\infty} V(g) \int_{t=-\infty}^{\infty} e^{i2\pi(g-f)t} dt dg df \]

\[ = \int_{f=-\infty}^{\infty} U^*(f) \int_{g=-\infty}^{\infty} V(g) \delta(g-f) dg df \] [lemma]

\[ = \int_{f=-\infty}^{\infty} U^*(f)V(f) df \]
Parseval’s Theorem: \[ \int_{t=-\infty}^{\infty} u^*(t)v(t)\,dt = \int_{f=-\infty}^{+\infty} U^*(f)V(f)\,df \]

For the special case \(v(t) = u(t)\), Parseval’s theorem becomes:
\[ \int_{t=-\infty}^{\infty} u^*(t)u(t)\,dt = \int_{f=-\infty}^{+\infty} U^*(f)U(f)\,df \]
\[ \Rightarrow E_u = \int_{t=-\infty}^{\infty} |u(t)|^2\,dt = \int_{f=-\infty}^{+\infty} |U(f)|^2\,df \]

Energy Conservation: The energy in \(u(t)\) equals the energy in \(U(f)\).

Example:
\[ u(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases} \Rightarrow E_u = \int |u(t)|^2\,dt = \left[ -\frac{e^{-2at}}{2a} \right]_0^\infty = \frac{1}{2a} \]

\[ U(f) = \frac{1}{a + i2\pi f} \quad \text{[from before]} \]
\[ \Rightarrow \int |U(f)|^2\,df = \int \frac{df}{a^2 + 4\pi^2 f^2} \]
\[ = \left[ \tan^{-1}\left( \frac{2\pi f}{a} \right) \right]_{-\infty}^{\infty} = \frac{\pi}{2\pi a} = \frac{1}{2a} \]
Example from before:

\[ w(t) = \begin{cases} e^{-at} \cos 2\pi F t & t \geq 0 \\ 0 & t < 0 \end{cases} \]

\[ W(f) = \frac{0.5}{a+i2\pi(f+F)} + \frac{0.5}{a+i2\pi(f-F)} \]

\[ = \frac{a+i2\pi f}{a^2+i4\pi af-4\pi^2(f^2-F^2)} \]

\[ |W(f)|^2 = \frac{a^2+4\pi^2f^2}{(a^2-4\pi^2(f^2-F^2))^2+16\pi^2a^2f^2} \]

- The units of \(|W(f)|^2\) are “energy per Hz” so that its integral, \(E_w = \int_{-\infty}^{\infty} |W(f)|^2 \, df\), has units of energy.

- The quantity \(|W(f)|^2\) is called the energy spectral density of \(w(t)\) at frequency \(f\) and its graph is the energy spectrum of \(w(t)\). It shows how the energy of \(w(t)\) is distributed over frequencies.

- If you divide \(|W(f)|^2\) by the total energy, \(E_w\), the result is non-negative and integrates to unity like a probability distribution.
Summary

- **Convolution:**
  - \( u(t) \ast v(t) \triangleq \int_{-\infty}^{\infty} u(\tau)v(t - \tau) d\tau \)
  - \( \triangleright \) Arguments of \( u(\cdot \cdot \cdot) \) and \( v(\cdot \cdot \cdot) \) sum to \( t \)
  - Acts like multiplication + time scaling/shifting formulae

- **Convolution Theorem:** multiplication \( \leftrightarrow \) convolution
  - \( w(t) = u(t)v(t) \Leftrightarrow W(f) = U(f) \ast V(f) \)
  - \( w(t) = u(t) \ast v(t) \Leftrightarrow W(f) = U(f)V(f) \)

- **Parseval’s Theorem:**
  - \( \int_{t=-\infty}^{\infty} u^*(t)v(t)dt = \int_{f=-\infty}^{+\infty} U^*(f)V(f)df \)

- **Energy Spectrum:**
  - Energy spectral density: \( |U(f)|^2 \) (energy/Hz)
  - Parseval: \( E_u = \int |u(t)|^2 dt = \int |U(f)|^2 df \)

