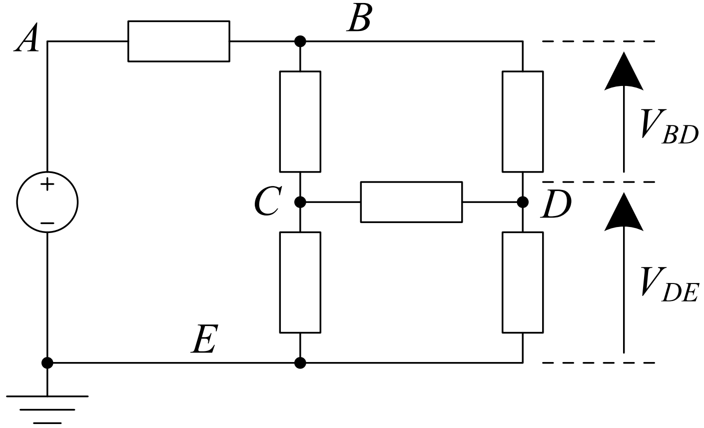
2: Resistor Circuits

Kirchoff's Voltage Law

The five nodes are labelled A, B, C, D, E where E is the reference node.

Each component that links a pair of nodes is called a *branch* of the network.



Kirchoff's Voltage Law (KVL) is a consequence of the fact that the work done in moving a charge from one node to another does not depend on the route you take; in particular the work done in going from one node back to the same node by any route is zero.

KVL: the sum of the voltage changes around any closed loop is zero.

Example: $V_{DE} + V_{BD} + V_{AB} + V_{EA} = 0$

Equivalent formulation:

$V_{XY} = V_{XE} - V_{YE} = V_X - V_Y$ for any nodes X and Y.

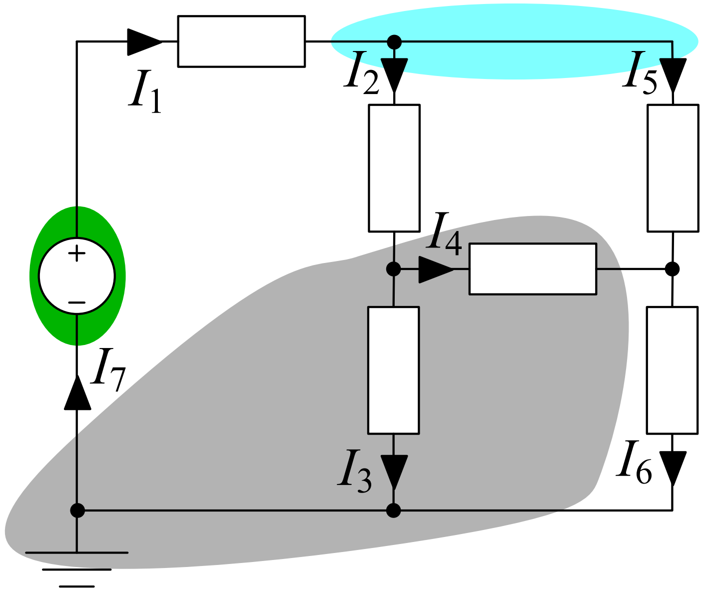
Kirchoff's Current Law

Wherever charges are free to move around, they will move to ensure charge neutrality everywhere at all times.
A consequence is Kirchoff's Current Law (KCL) which says that the current going into any closed region of a circuit must equal the current coming out.
KCL: The currents flowing out of any closed region of a circuit sum to zero.

Green: $I_1 = I_7$

Blue: $-I_1 + I_2 + I_5 = 0$

Gray: $-I_2 + I_4 - I_6 + I_7 = 0$



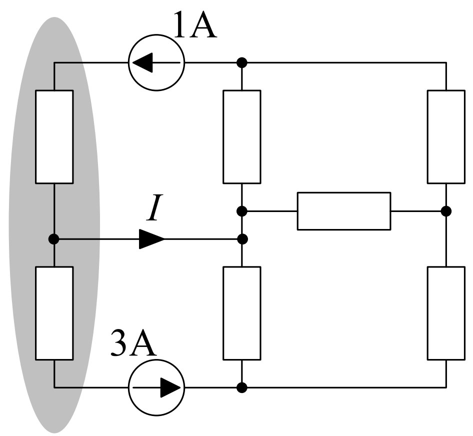
KCL Example

The currents and voltages in any linear circuit can be determined by using KCL, KVL and Ohm's law.

