3: Nodal Analysis

Aim of Nodal Analysis

The aim of nodal analysis is to determine the voltage at each node relative to the reference node (or ground). Once you have done this you can easily work out anything else you need.

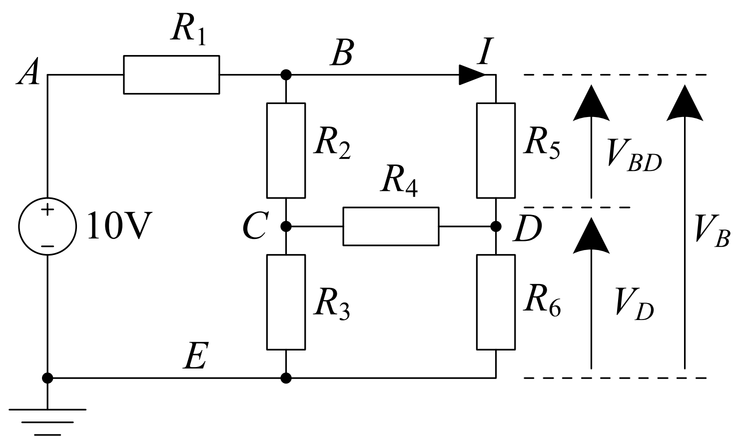
There are two ways to do this:

(1) Nodal Analysis - systematic; always works

(2) Circuit Manipulation - ad hoc; but can be less work and clearer

Reminders:

A node is all the points in a circuit that are directly interconnected. We assume the interconnections have zero resistance so all points within a node have the same voltage. Five nodes: $A, \cdots, E$.



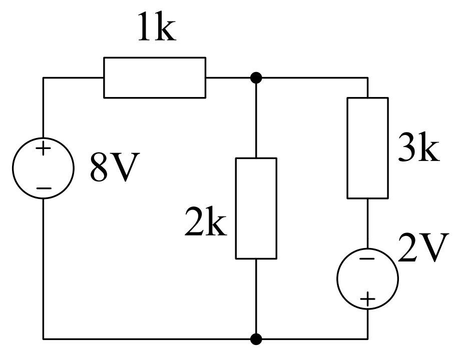
Ohm's Law: $V_{BD} = IR_5$

KVL: $V_{BD} = V_B - V_D$

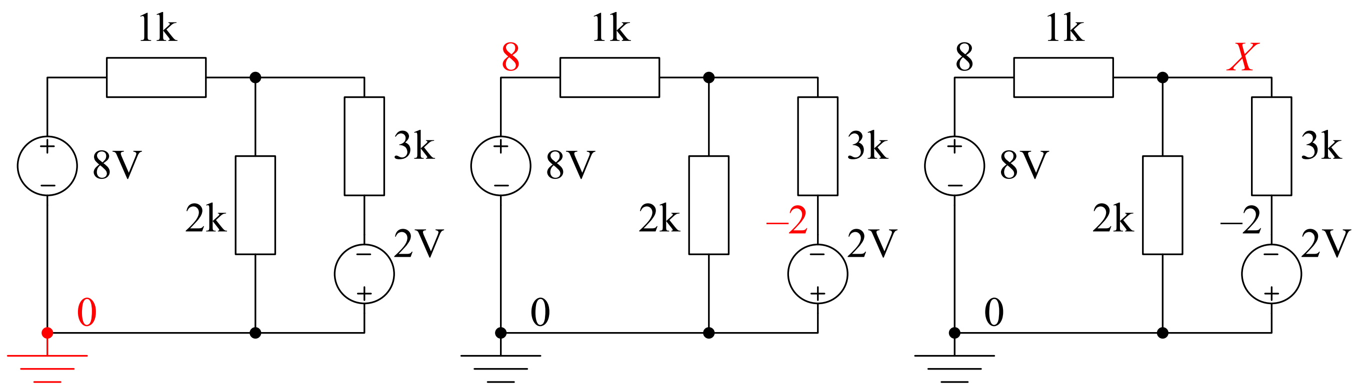
KCL: Total current exiting any closed region is zero.