For further details see RHB Chapter 13.1
8: Correlation

Cross-Correlation
Signal Matching
Cross-corr as Convolution
Normalized Cross-corr
Autocorrelation
Autocorrelation example
Fourier Transform Variants
Scale Factors
Summary
Spectrogram
The **cross-correlation** between two signals $u(t)$ and $v(t)$ is

$$
w(t) = u(t) \otimes v(t) \triangleq \int_{-\infty}^{\infty} u^*(\tau)v(\tau + t)d\tau
= \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)d\tau \quad \text{[sub: } \tau \to \tau - t]\]

The complex conjugate, $u^*(\tau)$ makes no difference if $u(t)$ is real-valued but makes the definition work even if $u(t)$ is complex-valued.

**Correlation versus Convolution:**

$$
u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau)v(\tau + t)d\tau \quad \text{[correlation]}
\quad u(t) \ast v(t) = \int_{-\infty}^{\infty} u(\tau)v(t - \tau)d\tau \quad \text{[convolution]}
$$

Unlike convolution, the integration variable, $\tau$, has the same sign in the arguments of $u(\cdots)$ and $v(\cdots)$ so the arguments have a constant difference instead of a constant sum (i.e. $v(t)$ is not time-flipped).

**Notes:**
(a) The argument of $w(t)$ is called the “lag” (\(\equiv\) delay of $u$ versus $v$).
(b) Some people write $u(t) \ast v(t)$ instead of $u(t) \otimes v(t)$.
(c) Some swap $u$ and $v$ and/or negate $t$ in the integral.

It is all rather inconsistent ☹.
Cross correlation is used to find where two signals match: $u(t)$ is the test waveform.

**Example 1:**
$v(t)$ contains $u(t)$ with an unknown delay and added noise.

$$w(t) = u(t) \otimes v(t) = \int u^*(\tau - t)v(\tau)d\tau$$

gives a peak at the time lag where $u(\tau - t)$ best matches $v(\tau)$; in this case at $t = 450$

**Example 2:**
$y(t)$ is the same as $v(t)$ with more noise
$z(t) = u(t) \otimes y(t)$ can still detect the correct time delay (hard for humans)

**Example 3:**
$p(t)$ contains $-u(t)$ so that
$q(t) = u(t) \otimes p(t)$ has a negative peak
Correlation: \[ w(t) = u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)d\tau \]

If we define \( x(t) = u^*(-t) \) then
\[
x(t) \ast v(t) \triangleq \int_{-\infty}^{\infty} x(t - \tau)v(\tau)d\tau = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)d\tau
= u(t) \otimes v(t)
\]

Fourier Transform of \( x(t) \):
\[
X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt = \int_{-\infty}^{\infty} u^*(-t)e^{-i2\pi ft}dt
= \int_{-\infty}^{\infty} u^*(t)e^{i2\pi ft}dt = \left( \int_{-\infty}^{\infty} u(t)e^{-i2\pi ft}dt \right)^{*}
= U^*(f)
\]

So \( w(t) = x(t) \ast v(t) \Rightarrow W(f) = X(f)V(f) = U^*(f)V(f) \)

Hence the Cross-correlation theorem:
\[
w(t) = u(t) \otimes v(t) \quad \Leftrightarrow \quad W(f) = U^*(f)V(f)
= u^*(-t) \ast v(t)
\]

Note that, unlike convolution, correlation is not associative or commutative:
\[
v(t) \otimes u(t) = v^*(-t) \ast u(t) = u(t) \ast v^*(-t) = w^*(-t)
\]
Normalized Cross-correlation

Correlation: \( w(t) = u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)\,d\tau \)

If we define \( y(t) = u(t - t_0) \) for some fixed \( t_0 \), then \( E_y = E_u \):

\[
E_y = \int_{-\infty}^{\infty} |y(t)|^2\,dt = \int_{-\infty}^{\infty} |u(t - t_0)|^2\,dt = \int_{-\infty}^{\infty} |u(\tau)|^2\,d\tau = E_u \quad [t \rightarrow \tau + t_0]
\]

