E1.10 Fourier Series and Transforms

Problem Sheet 1 (Lecture 1)

Key: [A]= easy ... [E]=hard

Questions from RBH textbook: 4.2, 4.8.

1. [B] Using the geometric progression formula, evaluate $\sum_{r=1}^{5} 3^r$.
2. [B] Determine expressions not involving a summation for
   (a) $\sum_{r=1}^{10} 3x^{2r}$, (b) $\sum_{r=0}^{10} \frac{2}{x^r}$, (c) $\sum_{r=0}^{R} x^r y^{r-2}$, (d) $\sum_{r=0}^{R} (-1)^r$.
3. [B] In the expression $\sum_{r=1}^{5} 3^r$, make the substitution $r = m + 1$ and then evaluate the resultant expression.
4. [C] Determine a simplified expression not involving a summation for $\sum_{r=-N}^{N} e^{j\omega r}$ for $N \geq 0$.
5. [C] Determine a simplified expression for $\sum_{r=0}^{R-1} e^{j2\pi r R^{-1}}$ for $R \geq 1$. Ensure your answer is correct even when $R = 1$.
6. [C] Determine the value of $\sum_{n=0}^{N} \sum_{m=1}^{M} 2x^{m-n}$.
7. [C] If $x(t) = \sin t$, determine (a) $\langle x(t) \rangle$, (b) $\langle |x(t)| \rangle$ and (c) $\left\langle x^2(t) \right\rangle$ where $\langle \cdots \rangle$ denotes the time-average.
8. [C] The first two normalized Legendre polynomials are $P_0(t) = 1$ and $P_1(t) = \sqrt{3}t$.
   (a) Show that $\left\langle P_0^2(t) \right\rangle_{[-1,1]} = \left\langle P_1^2(t) \right\rangle_{[-1,1]} = 1$ and $\langle P_0(t)P_1(t) \rangle_{[-1,1]} = 0$ where $\langle \cdots \rangle_{[-1,1]}$ denotes the average over the interval $-1 < t < 1$.
   (b) If $P_2(t) = at^2 + bt + c$, find the coefficients a, b and c such that
   $\langle P_0(t)P_2(t) \rangle_{[-1,1]} = \langle P_1(t)P_2(t) \rangle_{[-1,1]} = 0$ and $\left\langle P_2^2(t) \right\rangle_{[-1,1]} = 1$

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Problem Sheet 1 - Solutions

1. $\sum_{r=1}^{5} 3^r = 3 \times \frac{1-3^5}{1-3} = 3 \times \frac{-242}{-2} = 3 \times 121 = 363$. In the expression $3 \times \frac{1-3^5}{1-3}$, the "5" is the number of terms in the sum and the "$3\times$" is the first term (when $r = 1$).

2. (a) Each term is multiplied by a factor of x^2, so the standard formula gives $3x^2 \times \frac{1-x^{20}}{1-x^2}$ where $x^{20} = \left(x^2\right)^{10}$ since there are 10 terms.
(b) Each term is multiplied by a factor x^{-1} and, treating $x^0 = 1$, the first term equals 2, so the sum is $2 \times \frac{1-x^{-11}}{1-x^{-1}}$.
(c) Each term is multiplied by xy and the first term is y^{-2} so the sum is $y^{-2} \times \frac{1-(xy)^{R+1}}{1-xy}$.
(d) Each term is multiplied by -1 and the first term is $-1^0 = 1$ so the sum is

$$\frac{1-(-1)^{R+1}}{1-(-1)} = \frac{1+(-1)^R}{2} = \begin{cases} 1 & R \text{ even} \\ 0 & R \text{ odd} \end{cases}.$$

3. The substitution $r = m + 1 \Leftrightarrow m = r - 1$. So, making the substitution in both the limits and summand gives

$$\sum_{r=1}^{5} 3^r = \sum_{m=0}^{4} 3^{m+1} = 3\sum_{m=0}^{4} 3^m = 3 \times \frac{1-3^5}{1-3}.$$

So the answer is 363 as in question 1.

4. Each term is multiplied by $e^{j\omega}$ and the first of the $2N + 1$ terms is $e^{-j\omega N}$ so the sum is

$$e^{-j\omega N}\frac{1-e^{j\omega(2N+1)}}{1-e^{j\omega}} = \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1-e^{j\omega}}.$$

A very common trick when an expression includes the sum or difference of two exponentials is to take out a factor whose exponent is the average of the two original exponents; in this case the average exponent is $j0.5\omega$ in the denominator and also in the numerator since $\frac{-j\omega N + j\omega(N+1)}{2} = j0.5\omega$. This gives

$$\frac{e^{j0.5\omega}\left(e^{-j\omega(N+0.5)} - e^{j\omega(N+0.5)}\right)}{e^{j0.5\omega}\left(e^{-j0.5\omega} - e^{j0.5\omega}\right)} = \frac{e^{-j\omega(N+0.5)} - e^{j\omega(N+0.5)}}{e^{-j0.5\omega} - e^{j0.5\omega}} = \frac{-2j\sin((N+0.5)\omega)}{-2j\sin 0.5\omega} = \frac{\sin((N+0.5)\omega)}{\sin 0.5\omega}$$

5. Each term is multiplied by $e^{j2\pi R^{-1}}$ and the first term is $e^0 = 1$ so the sum formula gives

$$\frac{1-e^{j2\pi R R^{-1}}}{1-e^{j2\pi R^{-1}}} = \frac{1-e^{j2\pi}}{1-e^{j2\pi R^{-1}}} = \frac{0}{1-e^{j2\pi R^{-1}}} = 0.$$

However, when $R = 1$, the denominator is zero so the formula is invalid; in this case there is only one term in the summation and it equals $e^{j2\pi 0\times 1} = 1$. So the answer is 0 for all values of R except $R = 1$ when the answer is 1. We can write this compactly as

$$\delta[R-1] = \begin{cases} 1 & R = 1 \\ 0 & R > 1 \end{cases}$$

where the function $\delta[n]$ is the "Kroneker Delta function" and equals 1 if and only if its integer argument equals zero.

6. The summand in this question is "separable" because it can be expressed as the product of two factors that depend on m and n respectively. So we can write

$$\sum_{n=0}^{N}\sum_{m=1}^{M} 2x^{m-n} = 2\sum_{n=0}^{N} x^{-n} \sum_{m=1}^{M} x^m = 2\frac{1-x^{-(N+1)}}{1-x^{-1}} x \frac{1-x^M}{1-x} = \frac{2x^2\left(x^{-(N+1)}-1\right)\left(1-x^M\right)}{(1-x)^2}.$$

7. (a)The period is 2π, so we calculate the average by integrating over one period and dividing by the period: $\langle x(t)\rangle = \frac{1}{2\pi}\int_0^{2\pi}\sin t\,dt = \frac{1}{2\pi}\left[-\cos t\right]_0^{2\pi} = 0$.
(b) The period is now π, so we calculate the average as: $\langle |x(t)|\rangle = \frac{1}{\pi}\int_0^{\pi}|\sin t|\,dt = \frac{1}{\pi}\int_0^{\pi}\sin t\,dt = \frac{1}{\pi}\left[-\cos t\right]_0^{\pi} = \frac{2}{\pi}$.
(c) The period is still π:

$$\left\langle x^2(t)\right\rangle = \frac{1}{\pi}\int_0^{\pi}\sin^2 t\,dt = \frac{1}{\pi}\int_0^{\pi}\frac{1}{2}\left(1-\cos 2t\right)\,dt = \frac{1}{2\pi}\int_0^{\pi}\left(1-\cos 2t\right)\,dt = \frac{1}{2\pi}\left[t-\frac{1}{2}\sin 2t\right]_0^{\pi} = \frac{\pi}{2\pi} = \frac{1}{2}.$$

An easier way of getting this answer is to write

$$\left\langle \sin^2 t\right\rangle = \frac{1}{2}\left(\langle 1\rangle - \langle\cos 2t\rangle\right) = \frac{1}{2}\left(1-0\right) = \frac{1}{2}.$$