Nodal Analysis Stage 1: Label Nodes

To find the voltage at each node, the first step is to label each node with its voltage as follows

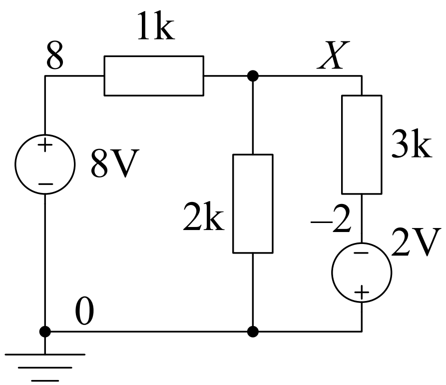


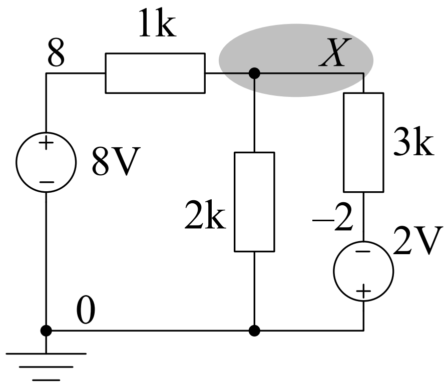
(1) Pick any node as the voltage reference. Label its voltage as $0\,\mathrm{V}$.
(2) If any fixed voltage sources are connected to a labelled node, label their other ends by adding the value of the source onto the voltage of the labelled end.
(3) Pick an unlabelled node and label it with X, Y, ..., then go back to step (2) until all nodes are labelled.



Nodal Analysis Stage 2: KCL Equations

The second step is to write down a KCL equation for each node labelled with a variable by setting the total current flowing out of the node to zero. For a circuit with N nodes and S voltage sources you will have $N-S-1$ simultaneous equations to solve.





We only have one variable:

$$\frac{X-8}{1\ \mathrm{k}}+\frac{X-0}{2\ \mathrm{k}}+\frac{X-(-2)}{3\ \mathrm{k}}=0 \quad\Rightarrow\quad (6X-48)+3X+(2X+4)=0$$

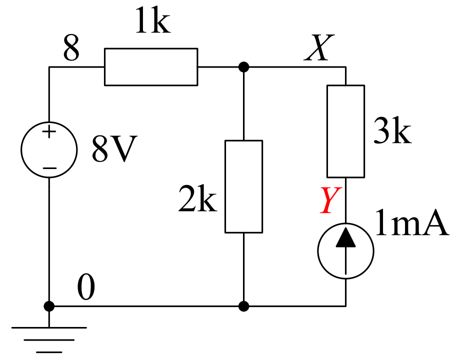
$$11X=44 \quad\Rightarrow\quad X=4$$

Numerator for a resistor is always of the form $X-V_N$ where V_N is the voltage on the other side of the resistor.

Current Sources

Current sources cause no problems.

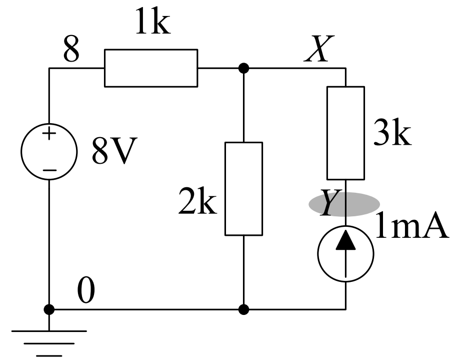
(1) Pick reference node.
(2) Label nodes: 8, X and Y.



(3) Write equations

$$\frac{X-8}{1}+\frac{X}{2}+\frac{X-Y}{3}=0$$

$$\frac{Y-X}{3}+(-1)=0$$



Ohm's law works OK if all resistors are in $\mathrm{k}\Omega$ and all currents in mA.
(4) Solve the equations: $X=6$, $Y=9$

Floating Voltage Sources

Floating voltage sources have neither end connected to a known fixed voltage. We have to change how we form the KCL equations slightly.

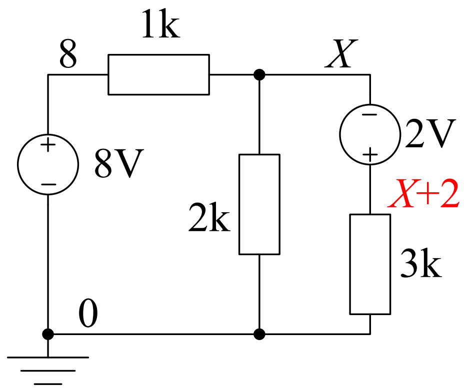
(1) Pick reference node.