Cauchy-Schwarz inequality: \( \left| \int_{-\infty}^{\infty} y^*(\tau)v(\tau)\,d\tau \right|^2 \leq E_y E_v \)

\[
\Rightarrow |w(t_0)|^2 = \left| \int_{-\infty}^{\infty} u^*(\tau - t_0)v(\tau)\,d\tau \right|^2 \leq E_y E_v = E_u E_v \]

but \( t_0 \) was arbitrary, so we must have \( |w(t)| \leq \sqrt{E_u E_v} \) for all \( t \)

We can define the normalized cross-correlation

\[
z(t) = \frac{u(t) \otimes v(t)}{\sqrt{E_u E_v}}
\]

with properties: (1) \( |z(t)| \leq 1 \) for all \( t \)

(2) \( |z(t_0)| = 1 \Leftrightarrow v(\tau) = \alpha u(\tau - t_0) \) with \( \alpha \) constant
You do not need to memorize this proof

We want to prove the Cauchy-Schwarz Inequality:

\[ \left| \int_{-\infty}^{\infty} u^*(t)v(t)dt \right|^2 \leq E_u E_v \]

where \( E_u \triangleq \int_{-\infty}^{\infty} |u(t)|^2 dt \).

Suppose we define \( w \triangleq \int_{-\infty}^{\infty} u^*(t)v(t)dt \). Then,

\[
0 \leq \int |E_v u(t) - w^* v(t)|^2 dt \leq \]

\[
\int (E_v u^*(t) - w v^*(t)) (E_v u(t) - w^* v(t)) dt \leq \]

\[
E_v^2 \int u^*(t)u(t)dt + |w|^2 \int v^*(t)v(t)dt - w^* E_v \int u^*(t)v(t)dt - w E_v \int u(t)v^*(t)dt \]

\[
= E_v^2 \int |u(t)|^2 dt + |w|^2 \int |v(t)|^2 dt - E_v w^* w - E_v w w^* \]

\[
= E_v^2 E_u + |w|^2 E_v - 2 |w|^2 E_v = E_v \left( E_u E_v - |w|^2 \right) \]

Unless \( E_v = 0 \) (in which case, \( v(t) \equiv 0 \) and the C-S inequality is true), we must have \( |w|^2 \leq E_u E_v \)

which proves the C-S inequality.

Also, \( E_u E_v = |w|^2 \) only if we have equality in the first line,

that is, \( \int |E_v u(t) - w^* v(t)|^2 dt = 0 \) which implies that the integrand is zero for all \( t \).

This implies that \( u(t) = \frac{w^*}{E_v} v(t) \).

So we have shown that \( E_u E_v = |w|^2 \) if and only if \( u(t) \) and \( v(t) \) are proportional to each other.
The correlation of a signal with itself is its *autocorrelation*: 

\[ w(t) = u(t) \otimes u(t) = \int_{-\infty}^{\infty} u^*(\tau - t)u(\tau)d\tau \]

The autocorrelation at zero lag:

\[ w(0) = \int_{-\infty}^{\infty} u^*(\tau - 0)u(\tau)d\tau = \int_{-\infty}^{\infty} u^*(\tau)u(\tau)d\tau = \int_{-\infty}^{\infty} |u(\tau)|^2 d\tau = E_u \]

The autocorrelation at zero lag, \( w(0) \), is the energy of the signal.

The normalized autocorrelation: 

\[ z(t) = \frac{u(t) \otimes u(t)}{E_u} \]

satisfies \( z(0) = 1 \) and \( |z(t)| \leq 1 \) for any \( t \).

**Wiener-Khinchin Theorem:** [Cross-correlation theorem when \( v(t) = u(t) \)]

\[ w(t) = u(t) \otimes u(t) \iff W(f) = U^*(f)U(f) = |U(f)|^2 \]

The Fourier transform of the autocorrelation is the energy spectrum.
Cross-correlation is used to find when two different signals are similar. Autocorrelation is used to find when a signal is similar to itself delayed.

First graph shows \( s(t) \) a segment of the microphone signal from the initial vowel of “early” spoken by me. The waveform is “quasi-periodic” = “almost periodic but not quite”.

Second graph shows normalized autocorrelation, \( z(t) = \frac{s(t) \otimes s(t)}{E_s} \).