8. (a)

$$\left\langle P_0^2(t)\right\rangle_{[-1,1]} = \frac{1}{2}\int_{-1}^{1}1^2 dt = \frac{1}{2}\left[t\right]_{-1}^{1} = 1$$

and

$$\left\langle P_1^2(t)\right\rangle_{[-1,1]} = \frac{1}{2}\int_{-1}^{1}3t^2 dt = \frac{1}{2}\left[t^3\right]_{-1}^{1} = 1.$$

Finally

$$\left\langle P_0(t)P_1(t)\right\rangle_{[-1,1]} = \frac{\sqrt{3}}{2}\int_{-1}^{1}t\,dt = \frac{\sqrt{3}}{4}\left[t^2\right]_{-1}^{1} = 0.$$

(b) The analysis is slightly easier if you do it in the right order.

$$\left\langle P_1(t)P_2(t)\right\rangle_{[-1,1]} = \frac{\sqrt{3}}{2}\int_{-1}^{1}at^3+bt^2+ct\,dt = \frac{\sqrt{3}}{2}\left[\frac{at^4}{4}+\frac{bt^3}{3}+\frac{ct^2}{2}\right]_{-1}^{1} = \frac{b}{\sqrt{3}} = 0$$

so $b=0$. Now

$$\left\langle P_0(t)P_2(t)\right\rangle_{[-1,1]} = \frac{1}{2}\int_{-1}^{1}at^2+c\,dt = \frac{1}{2}\left[\frac{at^3}{3}+ct\right] = \frac{a}{3}+c = 0$$

so $a=-3c$. Finally

$$\left\langle P_2^2(t)\right\rangle_{[-1,1]} = \frac{c^2}{2}\int_{-1}^{1}9t^4-6t^2+1\,dt = \frac{c^2}{2}\left[\frac{9t^5}{5}-2t^3+t\right]_{-1}^{1} = c^2\left(\frac{9}{5}-2+1\right) = \frac{4}{5}c^2 = 1$$

from which $c=\pm\frac{\sqrt{5}}{2}$. So the polynomial is $P_2(t) = \frac{\sqrt{5}}{2}\left(3t^2-1\right)$.

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Problem Sheet 2 (Lectures 2, 3)

Key: [A]= easy ... [E]=hard

Questions from RBH textbook: 12.1, 12.2, 12.3, 12.4, 12.5, 12.8, 12.9, 12.10, 12.11, 12.12, 12.13, 12.14, 12.15, 12.17, 12.20, 12.21, 12.22, 12.26.

1. [B] Give the fundamental period of (a) $\cos 1000\pi t$, (b) $\cos 1000\pi t + 0.01\cos 1250\pi t$, (c) $\cos 1000\pi t + \cos 1000t$.

2. [C] A sufficient condition for a periodic function, $u(t)$, to have a Fourier series is that it satisfies the Dirichlet conditions on page 2-5 of the notes. Determine which of the following functions satisfies these conditions. The notation $x \mod n$ means the remainder when x is divided by n.

   (a) $\sin^2 t$, (b) $\frac{1}{\sin t}$, (c) $\sqrt{\frac{1}{|\sin t|}}$, (d) $\frac{1}{1+t^2}$, (e) $t \mod 1$.

3. [B] Determine the fundamental frequency and the Fourier Series coefficients for $u(t) = 1+2\cos(6000\pi t)+3\sin(4000\pi t)$.

4. [B] The phasor $2 + 4i$ represents the waveform $2\cos\omega t - 4\sin\omega t$. Give (a) the Fourier coefficients and (b) the complex Fourier coefficients for this waveform.

5. [C] Determine the fundamental frequency and the Fourier Series coefficients for $u(t) = \cos^4(2000\pi t)$.

6. [C] (a) Determine the Fourier coefficients, $\{a_n, b_n\}$ for the waveform, $u(t)$, with period $T = 2$ defined by $u(t) = 3t$ for $-1 \le t < 1$.

   (b) Determine the complex Fourier coefficients, U_n, for the same waveform.

   (c) Determine the complex Fourier coefficients for $v(t) = u(t-1)$.

   (d) Determine the complex Fourier coefficients for $w(t) = 2v(t) + 4$.

7. [B] If $u(t)$ has period $T = \frac{1}{F}$ and Fourier coefficients $a_{0:2} = [5, 2, 3]$ and $b_1 = 1$ with all other coefficients zero. (a) Give an expression for $u(t)$, (b) Determine the complex Fourier coefficients, U_n.

8. [B] Each of the following waveforms has period $T = 2$ and equals the expression given for $-1 \le t < 1$. In each case say whether the complex Fourier coefficients will be (i) real-valued, (ii) purely imaginary or (iii) neither.

   (a) t^2 (b) t^3 (c) $2t+t^2$ (d) t^2+1 (e) t^3+1 (f) $t\sin t$ (g) $t\cos 2t$ (h) $t^2 \sin t$.

9. [C] Each of the following waveforms has period $T = 2$ and equals the expression given for $-1 \le t < 1$. In each case say whether or not all the even-numbered Fourier coefficients will equal zero.

   (a) $\sin \pi t$ (b) $\begin{cases} t+1 & t<0 \\ -t & t \ge 0 \end{cases}$ (c) $\begin{cases} t+1 & t<0 \\ 1-t & t \ge 0 \end{cases}$ (d) $t(1-|t|)$ (e) $t^3 - t$.

10. [C] $u(t)$ has period $T = 4$ and is defined by $u(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 4 \end{cases}$.

    (a) Find the complex Fourier coefficients, U_n expressing them in polar form: $r \times e^{i\theta}$. Identify which of the coefficients are equal to zero.

    (b) Find the complex Fourier coefficients of $v(t) = u(t + 0.5)$ and explain why they are necessarily real-valued. Explain the relation between the magnitudes $|V_n|$and $|U_n|$.

    (c) Find the complex Fourier coefficients of $w(t) = v(t) + v(t-2)$. Identify which of the coefficients are non-zero and explain how your answer relates to the symmetries of $w(t)$.

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Problem Sheet 2 - Solutions

1. (a) The fundamental frequency is $\frac{1000\pi}{2\pi} = 500\,\text{Hz}$ and the period is $\frac{2\pi}{1000\pi} = 2\,\text{ms}$.

   (b) For a mixture of cosine waves, the fundamental period is the lowest common multiple (LCM) of the periods of the constituent waves. In this case the frequencies of the two waves are 500 Hz and 625 Hz with periods 2 ms and 1.6 ms respectively. The LCM of 2 and 1.6 is 8 ms corresponding to a frequency of 125 Hz. Alternatively, you can obtain the same answer by finding the highest common factor (HCF) of the two frequencies. Notice that the addition of even a tiny amplitude at 625 Hz has quadrupled the fundamental period.

   (c) The periods of the two constituent waves are here $2\,ms$ and $\frac{2\pi}{1000} = 6.28\ldots\,ms$. Since the second of these is irrational, there is no LCM and the resultant waveform is not periodic at all (or equivalently its period is ∞).

2. (a) Yes. (b) No.
   We have $T = 2\pi$, so $\int_0^T \left|\frac{1}{\sin t}\right| dt = 2\int_0^\pi \frac{1}{\sin t} dt = 2\left[\ln\left(\tan\left(0.5t\right)\right)\right]_0^\pi = 2\ln\left(\frac{\tan 0.5\pi}{\tan 0}\right) = 2\ln\frac{\infty}{0} = \infty$.
   (c) Yes for a similar reason to the previous part since $\int_0^T \sqrt{\frac{1}{\tau}} d\tau = \left[\sqrt{\tau}\right]_0^T = \sqrt{T} < \infty$.
   (d) No since it is not periodic. (e) Yes; this function is a triangle wave with period 1.