Sometimes KCL allows you to determine currents very easily without having to solve any simultaneous equations:

How do we calculate I ?

KCL: $-1 + I + 3 = 0$
$\Longrightarrow I = -2\,\text{A}$



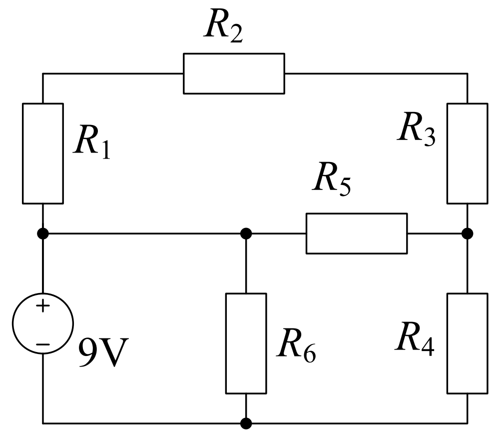
Note that here I ends up negative which means we chose the wrong arrow direction to label the circuit. This does not matter. You can choose the directions arbitrarily and let the algebra take care of reality.

Series: Components that are connected in a chain so that the same current flows through each one are said to be *in series*.

R_1, R_2, R_3 are in series and the same current always flows through each.

Within the chain, each internal node connects to only two branches.

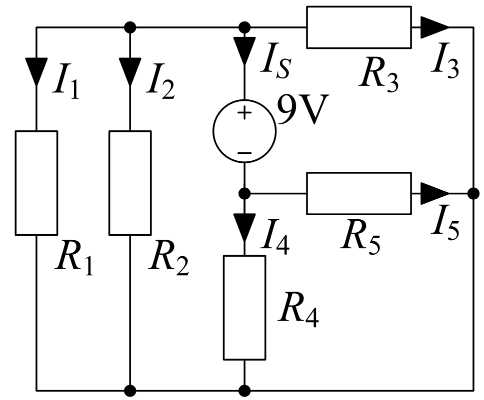
R_3 and R_4 are not in series and do not necessarily have the same current.



Parallel: Components that are connected to the same pair of nodes are said to be *in parallel*.

R_1, R_2, R_3 are in parallel and the same voltage is across each resistor (even though R_3 is not close to the others).

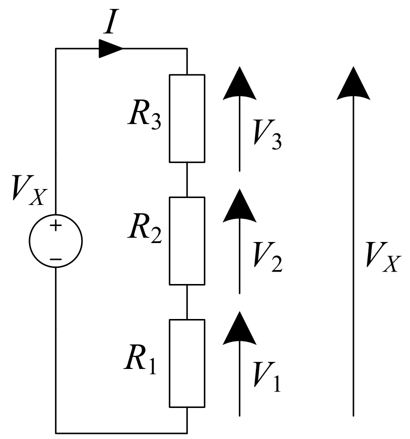
R_4 and R_5 are also in parallel.



$$V_X = V_1 + V_2 + V_3$$
$$= IR_1 + IR_2 + IR_3$$
$$= I(R_1 + R_2 + R_3)$$

$$\frac{V_1}{V_X} = \frac{IR_1}{I(R_1+R_2+R_3)}$$
$$= \frac{R_1}{R_1+R_2+R_3} = \frac{R_1}{R_T}$$

where $R_T = R_1 + R_2 + R_3$ is the total resistance of the chain.

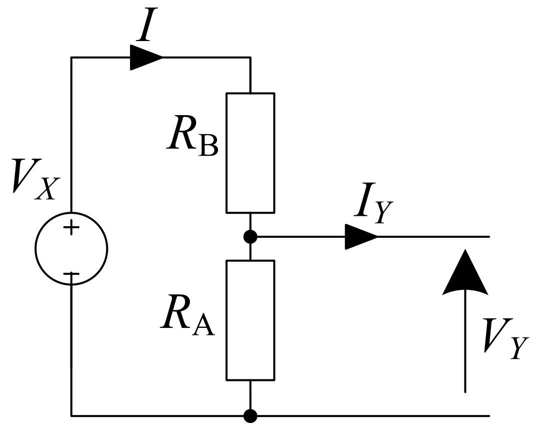


V_X is divided into $V_1 : V_2 : V_3$ in the proportions $R_1 : R_2 : R_3$.