\( z(0) = 1 \) for \( t = 0 \) since a signal always matches itself exactly.

\( z(t) = 0.82 \) for \( t = 6.2 \text{ ms} \) = one period lag (not an exact match).

\( z(t) = 0.53 \) for \( t = 12.4 \text{ ms} \) = two periods lag (even worse match).
There are three different versions of the Fourier Transform in current use.

1. **Frequency version** (we have used this in lectures)
   \[
   U(f) = \int_{-\infty}^{\infty} u(t) e^{-i2\pi ft} dt \quad u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df
   \]
   - Used in the communications/broadcasting industry and textbooks.
   - The formulae do not need scale factors of \(2\pi\) anywhere.

2. **Angular frequency version**
   \[
   \tilde{U}(\omega) = \int_{-\infty}^{\infty} u(t) e^{-i\omega t} dt \quad u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{U}(\omega) e^{i\omega t} d\omega
   \]
   Continuous spectra are unchanged: \(\tilde{U}(\omega) = U(f) = U\left(\frac{\omega}{2\pi}\right)\)
   However \(\delta\)-function spectral components are multiplied by \(2\pi\) so that
   \[
   U(f) = \delta(f - f_0) \Rightarrow \tilde{U}(\omega) = 2\pi \times \delta(\omega - 2\pi f_0)
   \]
   - Used in most signal processing and control theory textbooks.

3. **Angular frequency + symmetrical scale factor**
   \[
   \hat{U}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t) e^{-i\omega t} dt \quad u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{U}(\omega) e^{i\omega t} d\omega
   \]
   In all cases \(\hat{U}(\omega) = \frac{1}{\sqrt{2\pi}} \tilde{U}(\omega)\)
   - Used in many Maths textbooks (mathematicians like symmetry)
Fourier Transform using Angular Frequency:

\[ \tilde{U}(\omega) = \int_{-\infty}^{\infty} u(t) e^{-i\omega t} dt \quad u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{U}(\omega) e^{i\omega t} d\omega \]

Any formula involving \( \int df \) will change to \( \frac{1}{2\pi} \int d\omega \) \[\text{[since } d\omega = 2\pi df\]\n
Parseval’s Theorem:

\[ \int u^*(t)v(t) dt = \frac{1}{2\pi} \int \tilde{U}^*(\omega)\tilde{V}(\omega) d\omega \]

\[ E_u = \int |u(t)|^2 dt = \frac{1}{2\pi} \int |\tilde{U}(\omega)|^2 d\omega \]

Waveform Multiplication: (convolution implicitly involves integration)

\[ w(t) = u(t)v(t) \Rightarrow \tilde{W}(\omega) = \frac{1}{2\pi} \tilde{U}(\omega) \ast \tilde{V}(\omega) \]

Spectrum Multiplication: (multiplication \( \not\Rightarrow \) integration)

\[ w(t) = u(t) \ast v(t) \Rightarrow \tilde{W}(\omega) = \tilde{U}(\omega)\tilde{V}(\omega) \]

To obtain formulae for version (3) of the Fourier Transform, \( \hat{U}(\omega) \), substitute into the above formulae: \( \tilde{U}(\omega) = \sqrt{2\pi}\hat{U}(\omega) \).
Summary

Cross-Correlation: \( w(t) = u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)d\tau \)
- Used to find similarities between \( v(t) \) and a delayed \( u(t) \)
- Cross-correlation theorem: \( W(f) = U^*(f)V(f) \)
- Cauchy-Schwarz Inequality: \( |u(t) \otimes v(t)| \leq \sqrt{E_u E_v} \)
  - Normalized cross-correlation: \( \left| \frac{u(t) \otimes v(t)}{\sqrt{E_u E_v}} \right| \leq 1 \)

Autocorrelation: \( x(t) = u(t) \otimes u(t) = \int_{-\infty}^{\infty} u^*(\tau - t)u(\tau)d\tau \leq E_u \)
- Wiener-Khinchin: \( X(f) = \) energy spectral density, \( |U(f)|^2 \)
- Used to find periodicity in \( u(t) \)