3. The fundamental frequency is the highest common factor of the constituent frequencies, i.e. $2000\pi\,\text{rad/s} = 1\,\text{kHz}$. So, setting $F = 1000$, we have $u(t) = 1 + 2\cos\left(2\pi 3Ft\right) + 3\sin\left(2\pi 2Ft\right)$. So the Fourier coefficients are $a_0 = 2$, $b_2 = 3$, $a_3 = 2$ with all other coefficients zero.

4. (a) The non-zero Fourier coefficients are $a_1 = 2$ and $b_1 = -4$.

   (b) The non-zero complex Fourier coefficients are $U_{-1} = 1 - 2i$ and $U_1 = 1 + 2i$. We see that $U_1$is exactly half the value of the phasor and that $U_{-1}$is the complex conjugate of U_1.

5. We need to express $\cos^4\theta$ in terms of components of the form $\cos n\theta$. We can do this by writing

$$\begin{aligned}
\cos^4\theta &= \frac{1}{16}\left(e^{i\theta} + e^{-i\theta}\right)^4 \\
&= \frac{1}{16}\left(e^{i4\theta} + 4e^{i2\theta} + 6 + 4e^{-i2\theta} + e^{-i4\theta}\right) \\
&= \frac{1}{16}\left(2\cos 4\theta + 8\cos 2\theta + 6\right)
\end{aligned}$$

   From this we find that the fundamental frequency is actually $4000\pi\,\text{rad/s} = 2\,\text{kHz}$ and the non-zero Fourier coefficients are therefore $a_0 = \frac{3}{4}$, $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{8}$.

6. (a) From the formulae on slide 2-11 of the notes (and observing that $F = \frac{1}{T} = \frac{1}{2}$),

$$\begin{aligned}
a_n &= \frac{2}{T}\int_{-1}^{1} u(t)\cos\left(2\pi nFt\right) dt \\
&= \int_{-1}^{1} 3t\cos\left(\pi nt\right) dt \\
&= \frac{3}{\pi^2 n^2}\left[\pi nt\sin\left(\pi nt\right) + \cos\left(\pi nt\right)\right]_{t=-1}^{1} \\
&= 0 \\
b_n &= \int_{-1}^{1} 3t\sin\left(\pi nt\right) dt \\
&= \frac{3}{\pi^2 n^2}\left[-\pi nt\cos\left(\pi nt\right) + \sin\left(\pi nt\right)\right]_{t=-1}^{1} \\
&= \frac{-6}{\pi n}\cos\left(\pi n\right) = \frac{-6\left(-1\right)^n}{\pi n}
\end{aligned}$$

   Note that the b_n expression applies only for $n \geq 1$. Notice also that the a_n are all zero because $u(t)$ is a real-valued odd function and that the coefficient magnitudes are $\propto n^{-1}$ which is a characteristic of waveforms that include a discontinuity.

(b) We have $U_0 = \frac{1}{2}a_0 = 0$ and, for $n \geq 1$,

$$U_{\pm n} = \frac{1}{2}\left(a_{|n|} \mp ib_{|n|}\right) = \frac{\pm i3\,(-1)^n}{\pi\,|n|}$$

from which $U_n = \frac{i3(-1)^n}{\pi n}$. Again, we note that U_n is purely imaginary because $u(t)$ is a real-valued odd function.

(c) We can calculate the V_n directly by integrating over an interval that does not include a discontinuity.

$$\begin{aligned}
V_0 &= \frac{1}{2}\int_0^2 v(t)dt = \frac{1}{2}\int_0^2 (t-1)dt = 0 \\
\text{for } n \neq 0:\ V_n &= \frac{1}{2}\int_0^2 v(t)e^{-i\pi nt}dt \\
&= \frac{3}{2}\int_0^2 (t-1)\,e^{-i\pi nt}dt \\
&= \frac{3}{2\pi^2 n^2}\left[(1 + i\pi n\,(t-1))\,e^{-i\pi nt}\right]_{t=0}^2 \\
&= \frac{3i\pi n}{\pi^2 n^2} = \frac{3i}{\pi n}
\end{aligned}$$

An alternative way to calculate V_n is to use the time-shifting formula:

$$\begin{aligned}
v(t) = u(t-1) \Rightarrow V_n &= U_n e^{-i2\pi nF} \\
&= U_n e^{-in\pi} \\
&= (-1)^n\, U_n.
\end{aligned}$$

Notice that time-shifting a waveform changes the phases of the $V_n$but not the magnitudes.

(d) The complex Fourier transform of $x(t) = 4$ is iust $X_0 = 4$ with all other coefficients zero. So, since the Fourier transform is linear, if $w(t) = 2v(t) + x(t)$ we must have $W_n = 2V_n + X_n$ which means that $W_0 = 4$ and, for $n \neq 0$, $W_n = \frac{6i}{\pi n}$.

7. (a) $u(t) = 2.5 + 2\cos(2\pi Ft) + \sin(2\pi Ft) + 3\cos(4\pi Ft)$.

   (b) $U_{\pm n} = \frac{1}{2}\left(a_{|n|} \mp ib_{|n|}\right)$ from which $U_{-2:2} = [1.5,\ 1 + 0.5i,\ 2.5,\ 1 - 0.5i,\ 1.5]$. Notice that, since $u(t)$ is real, U_{-n} is the complex conjugate of U_{+n}.

8. The Fourier transform of a real-valued signal is purely real or purely imaginary if it is even or odd respectively. So we have the following: (a) real (b) imaginary (c) neither (d) real (e) neither (f) real (g) imaginary (g) imaginary.

9. All the even-numbered Fourier coefficients of a waveform are zero if it is anti-periodic which, in this case with $T = 2$, means that $u(t) = -u(t-1)$ for $0 \leq t < 1$; note that you only need to prove this relationship for half a period since the periodicity relationship, $u(t) = u(t+T)$, then means that it applies for the other half. So we have the following ("Yes" means it is anti-periodic):

   (a) Yes: $\sin \pi t = -\sin(\pi t - \pi)$ and of course the Fourier transform has only a single component with $b_1 = 1$ or, equivalently, $X_{\pm 1} = \mp i$.

   (b) Yes: for $0 \leq t < 1$, $u(t) = -t$ and $u(t-1) = (t-1) + 1 = t = -u(t)$

   (c) No: for $0 \leq t < 1$, $u(t) = 1 - t$ but $u(t-1) = (t-1) + 1 = t \neq -u(t)$. Note however that $v(t) = u(t) - 0.5$ is anti-periodic since, for $0 \leq t < 1$, $v(t) = u(t) - 0.5 = t + 0.5 = -((1-(t-1)) - 0.5) = -(u(t-1) - 0.5) = -v(t-1)$. Thus the only non-zero even harmonic of $u(t)$ is $U_0 = 0.5$.

   (d) Yes: for $0 \leq t < 1$, $u(t) = t\,(1-t)$ and $u(t-1) = (t-1)\,(1 + (t-1)) = (t-1)\,t = -u(t)$

   (e) No: for $0 \leq t < 1$, $u(t) = t\left(t^2 - 1\right)$ but $u(t-1) = (t-1)^3 - (t-1) = t\left(t^2 - 3t + 2t\right) \neq -u(t)$.

10. (a) We note that the fundamental frequency is $F = \frac{1}{4}$.

$$\begin{aligned} U_n = \frac{1}{4}\int_0^1 e^{-i0.5\pi n t}dt &= \frac{i}{4\times 0.5\pi n}\left(e^{-i0.5\pi n} - 1\right) \\ &= \frac{ie^{-i0.25\pi n}}{2\pi n}\left(e^{-i0.25\pi n} - e^{i0.25\pi n}\right) \\ &= \frac{ie^{-i0.25\pi n}}{2\pi n} \times -2i\sin 0.25\pi n \\ &= \frac{\sin 0.25\pi n}{\pi n} \times e^{-i0.25\pi n} \end{aligned}$$

We know that $\sin\theta = 0$ whenever θ is a multiple of π, so $U_n = 0$ whenever n is a non-zero multiple of 4.