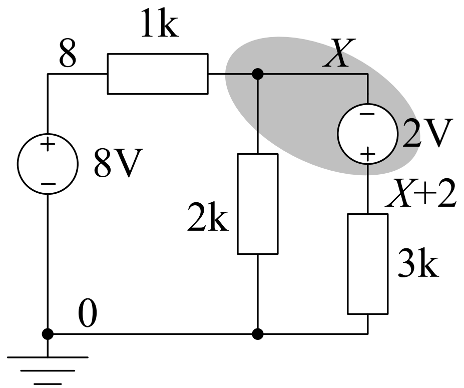
(2) Label nodes: 8, X and $X+2$ since it is joined to X via a voltage source.

(3) Write KCL equations but count all the nodes connected via floating voltage sources as a single "super-node" giving one equation

$$\frac{X-8}{1}+\frac{X}{2}+\frac{(X+2)-0}{3}=0$$

(4) Solve the equations: $X=4$





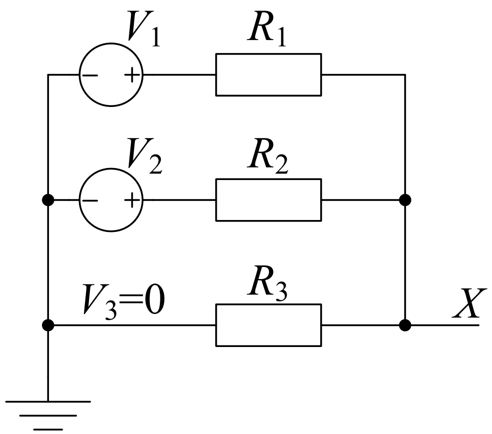
Ohm's law always involves the difference between the voltages at either end of a resistor. (Obvious but easily forgotten)

A very useful sub-circuit that calculates the weighted average of any number of voltages.

KCL equation for node X:

$$\frac{X-V_1}{R_1} + \frac{X-V_2}{R_2} + \frac{X-V_3}{R_3} = 0$$

Still works if $V_3 = 0$.



Or using conductances:

$$(X - V_1)G_1 + (X - V_2)G_2 + (X - V_3)G_3 = 0$$

$$X(G_1 + G_2 + G_3) = V_1G_1 + V_2G_2 + V_3G_3$$

$$X = \frac{V_1G_1+V_2G_2+V_3G_3}{G_1+G_2+G_3} = \frac{\sum_{i=1}^{3} V_iG_i}{\sum_{i=1}^{3} G_i}$$

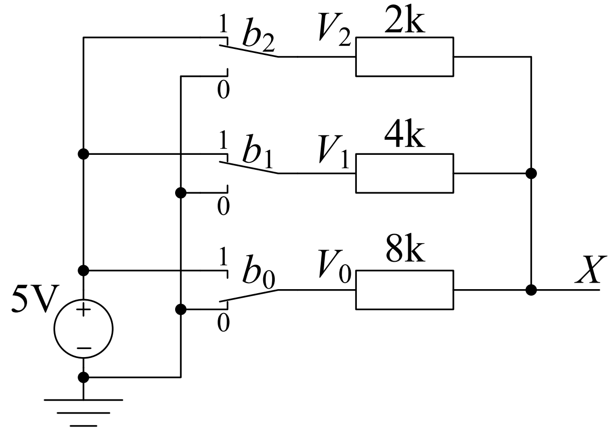
Voltage X is the average of V_1, V_2, V_3 weighted by the conductances.

A 3-bit binary number, b, has bit-weights of 4, 2 and 1. Thus 110 has a value 6 in decimal. If we label the bits $b_2 b_1 b_0$, then $b = 4b_2 + 2b_1 + b_0$.

We use $b_2 b_1 b_0$ to control the switches which determine whether $V_i = 5\,\text{V}$ or $V_i = 0\,\text{V}$. Thus $V_i = 5b_i$. Switches shown for $b = 6$.

$$X = \frac{\frac{1}{2}V_2 + \frac{1}{4}V_1 + \frac{1}{8}V_0}{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}}$$

$$= \frac{1}{7}\left(4V_2 + 2V_1 + V_0\right)$$



but $V_i = 5 \times b_i$ since it connects to either $0\,\text{V}$ or $5\,\text{V}$

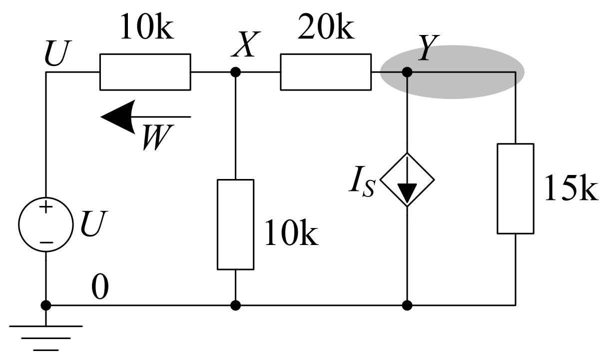
$$= \frac{5}{7}\left(4b_2 + 2b_1 + b_0\right) = \frac{5}{7}b$$

$$G_2 = \frac{1}{R_2} = \frac{1}{2}\,\text{mS}, \ \ldots$$

So we have made a circuit in which X is proportional to a binary number b.