Fourier Transform using \( \omega \):
- Continuous spectra unchanged; spectral impulses multiplied by \( 2\pi \)
- In formulae: \( \int df \rightarrow \frac{1}{2\pi} \int d\omega \); \( \omega \)-convolution involves an integral

For further details see RHB Chapter 13.1
Spectrogram of “Merry Christmas” spoken by Mike Brookes
All waveforms have period $T = 1$. $\delta_{\text{condition}}$ is 1 whenever “condition” is true and otherwise 0.

| Waveform         | $x(t)$ for $|t| < 0.5$                                      | $X_n$                                                                 |
|------------------|----------------------------------------------------------|----------------------------------------------------------------------|
| Square wave      | $2\delta_{|t|<0.25} - 1$                                | $\frac{2 \sin 0.5\pi n}{\pi n} \times \delta_{n\neq0}$            |
| Pulse of width $d$ | $\delta_{|t|<0.5d}$                                     | $\frac{\sin \pi dn}{\pi n}$                                        |
| Sawtooth wave    | $2t$                                                     | $\frac{i(-1)^n}{\pi n} \times \delta_{n\neq0}$                    |
| Triangle wave    | $1 - 4|t|$                                               | $\frac{2(1-(-1)^n)}{\pi^2 n^2}$                                   |
You need not memorize these properties. All integrals are $\int_{-\infty}^{\infty}$

<table>
<thead>
<tr>
<th>Property</th>
<th>$x(t)$</th>
<th>$X(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forward</td>
<td>$x(t)$</td>
<td>$\int x(t)e^{-i2\pi ft}dt$</td>
</tr>
<tr>
<td>Inverse</td>
<td>$\int X(f)e^{i2\pi ft}df$</td>
<td>$X(f)$</td>
</tr>
<tr>
<td>Spectral Zero</td>
<td>$\int x(t)dt$</td>
<td>$= X(0)$</td>
</tr>
<tr>
<td>Temporal Zero</td>
<td>$x(0)$</td>
<td>$= \int X(f)df$</td>
</tr>
<tr>
<td>Duality</td>
<td>$X(t)$</td>
<td>$x(-f)$</td>
</tr>
<tr>
<td>Reversal</td>
<td>$x(-t)$</td>
<td>$X(-f)$</td>
</tr>
<tr>
<td>conjugate</td>
<td>$x^*(t)$</td>
<td>$X^*(-f)$</td>
</tr>
<tr>
<td>Temporal Derivative</td>
<td>$\frac{d^n}{dt^n}x(t)$</td>
<td>$(i2\pi f)^n X(f)$</td>
</tr>
<tr>
<td>Spectral Derivative</td>
<td>$(-i2\pi t)^n x(t)$</td>
<td>$\frac{d^n}{df^n}X(f)$</td>
</tr>
<tr>
<td>Integral</td>
<td>$\int_{-\infty}^{t} x(\tau)d\tau$</td>
<td>$\frac{1}{i2\pi f} X(f) + \frac{1}{2}X(0)\delta(f)$</td>
</tr>
<tr>
<td>Scaling</td>
<td>$x(\alpha t + \beta)$</td>
<td>$\frac{1}{</td>
</tr>
<tr>
<td>Time Shift</td>
<td>$x(t - T)$</td>
<td>$X(f)e^{-i2\pi ft}$</td>
</tr>
<tr>
<td>Frequency Shift</td>
<td>$x(t)e^{i2\pi Ft}$</td>
<td>$X(f - F)$</td>
</tr>
</tbody>
</table>
You need not memorize these properties. All integrals are $\int_{-\infty}^{\infty}$