(b) By the time-shift formula $V_n = U_n e^{i2\pi n F 0.5} = U_n e^{i0.25\pi n} = \frac{\sin 0.25\pi n}{\pi n}$. Since $v(t)$ is real and symmetric, $V_n$will also be real and symmetric. The time-shifting affects only the phase and so $|V_n| = |U_n|$.

(c) By linearity and the time-shift formula, $W_n = V_n\left(1 + e^{i2\pi n F 2}\right) = V_n\left(1 + e^{i\pi n}\right) = V_n\left(1 + (-1)^n\right)$. The quantity $\left(1 + (-1)^n\right)$ equals 2 for even values of n and 0 for odd values of n. Since in addition, $V_n = 0$ when n is a non-zero multiple of 4, W_n is only non-zero for $n = 0$ and for odd multiplies of 2. In fact, the period of $w(t)$ is 2 rather than 4 which explains why $W_n = 0$ for all odd values of n. In addition, when considered with a period of 2, $(w(t) - W_0)$ is anti-periodic and so all its even Fourier coefficients will be zero.

E1.10 Fourier Series and Transforms

Problem Sheet 3 (Lectures 4, 5)

Key: [A]= easy ... [E]=hard

Questions from RBH textbook: 12.19, 12.23, 12.25.

1. [C] (a) Determine the fundamental frequency, the Fourier coefficients and the complex Fourier coefficients of $u(t) = \cos^2 t$.

   (b) Determine the power, $P_u = \langle u^2(t) \rangle$ where $\langle \cdots \rangle$ denotes the time average. Hint: $\cos^4 t = \frac{1}{8}\cos 4t + \frac{1}{2}\cos 2t + \frac{3}{8}$.

   (c) Show that Parseval's theorem applies: $P_u = \sum_{n=-\infty}^{\infty} |U_n|^2 = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$.

2. [C] The even function $u(t)$ with period $T = 1$ is defined in the region $|t| \leq \frac{1}{2}$ by $u(t) = \begin{cases} a^{-1} & |t| \leq \frac{a}{2} \\ 0 & |t| > \frac{a}{2} \end{cases}$ where $0 < a < 1$.

   (a) Determine the complex Fourier coefficients, U_n.

   (b) Explain why U_0 does not depend on a.

   (c) Show that $\sum_{n=-\infty}^{\infty}\left(\frac{\sin an\pi}{an\pi}\right)^2 = \frac{1}{a}$.

3. [C] Determine the fundamental frequency and the complex Fourier Series coefficients of $x(t) = (6 + 4\cos 8\pi t)\cos 20\pi t$ in two ways: (a) by expanding our the product using trigonometrical formulae and (b) by convolving the Fourier coefficients of the two factors.

4. [C] (a) Give the complex Fourier coefficients, U_n, if $u(t) = \cos t$. (b) Show, by using the convolution theorem, that $v(t) = u^2(t) = \frac{1}{2}\cos 2t + \frac{1}{2}$. (c) Show, by using the convolution theorem again that $w(t) = v^2(t) = u^4(t) = \frac{1}{8}\cos 4t + \frac{1}{2}\cos 2t + \frac{3}{8}$.

5. [C] Suppose $u(t) = \sin t$ and $v(t) = \begin{cases} 1 & 0 \leq t < \pi \\ 0 & \pi \leq t < 2\pi \end{cases}$ both with period $T = 2\pi$.

   (a) Determine the complex Fourier coefficients U_n and V_n.

   (b) If $w(t) = u(t)v(t)$, determine $W_n = U_n * V_n$ by convolving U_n and V_n.

6. [B] The waveform $u(t)$ has period $T = 1$ and equals $u(t) = 4t - 1$ for $0 \leq t < 1$. If $u_N(t) = \sum_{n=-N}^{N} U_n e^{i2\pi nt}$ estimate, for large N, the minimum value and maximum value of $u_N(t)$ and also the value of $u_N(0)$.

7. [B] The waveform $u(t)$ has period $T = 1$. Estimate how rapidly U_n will decrease with $|n|$ when $u(t)$ in the range $0 \leq t < 1$ is given by

   (a) t, (b) t^2, (c) $t\,(1-t)$, (d) $t^2\,(1-t)^2$, (e) $2t^3 - 3t^2 + t + 1$

8. [C] The waveform $u(t)$ has period $T_u = 1$ and satisfies $u(t) = \exp t$ for $0 \leq t < 1$. The waveform $v(t)$ has period $T_v = 2$ and satisfies $v(t) = \exp|t|$ for $-1 \leq t < 1$.

   (a) Find expressions for the complex Fourier coefficients U_n and V_n.
   (b) Calculate the average powers $\langle u^2(t) \rangle$ and $\langle v^2(t) \rangle$ and also those of $\langle u_2^2(t) \rangle$ and $\langle v_2^2(t) \rangle$ where $u_N(t)$ is the waveform formed by summing harmonics $-N$ to $+N$.

   (c) Determine the average error powers $\left\langle (u(t) - u_2(t))^2 \right\rangle$ and $\left\langle (v(t) - v_2(t))^2 \right\rangle$.

E1.10 Fourier Series and Transforms

Problem Sheet 3 - Solutions

1. (a) We have $u(t) = \cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$. So the fundamental period is $T = \pi$ and the fundamental frequency is $F = \frac{1}{T} = \frac{1}{\pi}$. The Fourier coefficients are $a_0 = 1$ and $a_1 = \frac{1}{2}$, so the complex Fourier coefficients are $U_0 = \frac{1}{2}$, $U_{-1} = U_1 = \frac{1}{4}$.

   (b) $P_u = \frac{1}{\pi}\int_0^\pi \cos^4 t\, dt = \frac{1}{32\pi}\left[12t + 8\sin 2t + \sin 4t\right]_0^\pi = \frac{1}{32\pi}(12\pi + 0 + 0) = \frac{3}{8}$.

   (c) $\sum_{n=-\infty}^{\infty} |U_n|^2 = \left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{3}{8}$. Also $\frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \frac{1}{4}\times 1^2 + \frac{1}{2}\times\left(\frac{1}{2}\right)^2 = \frac{3}{8}$. Note that the formula for Parseval's theorem is much more elegant and memorable when using complex Fourier coefficients.

2. (a) We have

$$\begin{aligned} U_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)e^{-i2\pi nFt}dt \\ &= \frac{1}{1}\int_{-\frac{a}{2}}^{\frac{a}{2}} a^{-1}e^{-i2\pi nt}dt \\ &= \frac{i}{2an\pi}\left[e^{-i2\pi nt}\right]_{t=-\frac{a}{2}}^{\frac{a}{2}} \\ &= \frac{-i}{2an\pi}\left(e^{i\pi na} - e^{-i\pi na}\right) \\ &= \frac{\sin an\pi}{an\pi} \end{aligned}$$

   Note that U_n is real-valued and even as expected since $u(t)$ is real-valued and even.

   (b) From the formula $U_0 = \left.\frac{\sin an\pi}{an\pi}\right|_{n=0}$ but this is not defined so we either determine U_0 directly from the original integral as $U_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)\,dt = 1$ or else as a limit: $U_0 = \lim_{n\to 0}\frac{\sin an\pi}{an\pi}$. We can find this limit using L'Hôpital's rule: $\lim_{n\to 0}\frac{\sin an\pi}{an\pi} = \left.\frac{a\pi\cos an\pi}{a\pi}\right|_{n=0} = 1$ or, equivalently, by using the small angle approximation, $\sin x \approx x$, which is exact for $x = 0$ and gives $U_0 = \lim_{n\to 0}\frac{\sin an\pi}{an\pi} = \frac{an\pi}{an\pi} = 1$. It is always true that $U_0 = \langle u(t)\rangle$ so since the average value of $u(t)$ is 1 for all values of a, it follows that U_0 will not depend on a.