A *dependent* voltage or current source is one whose value is determined by voltages or currents elsewhere in the circuit. These are most commonly used when modelling the behaviour of transistors or op-amps. Each dependent source has a defining equation.

In this circuit: $I_S = 0.2W$ mA where W is in volts.



(1) Pick reference node.

(2) Label nodes: 0, U, X and Y .

(3) Write equation for the dependent source, I_S, in terms of node voltages:

$$I_S = 0.2\,(U - X)$$

(4) Write KCL equations:

$$\frac{X-U}{10} + \frac{X}{10} + \frac{X-Y}{20} = 0 \qquad \frac{Y-X}{20} + I_S + \frac{Y}{15} = 0$$

(5) Solve all three equations to find X, Y and I_S in terms of U:

$$X = 0.1U,\ Y = -1.5U,\ I_S = 0.18U$$

Note that the value of U is assumed to be known.

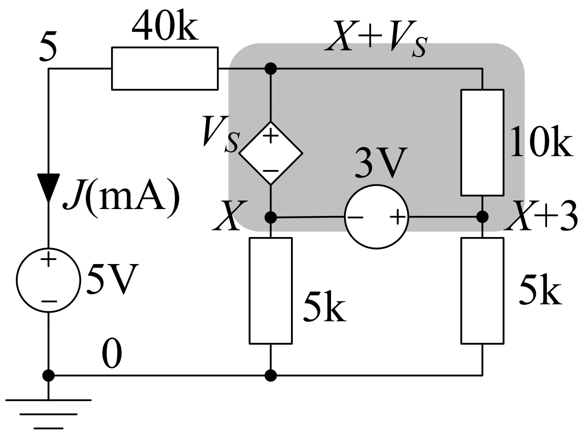
Dependent Voltage Sources

The value of the highlighted dependent voltage source is $V_S = 10J$ Volts where J is the indicated current in mA.

(1) Pick reference node.

(2) Label nodes: $0,\ 5,\ X,\ X+3$ and $X+V_S$.

(3) Write equation for the dependent source, V_S, in terms of node voltages:



$$V_S = 10J = 10 \times \tfrac{X+V_S-5}{40} \Rightarrow 3V_S = X - 5$$

(4) Write KCL equations: all nodes connected by floating voltage sources and all components connecting these nodes are in the same "super-node"

$$\tfrac{X+V_S-5}{40} + \tfrac{X}{5} + \tfrac{X+3}{5} = 0$$

(5) Solve the two equations: $X = -1$ and $V_S = -2$

Universal Nodal Analysis Algorithm

(1) Pick any node as the voltage reference. Label its voltage as $0\,\mathrm{V}$. Label any dependent sources with V_S, I_S, $\ldots$.

(2) If any voltage sources are connected to a labelled node, label their other ends by adding the value of the source onto the voltage of the labelled end. Repeat as many times as possible.

(3) Pick an unlabelled node and label it with X, Y, $\ldots$, then loop back to step (2) until all nodes are labelled.

(4) For each **dependent source**, write down an equation that expresses its value in terms of other node voltages.

(5) Write down a KCL equation for each "normal" node (i.e. one that is not connected to a floating voltage source).

(6) Write down a KCL equation for each "super-node". A super-node consists of a set of nodes that are joined by floating voltage sources and includes any other components joining these nodes.

(7) Solve the set of simultaneous equations that you have written down.

Summary

- Nodal Analysis
  - Simple Circuits (no floating or dependent voltage sources)
  - Floating Voltage Sources
    - use supernodes: all the nodes connected by floating voltage sources (independent or dependent)
  - Dependent Voltage and Current Sources
    - Label each source with a variable
    - Write extra equations expressing the source values in terms of node voltages
    - Write down the KCL equations as before
- Mesh Analysis (in most textbooks)
  - Alternative to nodal analysis but doesn't work for all circuits
  - No significant benefits $\Rightarrow$ ignore it

For further details see Hayt Ch 4 or Irwin Ch 3.