Approximate Voltage Divider:

If $I_Y = 0$, then $V_Y = \frac{R_A}{R_A+R_B} V_X$.

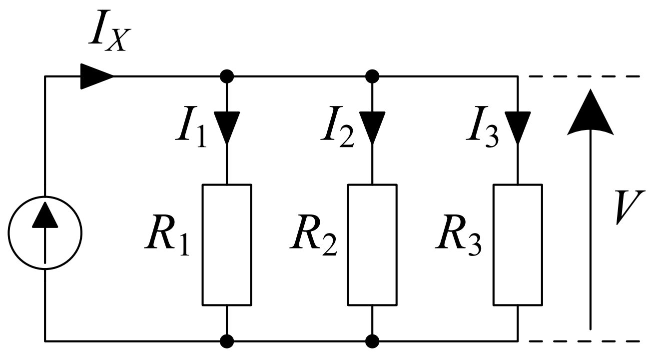
If $I_Y \ll I$, then $V_Y \approx \frac{R_A}{R_A+R_B} V_X$.



Parallel resistors all share the same V.

$I_1 = \frac{V}{R_1} = VG_1$ where $G_1 = \frac{1}{R_1}$ is the *conductance* of R_1.

$$\begin{aligned} I_X &= I_1 + I_2 + I_3 \\ &= VG_1 + VG_2 + VG_3 \\ &= V(G_1 + G_2 + G_3) \end{aligned}$$



$$\frac{I_1}{I_X} = \frac{VG_1}{V(G_1+G_2+G_3)} = \frac{G_1}{G_1+G_2+G_3} = \frac{G_1}{G_P}$$

where $G_P = G_1 + G_2 + G_3$ is the total conductance of the resistors.

I_X is divided into $I_1 : I_2 : I_3$ in the proportions $G_1 : G_2 : G_3$.

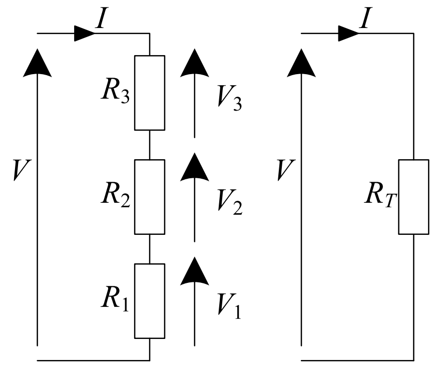
Special case for only two resistors:

$$I_1 : I_2 = G_1 : G_2 = R_2 : R_1 \Rightarrow I_1 = \frac{R_2}{R_1+R_2} I_X.$$

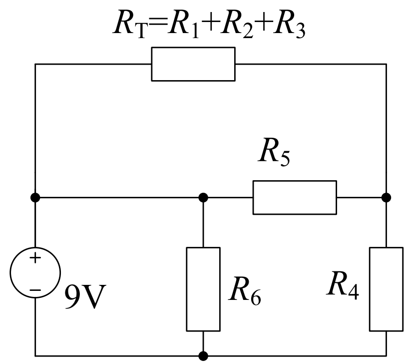
Equivalent Resistance: Series

We know that $V = V_1 + V_2 + V_3 = I(R_1 + R_2 + R_3) = IR_T$

So we can replace the three resistors by a single *equivalent resistor* of value R_T without affecting the relationship between V and I.



Replacing series resistors by their equivalent resistor will not affect any of the voltages or currents in the rest of the circuit.