   (c) We can calculate

$$\begin{aligned} \left\langle |u(t)|^2 \right\rangle &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u^2(t)\,dt \\ &= \int_{-\frac{w}{2}}^{\frac{w}{2}} w^{-2}dt \\ &= \frac{1}{w} \end{aligned}$$

   So, by Parseval's theorem, we know that

$$\begin{aligned} \sum_{n=-\infty}^{\infty}|U_n|^2 &= \sum_{n=-\infty}^{\infty}\left(\frac{\sin wn\pi}{wn\pi}\right)^2 \\ &= \left\langle |u(t)|^2 \right\rangle = \frac{1}{w} \end{aligned}$$

3. (a) Expanding the product gives $x(t) = 6\cos 20\pi t + 4\cos 8\pi t\cos 20\pi t = 6\cos 20\pi t + 2\cos 12\pi t + 2\cos 28\pi t$. The fundamental frequency is the HCF of the frequencies of these three components (or, equivalently, of the original two components) and equals 2 Hz (or 4π rad/s). The three frequency components ar therefore at 5, 3 and 7 times the fundamental frequency giving the coefficient set: $X_{-7:+7} = [1, 0, 3, 0, 1, 0, 0, \underline{0}, 0, 0, 1, 0, 3, 0, 1]$. Note that since $x(t)$ is even, the coefficients are

are symmetrical around X_0 which is underlined.

(b) We can write $x(t) = u(t)v(t)$ where $u(t) = 6 + 4\cos 8\pi t$ and $v(t) = \cos 20\pi t$. Using the fundamental frequency of the output (i.e. 2 Hz), the coefficients of $u(t)$ and $v(t)$ are $U_{-2:2} = [2,\, 0,\, \underline{6},\, 0,\, 2]$ and $V_{-5:5} = [0.5,\, 0,\, 0,\, 0,\, 0,\, \underline{0},\, 0,\, 0,\, 0,\, 0,\, 0.5]$. To convolve these, we replace each non-zero entry in $V_{-5:5}$ with a complete copy of $U_{-2:2}$ scaled by the corresponding entry of $V_{-5:5}$. This gives the same coefficients as in the previous part.

4. (a) The only non-zero coefficients are $U_{\pm 1} = 0.5$. (b) Convolving U_n with itself gives $V_{\pm 2} = 0.25$ and $V_0 = 0.25 + 0.25 = 0.5$. Thinverse Fourier transform gives $v(t) = \frac{1}{2}\cos 2t + \frac{1}{2}$ as required. (c) Convolving $V_n$with itself gives $W_{\pm 4} = 0.25^2 = 0.0625$, $W_{\pm 2} = 0.5 \times 0.25 + 0.25 \times 0.5 = 0.25$ and $W_0 = 0.25^2 + 0.5^2 + 0.25^2 = 0.375$. Taking the inverse Fourier transform gives the required answer.

5. (a) $U_{-1} = \frac{i}{2}$ and $U_1 = \frac{-i}{2}$. For V_n we write

$$
\begin{aligned}
V_0 &= \frac{1}{2\pi}\int_0^{\pi} e^{-i0t}dt = \frac{1}{2} \\
\text{for}\, n \neq 0:\ V_n &= \frac{1}{2\pi}\int_0^{\pi} e^{-int}dt \\
&= \frac{i}{2n\pi}\left[e^{-int}\right]_0^{\pi} \\
&= \frac{i}{2n\pi}\left(e^{-in\pi} - 1\right) \\
&= \frac{i}{2n\pi}\left((-1)^n - 1\right) \\
&= \begin{cases} \frac{-i}{n\pi} & n\,\text{odd} \\ 0 & n\,\text{even},\, n \neq 0 \\ \frac{1}{2} & n = 0 \end{cases}
\end{aligned}
$$

Note that, except for its DC component of $V_0 = \frac{1}{2}$, $v(t)$ is a real-valued, odd, anti-periodic function and therefore has purely imaginary coefficients with all even coefficients (except V_0) equal to zero.

(b) From the notes (slide 4-5) the convolution is defined by $W_n = U_n * V_n = V_n * U_n = \sum_{m=-\infty}^{\infty} V_{n-m}U_m$. Since $U_m = 0$ except for $m = \pm 1$, the infinite sum actually only has two non-zero terms and $W_n = U_1 V_{n-1} + U_{-1}V_{n+1} = \frac{i}{2}\left(V_{n+1} - V_{n-1}\right)$. If n is even, then $n+1$ and $n-1$ are both odd so, using the formula for V_n given above, $W_n = \frac{i}{2}\left(V_{n+1} - V_{n-1}\right) = \frac{i}{2}\left(\frac{-i}{(n+1)\pi} - \frac{-i}{(n-1)\pi}\right) = \frac{1}{2\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = \frac{1}{2\pi}\left(\frac{-2}{n^2-1}\right) = \frac{-1}{(n^2-1)\pi}$. If n is odd then $n+1$ and $n-1$ are both even and V_{n+1} and V_{n-1} are both zero unless $n+1$ or $n-1$ equals zero, i.e. unless $n = \pm 1$. So we have $W_1 = \frac{i}{2}\left(-V_0\right) = \frac{-i}{4}$ and $W_{-1} = \frac{i}{2}\left(V_0\right) = \frac{i}{4}$. We can combine all these results to give

$$
W_n = \begin{cases} 0 & n\,\text{odd},\, n \neq \pm 1 \\ \frac{-i}{4n} & n = \pm 1 \\ \frac{-1}{(n^2-1)\pi} & n\,\text{even} \end{cases}
$$

6. We have $u(0-) = u(1-) = 3$ but $u(0+) = -1$ so there is a discontinuity at $t = 0$. Therefore $u_N(0) \to \frac{3+(-1))}{2} = 1$. Notice that the actual value defined for $u(0) = 0$ has no affect on this answer. Due to Gibbs phenomenon, $u_N(t)$ will undershoot and overshoot the discontinuity by about 9% of the discontinuity height: $3 - (-1) = 4$. So $0.09 * 4 = 0.36$. So the maximum value of $u_N(t)$ will be 3.36 and the minimum value will be -1.36.

7. (a) $u(0) = 0$ but $u(1) = 1$ so the waveform has a discontinuity and the coefficients, U_n, will decrease $\propto |n|^{-1}$.

   (b) $u(0) = 0$ but $u(1) = 1$ so the waveform again has a discontinuity and the coefficients, U_n, will decrease $\propto |n|^{-1}$.

   (c) $u(0) = u(1) = 0$ but $u'(0) \neq u'(1)$ so coefficients, U_n, will decrease $\propto |n|^{-2}$.

(d) The first non-equal derivative is $u''(0) \neq u''(1)$ so coefficients, U_n, will decrease $\propto |n|^{-3}$.

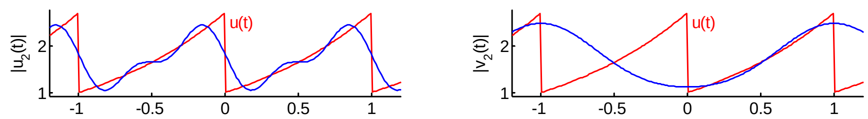
(e) $u(0) = u(1) = 1$ and $u'(0) = u'(1) = 1$. The first non-equal derivative is $-6 = u''(0) \neq u''(1) = 6$ so coefficients, U_n, will decrease $\propto |n|^{-3}$.

8. (a) $U_n = \frac{1}{T_u}\int_0^1 e^t e^{-i2\pi n F_u t} dt = \int_0^1 e^{(1-i2\pi n)t} dt = \frac{1}{1-i2\pi n}\left[e^{(1-i2\pi n)t}\right]_{t=0}^1 = \frac{1}{1-i2\pi n}\left(e^{(1-i2\pi n)} - 1\right)$ $= \frac{1}{1-i2\pi n}\left(e \times e^{-i2\pi n} - 1\right) = \frac{1}{1-i2\pi n}(e-1) = \frac{e-1}{1-i2\pi n}$. Note that we use the fact that $e^{-i2\pi n} = 1$ for any integer n.