<table>
<thead>
<tr>
<th>Property</th>
<th>$x(t)$</th>
<th>$X(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$\alpha x(t) + \beta y(t)$</td>
<td>$\alpha X(f) + \beta Y(f)$</td>
</tr>
<tr>
<td>Multiplication</td>
<td>$x(t)y(t)$</td>
<td>$X(f) * Y(f)$</td>
</tr>
<tr>
<td>Convolution</td>
<td>$x(t) * y(t)$</td>
<td>$X(f)Y(f)$</td>
</tr>
<tr>
<td>Correlation</td>
<td>$x(t) \otimes y(t)$</td>
<td>$X^*(f)Y(f)$</td>
</tr>
<tr>
<td>Autocorrelation</td>
<td>$x(t) \otimes x(t)$</td>
<td>$</td>
</tr>
<tr>
<td>Parseval or Plancherel</td>
<td>$\int x^<em>(t)y(t)dt$ = $\int X^</em>(f)Y(f)df$</td>
<td>$\int</td>
</tr>
<tr>
<td>Repetition</td>
<td>$\sum_n x(t-nT)$</td>
<td>$\left</td>
</tr>
<tr>
<td>Sampling</td>
<td>$\sum_n x(nT)\delta(t-nT)$</td>
<td>$\left</td>
</tr>
<tr>
<td>Modulation</td>
<td>$x(t) \cos(2\pi F t)$</td>
<td>$\frac{1}{2} X(f-F) + \frac{1}{2} X(f+F)$</td>
</tr>
</tbody>
</table>

Convolution: $x(t) * y(t) = \int x(\tau)y(t-\tau)d\tau$

Cross-correlation: $x(t) \otimes y(t) = \int x^*(\tau)y(\tau + t)d\tau = \int x^*(\tau - t)y(\tau)d\tau$
You need not memorize these pairs.

<table>
<thead>
<tr>
<th>$x(t)$</th>
<th>$X(f)$</th>
<th>$x(t)$</th>
<th>$X(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(t)$</td>
<td>$\frac{1}{\pi f}$</td>
<td>$1$</td>
<td>$\delta(f)$</td>
</tr>
<tr>
<td>$\text{rect}(t)$</td>
<td>$\frac{\sin(\pi f)}{\pi f}$</td>
<td>$\frac{\sin(t)}{t}$</td>
<td>$\pi \text{rect}(\pi f)$</td>
</tr>
<tr>
<td>$\text{tri}(t)$</td>
<td>$\frac{\sin^2(\pi f)}{\pi^2 f^2}$</td>
<td>$\frac{\sin^2(t)}{t^2}$</td>
<td>$\pi \text{tri}(\pi f)$</td>
</tr>
<tr>
<td>$\cos(2\pi \alpha t)$</td>
<td>$\frac{1}{2} \delta(f + \alpha) + \frac{1}{2} \delta(f - \alpha)$</td>
<td>$\sin(2\pi \alpha t)$</td>
<td>$\frac{i}{2} \delta(f + \alpha) - \frac{i}{2} \delta(f - \alpha)$</td>
</tr>
<tr>
<td>$e^{-\alpha t} u(t)$</td>
<td>$\frac{1}{\alpha + 2\pi i f}$</td>
<td>$\frac{1}{(\alpha + 2\pi i f)^2}$</td>
<td>$\frac{1}{e^{-\alpha t}} u(t)$</td>
</tr>
<tr>
<td>$e^{-\alpha</td>
<td>t</td>
<td>}$</td>
<td>$\frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$</td>
</tr>
<tr>
<td>$\text{sgn}(t)$</td>
<td>$\frac{1}{i\pi f}$</td>
<td>$u(t)$</td>
<td>$\frac{1}{2} \delta(f) + \frac{1}{2\pi i f}$</td>
</tr>
<tr>
<td>$\sum_{n=-\infty}^{\infty} \delta(t - nT)$</td>
<td>$\frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{T})$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Elementary Functions:

$$
\text{rect}(t) = \begin{cases} 
1, & |t| < 0.5 \\
0, & \text{elsewhere}
\end{cases}
\quad \text{tri}(t) = \begin{cases} 
1 - |t|, & |t| < 1 \\
0, & \text{elsewhere}
\end{cases}
$$

$$
\text{sgn}(t) = \begin{cases} 
-1, & t < 0 \\
0, & t = 0 \\
1, & t > 0
\end{cases}
\quad u(t) = \frac{1}{2} (1 + \text{sgn}(t)) = \begin{cases} 
0, & x < 0 \\
0.5, & x = 0 \\
1, & x > 0
\end{cases}
$$