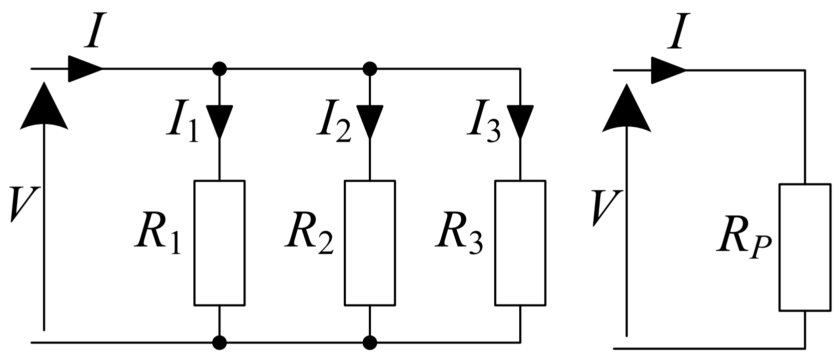
However the individual voltages V_1, V_2 and V_3 are no longer accessible.

Equivalent Resistance: Parallel

Similarly we known that $I = I_1 + I_2 + I_3 = V(G_1 + G_2 + G_3) = VG_P$.

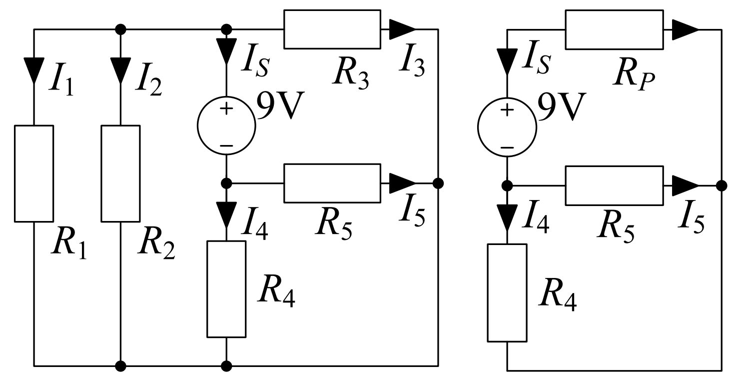
So $V = IR_P$ where $R_P = \frac{1}{G_P} = \frac{1}{G_1+G_2+G_3} = \frac{1}{1/R_1+1/R_2+1/R_3}$

We can use a single *equivalent resistor* of resistance R_P without affecting the relationship between V and I.



Replacing parallel resistors by their equivalent resistor will not affect any of the voltages or currents in the rest of the circuit.

$R_4$and R_5 are also in parallel.



Much simpler - although none of the original currents $I_1, \cdots, I_5$ are now accessible. Current I_S and the three node voltages are identical.

Equivalent Resistance: Parallel Formulae

For parallel resistors $G_P = G_1 + G_2 + G_3$

or equivalently $R_P = R_1||R_2||R_3 = \frac{1}{1/R_1 + 1/R_2 + 1/R_3}$.

These formulae work for any number of resistors.

- For the special case of two parallel resistors

$$R_P = \frac{1}{1/R_1 + 1/R_2} = \frac{R_1 R_2}{R_1 + R_2} \text{ ("product over sum")}$$

- If one resistor is a multiple of the other

Suppose $R_2 = kR_1$, then

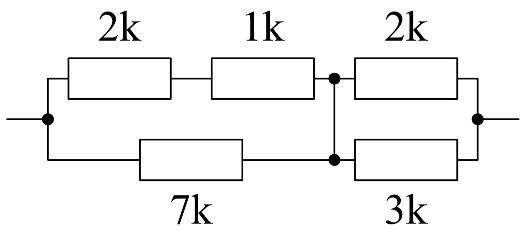
$$R_P = \frac{R_1 R_2}{R_1 + R_2} = \frac{kR_1^2}{(k+1)R_1} = \frac{k}{k+1} R_1 = \left(1 - \frac{1}{k+1}\right) R_1$$

Example: $1\ \text{k}\Omega\ ||\ 99\ \text{k}\Omega = \frac{99}{100}\ \text{k}\Omega = \left(1 - \frac{1}{100}\right)\ \text{k}\Omega$

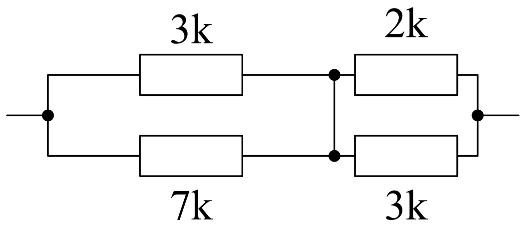
Important: The equivalent resistance of parallel resistors is always less than any of them.