$V_n = \frac{1}{T_v}\int_{-1}^1 e^{|t|} e^{-i2\pi n F_v t} dt = \frac{1}{2}\left(\int_{-1}^0 e^{-t} e^{-i\pi n t} dt + \int_0^1 e^t e^{-i\pi n t} dt\right)$
$= \frac{1}{2}\left(\frac{1}{-1-i\pi n}\left(1 - e^{-(-1-i\pi n)}\right) + \frac{1}{1-i\pi n}\left(e^{(1-i\pi n)} - 1\right)\right)$
$= \frac{1}{2}\left(\frac{1}{-1-i\pi n}\left(1 - e \times (-1)^n\right) + \frac{1}{1-i\pi n}\left(e \times (-1)^n - 1\right)\right)$
$= \frac{(-1)^n e - 1}{2}\left(\frac{1}{1+i\pi n} + \frac{1}{1-i\pi n}\right) = \frac{(-1)^n e - 1}{2} \times \frac{2}{1+\pi^2 n^2} = \frac{(-1)^n e - 1}{1+\pi^2 n^2}$. We see that this is real-symmetric (because $v(t)$ is real-symmetric) and that it decays $\propto n^{-2}$ because $v(t)$ is continuous but has gradient discontinuities at $t = 0$ and $t = 1$.

(b) $\langle u^2(t)\rangle = \frac{1}{T_u}\int_0^1 \left(e^t\right)^2 dt = \int_0^1 e^{2t} dt = \frac{1}{2}\left[e^{2t}\right]_{t=0}^1 = \frac{e^2-1}{2} = 3.1945$.
$\langle v^2(t)\rangle = \langle u^2(t)\rangle = \frac{e^2-1}{2}$ since reflecting a waveform in time does not affect its power.
$\langle u_2^2(t)\rangle = \sum_{-2}^{2} |U_n|^2 = U_0^2 + 2|U_1|^2 + 2|U_2|^2$
$= 1.7183^2 + 2\left(0.2701^2 + 0.1363^2\right) = 2.9525 + 0.1459 + 0.0372 = 3.1355$.

$\langle v_2^2(t)\rangle = \sum_{-2}^{2} |V_n|^2 = V_0^2 + 2|V_1|^2 + 2|V_2|^2$
$= 1.7183^2 + 2\left(0.3421^2 + 0.0424^2\right) = 2.9525 + 0.2340 + 0.0036 = 3.1901$.



We see that, for the same number of harmonics, $v_2(t)$ fits the exponential much better than $u_2(t)$ over the range $0 \le t < 1$ and that it includes much more of the energy of $u(t)$.

(c) We can use Parseval's theorem to calculate the power of the error, $\left\langle (u(t) - u_2(t))^2\right\rangle$. We know that $u(t) = \sum_{-\infty}^{+\infty} U_n e^{i2\pi nt}$ and that $u_2(t) = \sum_{-2}^{+2} U_n e^{i2\pi nt}$, so it follows that $u(t) - u_2(t) = \sum_{|n|>2} U_n e^{i2\pi nt}$. Applying Parseval's theorem to these threee expressions gives $\langle u^2(t)\rangle = \sum_{-\infty}^{+\infty} |U_n|^2$, $\langle u_2^2(t)\rangle = \sum_{-2}^{+2} |U_n|^2$ and $\left\langle (u(t) - u_2(t))^2\right\rangle = \sum_{|n|>2} |U_n|^2$. By subtracting the first two of these equations, we can see that $\langle u^2(t)\rangle - \langle u_2^2(t)\rangle = \left\langle (u(t) - u_2(t))^2\right\rangle$ and so, from part (b), $\left\langle (u(t) - u_2(t))^2\right\rangle = \langle u^2(t)\rangle - \langle u_2^2(t)\rangle = 3.1945 - 3.1355 = 0.0590$. Likewise $\left\langle (v(t) - v_2(t))^2\right\rangle = 3.1945 - 3.1901 = 0.0044$. Note that, for arbitrary functions $x(t)$ and $y(t)$ having the same period, the relationship $\left\langle (x(t) - y(t))^2\right\rangle = \langle x^2(t)\rangle - \langle y^2(t)\rangle$ is only true if $\langle x(t)y(t)\rangle = 0$ or, equivalently, if they have non-overlapping Fourier series (i.e. X_n and Y_n are never both non-zero for any n).

E1.10 Fourier Series and Transforms

Problem Sheet 4 (Lectures 6, 7, 8)

Key: [A]= easy ... [E]=hard

Fourier Transform: $X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt$ **Inverse Transform:** $x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}df$

Questions from RBH textbook: 13.1, 13.2, 13.3, 13.5, 13.7, 13.9, 13.19, 13.20.

1. [B] Evaluate $\int_{-\infty}^{\infty} \delta(t-3)t^3e^{-t}dt$.

2. [B] (a) Evaluate $\int_{-\infty}^{\infty} \delta(t-6)t^2\,dt$. (b) Now make the substitution $t = 3\tau$ for the integration variable and show that the integral remains unchanged.

3. [B] Express $2x^2\delta(8-2x)$ in the form $a\delta(x-b)$

4. [C] (a) If $v(t) = e^{-|t|}$, show that its Fourier transform is $V(f) = \frac{2}{1+4\pi^2f^2}$.

   (b) Using the time shifting and scaling formulae from slides 6-9 and 6-10 and without doing any additional integrations, determine the Fourier transforms of (i) $v_1(t) = e^{-|at|}$, (ii) $v_2(t) = e^{-|t-b|}$, (iii) $v_3(t) = \frac{1}{1+t^2}$.

5. [D] Determine the Fourier transform, $X(f)$, when $x(t) = t^2e^{-|t|}$.

6. [C] If $x(t) = \delta(t)$ determine the Fourier transform, $X(f)$. Hence, by considering the inverse transform, show that $\int_{-\infty}^{\infty} e^{i\alpha ft}df = \frac{2\pi}{|\alpha|}\delta(t)$ where $\alpha \neq 0$ is a real constant.

7. [B] Determine the Fourier transform, $X(f)$, when $x(t)$ is a DC voltage: $x(t) = 10$.

8. [B] Determine the Fourier transform, $X(f)$, when $x(t) = 12\cos 200\pi t + 8\sin 400\pi t$.

9. [C] If $v(t)$ is a periodic signal with frequency F for which $v(t) = \delta(t)$ for $-\frac{1}{2F} \le t < \frac{1}{2F}$, determine the coefficients, V_n, of its complex Fourier series. Hence find the Fourier transform, $X(f)$, of the "impulse train" given by $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - \frac{n}{F})$.

10. [C] If the Fourier transform of $x(t)$ is $X(f) = \cos 100f$, determine $x(t)$ in two ways: (a) using the duality relation: $v(t) = U(t) \Leftrightarrow V(f) = u(-f)$ and (b) by directly evaluating the inverse transform integral and using the result of question 6.

11. [B] If $x(t) = \begin{cases} 1 & |t| \le 0.5 \\ 0 & |t| > 0.5 \end{cases}$ show that $X(f) = \frac{\sin \pi f}{\pi f}$. This function is often called a top-hat function or $\mathrm{rect}(t)$.

12. [B] If $x(t) = \begin{cases} e^{-at} & t \ge 0 \\ 0 & t < 0 \end{cases}$ show that $X(f) = \frac{1}{i2\pi f + a}$ for $a > 0$.

13. [C] An electronic circuit, whose input and output signals are $x(t)$ and $y(t)$ respectively, has a frequency response given by $\frac{Y}{X}(i\omega) = \frac{2000}{i\omega+1000}$.

    (a) If $x(t) = \cos^2(1000t)$, use phasors to find an expression for $y(t)$. Give expressions for the Fourier transforms $X(f)$ and $Y(f)$.

    (b) If $x(t) = \begin{cases} e^{-500t} & t \ge 0 \\ 0 & t < 0 \end{cases}$ give an expression for $Y(f)$ (you may use without proof the result of question 12). Show that $Y(f)$ may be written as $\frac{c}{i2\pi f+500} + \frac{d}{i2\pi f+1000}$ and find the values of the constants c and d. Hence give an expression for $y(t)$.

14. [C] The triangle function is given by $y(t) = \begin{cases} 1-|t| & |t| \le 1 \\ 0 & |t| > 1 \end{cases}$. Show that $y(t)$ may be obtained by convolving $x(t)$ with itself where $x(t) = \mathrm{rect}(t)$ as defined in question 11, i.e. $y(t) = x(t) * x(t) \triangleq \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$. Hence use the convolution theorem and the result of question 11 to give the Fourier transform $Y(f)$.

15. [B] An "energy signal" has finite energy: $E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$. A "power signal" has infinite energy but finite power: $\left\langle |x(t)|^2 \right\rangle = \lim_{A,B\to\infty} \frac{1}{B-A} \int_{-A}^{B} |x(t)|^2 \, dt < \infty$. Say whether each of the following functions of time, t, is (i) an energy signal, (ii) a power signal or (iii) neither: (a) $2\cos\omega t$, (b) 10, (c) t, (d) $\sqrt{|t|}$, (e) e^t, (f) e^{-t}, (g) $e^{-|t|}$, (h) $\frac{1}{1+t^2}$, (i) $\cos t^2$, (i) $\frac{1}{1+|t|}$, (k) $\frac{1}{\sqrt{|t|}}$.

16. [C] Suppose the Fourier transform of $x(t)$ is $X(f) = \frac{1}{1+(2\pi f)^2} + 2i\left(\delta(f+4) - \delta(f-4)\right)$. Give expressions for the alternative versions of the Fourier transform: (a) $\widetilde{X}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$ and (b) $\widehat{X}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$. State the general formulae for the inverse transform integrals that give $x(t)$ in terms of $\widehat{X}(\omega)$ and $\widetilde{X}(\omega)$.

E1.10 Fourier Series and Transforms

Problem Sheet 4 - Solutions

1. $\int_{-\infty}^{\infty} \delta(t-3)t^3 e^{-t} dt = \left[t^3 e^{-t}\right]_{t=3} = 3^3 e^{-3} = 27 \times 0.498 = 1.344$.

2. (a) $\int_{-\infty}^{\infty} \delta(t-6)t^2\, dt = \left[t^2\right]_{t=6} = 36$

   (b) Substituting $t = 3\tau$ gives $\int_{-\infty}^{\infty} \delta(3\tau - 6) 9\tau^2\, 3d\tau = 27 \int_{-\infty}^{\infty} \delta(3(\tau-2))\tau^2\, d\tau$ $= 27 \int_{-\infty}^{\infty} \frac{1}{|3|}\delta(\tau-2)\tau^2\, d\tau = 9\left[\tau^2\right]_{\tau=2} = 36$. We here use the relation that $|c|\,\delta(cx) = \delta(x)$.

3. $2x^2\delta(8-2x) = 2x^2\delta\left(-2\left(x-4\right)\right) = \frac{2x^2}{|-2|}\delta\left(x-4\right) = x^2\delta\left(x-4\right) = 16\delta\left(x-4\right)$.

4. (a) $V(f) = \int_{-\infty}^{\infty} e^{-|t|}e^{-i2\pi ft}dt = \int_{-\infty}^{0} e^{t}e^{-i2\pi ft}dt + \int_{0}^{\infty} e^{-t}e^{-i2\pi ft}dt$ $= \frac{1}{1-i2\pi f}\left[e^{(1-i2\pi f)t}\right]_{t=-\infty}^{0} + \frac{1}{-1-i2\pi f}\left[e^{(-1-i2\pi f)t}\right]_{t=0}^{\infty} = \frac{1}{1-i2\pi f} - \frac{1}{-1-i2\pi f} = \frac{2}{1+4\pi^2 f^2}$. Notice that in the first step we split the integral up into the two ranges of t for which the quantity $|t|$ is equal to $-t$ and $+t$ respectively; this is necessary for any integral involving absolute values. Also notice that $e^{(a+bi)t}$ is zero at $t = +\infty$ if $a < 0$ and zero at $t = -\infty$ if $a > 0$.

   (b) If $v_1(t) = v(at)$ then $V_1(f) = \frac{1}{|a|}V\left(\frac{f}{a}\right) = \frac{2a^2}{a^2+4\pi^2 f^2}$.

   If $v_2(t) = v(t-b)$ then $V_2(f) = e^{-i2\pi fb}V(f) = \frac{2e^{-i2\pi fb}}{1+4\pi^2 f^2}$.

   If $w(t) = V(t) = \frac{2}{1+4\pi^2 t^2}$ then $W(f) = v(-f) = e^{-|f|}$. However we want $v_3(t) = 0.5w\left(\frac{t}{2\pi}\right)$ so $V_3(f) = 0.5 \times 2\pi \times W(2\pi f) = \pi e^{-|2\pi f|}$.

5.

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} t^2 e^{-|t|}e^{-i2\pi ft}dt \\
&= \int_{-\infty}^{0} t^2 e^{t}e^{-i2\pi ft}dt + \int_{0}^{\infty} t^2 e^{-t}e^{-i2\pi ft}dt \\
&= \int_{-\infty}^{0} t^2 e^{(1-i2\pi f)t}dt + \int_{0}^{\infty} t^2 e^{(-1-i2\pi f)t}dt \\
&= \left[\left(\left(1-i2\pi f\right)^2 t^2 - 2\left(1-i2\pi f\right)t + 2\right)\frac{e^{(1-i2\pi f)t}}{\left(1-i2\pi f\right)^3}\right]_{t=-\infty}^{0} \\
&\quad + \left[\left(\left(-1-i2\pi f\right)^2 t^2 - 2\left(-1-i2\pi f\right)t + 2\right)\frac{e^{(-1-i2\pi f)t}}{\left(-1-i2\pi f\right)^3}\right]_{t=0}^{\infty} \\
&= 2\left(\frac{1}{\left(1-i2\pi f\right)^3} - \frac{1}{\left(-1-i2\pi f\right)^3}\right) \\
&= \frac{4+48\pi^2 f^2}{\left(1+4\pi^2 f^2\right)^3}
\end{aligned}
$$

6. $X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt = \int_{-\infty}^{\infty} \delta(t)e^{-i2\pi ft}dt = \left[e^{-i2\pi ft}\right]_{t=0} 1$. Note that this is the same for all values of f and is called a "flat" or "white" spectrum. The inverse transform is

$$\delta(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}df = \int_{-\infty}^{\infty} e^{i2\pi ft}df.$$

   If we now substitute $\tau = \frac{2\pi}{\alpha}t$, we obtain $\int_{-\infty}^{\infty} e^{i\alpha f\tau}df = \delta\left(\frac{\alpha}{2\pi}\tau\right) = \frac{2\pi}{|\alpha|}\delta(\tau)$. Alternatively, we could substitute $\nu = \frac{2\pi}{\alpha}f$ to obtain $\delta(t) = \frac{2\pi}{\alpha}\int_{f=-\infty}^{\infty} e^{i\alpha\nu t}d\nu$. The new limits (in terms of ν) are either $\nu = \mp\infty$ if $\alpha > 0$ or else $\nu = \pm\infty$ if $\alpha < 0$ and in the latter case we need to reverse the order of the limits and multiply by -1. Thus we end up with $\delta(t) = \frac{2\pi}{|\alpha|}\int_{f=-\infty}^{\infty} e^{i\alpha\nu t}d\nu$ which is the same result as before.

7. $X(f) = \int_{-\infty}^{\infty} 10e^{-i2\pi ft}dt\ = 10\delta(f)$. This follows from the answer to question 6 with $\alpha = -2\pi$.