Simplifying Resistor Networks

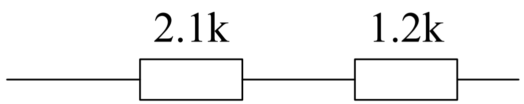
Many resistor circuits can be simplified by alternately combining series and parallel resistors.



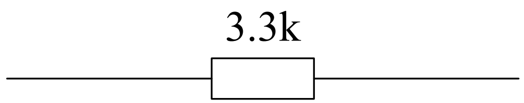
Series: $2\,\text{k} + 1\,\text{k} = 3\,\text{k}$



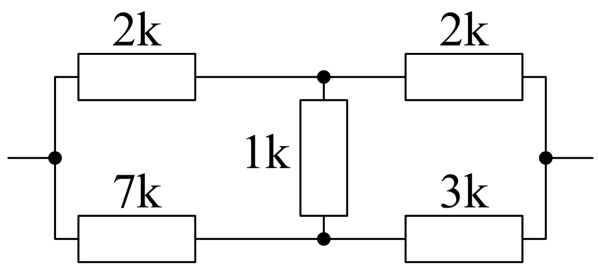
Parallel: $3\,\text{k} \,||\, 7\,\text{k} = 2.1\,\text{k}$
Parallel: $2\,\text{k} \,||\, 3\,\text{k} = 1.2\,\text{k}$



Series: $2.1\,\text{k} + 1.2\,\text{k} = 3.3\,\text{k}$

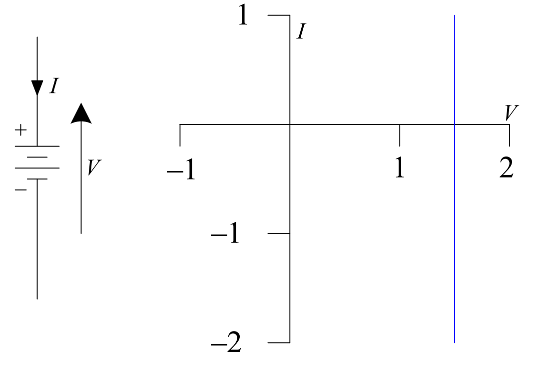


Sadly this method does not always work: there are no series or parallel resistors here.



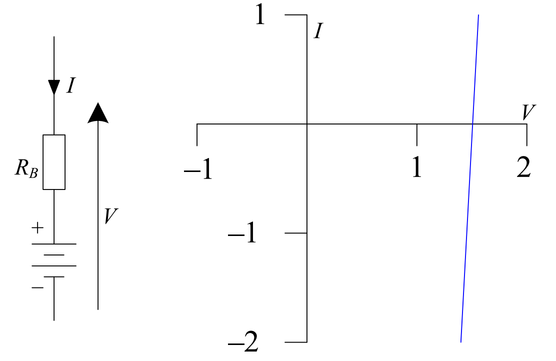
Non-ideal Voltage Source

An ideal battery has a characteristic that is vertical: battery voltage does not vary with current.
Normally a battery is supplying energy so V and I have opposite signs, so $I \leq 0$.

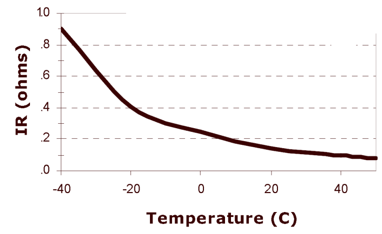


An real battery has a characteristic that has a slight positive slope: battery voltage decreases as the (negative) current increases.

Model this by including a small resistor in series. $V = V_B + IR_B$.



The equivalent resistance for a battery increases at low temperatures.



Summary

- Kichoff's Voltage and Current Laws
- Series and Parallel components
- Voltage and Current Dividers
- Simplifying Resistor Networks
- Battery Internal Resistance

For further details see Hayt Ch 3 or Irwin Ch 2.