8. The Fourier transform of a periodic waveform is iust the complex Fourier series coefficients multiplied by delta functions at the appropriate positive and negative frequencies. So $X(f) = 6\delta(f+100) + 6\delta(f-100) + 4i\delta(f+200) - 4i\delta(f-200)$.

9. The complex Fourier series coefficients are $V_n = F\int_{-0.5T}^{0.5t} \delta(t)e^{-i2\pi Ft}dt = F\left[e^{-i2\pi Ft}\right]_{t=0} = F$ (i.e. the same for all n). In fact, $x(t)$ is equal to $v(t)$ but iust written in a different way. So, from the theorem on page 6-8 of the notes, $X(f) = \sum_{n=-\infty}^{\infty} X_n\delta(f-nF) = F\sum_{n=-\infty}^{\infty}\delta(f-nF)$. Thus the Fourier transform of an impulse train with spacing $\frac{1}{F}$ is another impulse train with spacing F.

10. (a) If $v(t) = X(t) = \cos 100t$, then $V(f) = \frac{1}{2}\delta(f+\frac{50}{\pi}) + \frac{1}{2}\delta(f-\frac{50}{\pi})$. So, from the duality theorem, $x(f) = V(-f)$, so $x(t) = \frac{1}{2}\delta(t+\frac{50}{\pi}) + \frac{1}{2}\delta(t-\frac{50}{\pi})$.

    (b) $x(t) = \int_{-\infty}^{\infty} \cos(100f)\, e^{i2\pi ft}df = \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{i100f} + e^{-i100f}\right)e^{i2\pi ft}df$
    $= \frac{1}{2}\int_{-\infty}^{\infty} e^{i\left(2\pi\left(t+\frac{50}{\pi}\right)\right)f}df + \frac{1}{2}\int_{-\infty}^{\infty} e^{i\left(2\pi\left(t-\frac{50}{\pi}\right)\right)f}df == \frac{1}{2}\delta(t+\frac{50}{\pi}) + \frac{1}{2}\delta(t-\frac{50}{\pi})$.

11. $X(f) = \int_{-0.5}^{0.5} e^{-i2\pi ft}dt = \frac{1}{-i2\pi f}\left[e^{-i2\pi ft}\right]_{t=-0.5}^{0.5} = \frac{1}{-i2\pi f} \times -2i\sin\pi f = \frac{\sin\pi f}{\pi f}$.

12. $X(f) = \int_0^\infty e^{-at}e^{-i2\pi ft}dt = \int_0^\infty e^{(-a-i2\pi f)t}dt = \frac{1}{-a-i2\pi f}\left[e^{(-a-i2\pi f)t}\right]_{t=0}^{\infty} = \frac{-1}{-a-i2\pi f} = \frac{1}{a+i2\pi f}$. Note that the value of $e^{(-a-i2\pi f)t}$ is zero at $t=\infty$ provided that $a>0$.

13. (a) $x(t) = \cos^2(1000t) = 0.5 + 0.5\cos(2000t)$. The gains at these component frequencies are $\frac{Y}{X}(i0) = 2$ and $\frac{Y}{X}(i2000) = \frac{2}{1+2i} = 0.4 - 0.8i$. It follows (from phasors) that

$$y(t) = 1 + 0.2\cos(2000t) + 0.4\sin(2000t).$$

The Fourier transforms are $X(f) = 0.5\delta(f) + 0.25\delta\left(f+\frac{1000}{\pi}\right) + 0.25\delta\left(f-\frac{1000}{\pi}\right)$ and $Y(f) = \delta(f) + (0.1+0.2i)\,\delta\left(f+\frac{1000}{\pi}\right) + (0.1-0.2i)\,\delta\left(f-\frac{1000}{\pi}\right)$. Note that the positive frequency term, $\delta\left(f-\frac{1000}{\pi}\right)$, is multiplied by $\frac{Y}{X}(i2\pi f)$ while the negative frequency term, $\delta\left(f+\frac{1000}{\pi}\right)$, is multiplied by its complex conjugate, $\frac{Y}{X}(-i2\pi f)$.

(b) From question 12 we know that $X(f) = \frac{1}{i2\pi f+500}$. So it follows that

$$Y(f) = X(f)\times\frac{Y}{X}(i2\pi f) = \frac{1}{i2\pi f+500}\times\frac{2000}{i2\pi f+1000} = \frac{2000}{(i2\pi f+500)(i2\pi f+1000)}$$

We can put the given expression over a common denominator: $\frac{c}{i2\pi f+500} + \frac{d}{i2\pi f+1000} = \frac{i2\pi f(c+d)+1000c+500d}{(i2\pi f+500)(i2\pi f+1000)}$.
Equating the numerator to 2000 gives $c=4$ and $d=-4$. Hence $y(t) = \begin{cases} 4\left(e^{-500t} - e^{-1000t}\right) & t\ge 0 \\ 0 & t<0 \end{cases}$.

14. $y(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$. The integrand is only non-zero when the arguments of both top-hat functions lie in the range ± 0.5. Thus we must have $-0.5<\tau<0.5$ and also
$-0.5 < t-\tau < 0.5 \Leftrightarrow t-0.5<\tau<t+0.5$.
We can therefore write $y(t) = \int_{\max(-0.5,\,t-0.5)}^{\min(0.5,\,t+0.5)} d\tau = \begin{cases}\int_{-0.5}^{t+0.5} d\tau & t<0 \\ \int_{t-0.5}^{0.5} d\tau & t\ge 0\end{cases}$. The integration range is empty if $|t|>1$ and so we can write $y(t) = \begin{cases} 1+t & t<0 \\ 1-t & t\ge 0\end{cases}$ which also equals $y(t) = \begin{cases} 1-|t| & |t|\le 1 \\ 0 & |t|>1\end{cases}$ as requested.
From the convolution theorem, $Y(f) = X^2(f) = \frac{\sin^2\pi f}{\pi^2 f^2}$.

15. [B] An "energy signal" has finite energy: $\int_{-\infty}^{\infty}|x(t)|^2\,dt < \infty$. A "power signal" has infinite energy but finite power: $\lim_{A,B\to\infty}\frac{1}{B-A}\int_{-A}^{B}|x(t)|^2\,dt < \infty$. The answers are therefore (a) P, (b) P, (c) N, (d) N, (e) N, (f) N, (g) E, (h) E, (i) P, (i) E, (i) P. The final example has zero average power but is not an energy signal because it has infinite energy.

16. (a) We substitute $\omega = 2\pi f$ to obtain:

$$\begin{aligned}\widetilde{X}(\omega) &= \frac{1}{1+\omega^2} + 2i\left(\delta(\frac{\omega}{2\pi}+4) - \delta(\frac{\omega}{2\pi}-4)\right)\\ &= \frac{1}{1+\omega^2} + 2i\left(\delta(\frac{\omega+8\pi}{2\pi}) - \delta(\frac{\omega-8\pi}{2\pi})\right)\\ &= \frac{1}{1+\omega^2} + 4\pi i\left(\delta(\omega+8\pi) - \delta(\omega-8\pi)\right).\end{aligned}$$

The final line is obtained using the scaling formula for delta functions: $|c|\,\delta(cx) = \delta(x)$. Thus we see that in the angular-frequency version of the Fourier transform, any continuous functions of f remain the same amplitude but delta functions are multiplied by 2π. The inverse transform is given by $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{X}(\omega)e^{i\omega t}d\omega$; this can be obtained by changing the variable in the normal inverse transform from f to ω.

(b) $\widehat{X}(\omega)$ is exactly the same as $\widetilde{X}(\omega)$ but divided by $\sqrt{2\pi}$. So

$$\widetilde{X}(\omega) = \frac{1}{\sqrt{2\pi}\,(1+\omega^2)} + \sqrt{8\pi}i\left(\delta(\omega+8\pi) - \delta(\omega-8\pi)\right).$$

The inverse transform is the same as in the previous part but multiplied by $\sqrt{2\pi}$, i.e.

$$x(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{X}(\omega)e^{i\omega t}d\omega.$$