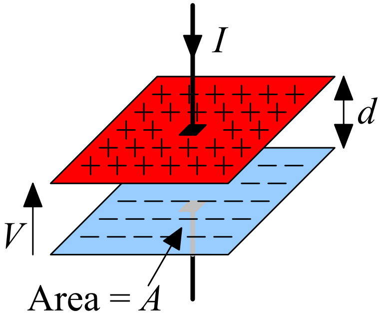
9: Capacitors and Inductors

Capacitors

A capacitor is formed from two conducting plates separated by a thin insulating layer.

If a current i flows, positive change, q, will accumulate on the upper plate. To preserve charge neutrality, a balancing negative charge will be present on the lower plate.



There will be a potential energy difference (or voltage v) between the plates proportional to q.
$v = \frac{d}{A\epsilon}q$ where A is the area of the plates, d is their separation and ϵ is the permittivity of the insulating layer ($\epsilon_0 = 8.85\,{}^{pF}\!/_{\mathrm{m}}$ for a vacuum).

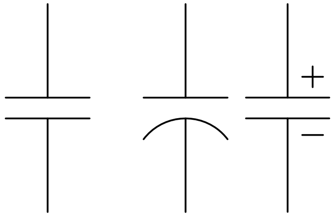
The quantity $C = \frac{A\epsilon}{d}$ is the *capacitance* and is measured in Farads (F), hence $q = Cv$.

The current, i, is the rate of charge on the plate, hence the

capacitor equation: $i = \frac{dq}{dt} = C\frac{dv}{dt}$.

Types of Capacitor

Capacitor symbol represents the two separated plates. Capacitor types are distinguished by the material used as the insulator.



Polystyrene: Two sheets of foil separated by a thin plastic film and rolled up to save space. Values: $10\,\mathrm{pF}$ to $1\,\mathrm{nF}$.

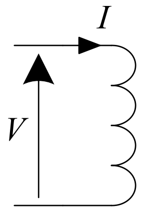
Ceramic: Alternate layers of metal and ceramic (a few $\mu\mathrm{m}$ thick). Values: $1\,\mathrm{nF}$ to $1\,\mu\mathrm{F}$.

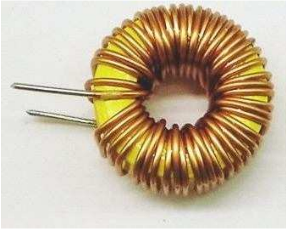
Electrolytic: Two sheets of aluminium foil separated by paper soaked in conducting electrolyte. The insulator is a thin oxide layer on one of the foils. Values: $1\,\mu\mathrm{F}$ to $10\,\mathrm{mF}$.

Electrolytic capacitors are polarised: the foil with the oxide layer must always be at a positive voltage relative to the other (else explosion). Negative terminal indicated by a curved plate in symbol or “-”.

Inductors

Inductors are formed from coils of wire, often around a steel or ferrite core.





The magnetic flux within the coil is $\Phi = \frac{\mu N A}{l} i$ where N is the number of turns, A is the cross-sectional area of the coil and l is the length of the coil (around the toroid).

μ is a property of the material that the core is made from and is called its *permeability*. For free space (or air): $\mu_0 = 4\pi \times 10^{-7} = 1.26\,{}^{\mu H}\!/\mathrm{m}$, for steel, $\mu \approx 4000\mu_0 = 5\,{}^{mH}\!/\mathrm{m}$.

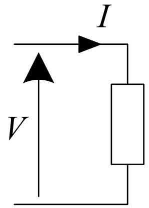
From Faraday's law: $v = N\frac{d\Phi}{dt} = \frac{\mu N^2 A}{l}\frac{di}{dt} = L\frac{di}{dt}$.

We measure the *inductance*, $L = \frac{\mu N^2 A}{l}$, in Henrys (H).

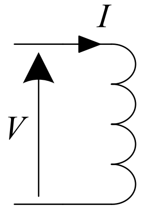
Passive Components

We can describe all three types of passive component by the relationship between V and I using, in each case, the passive sign convention.

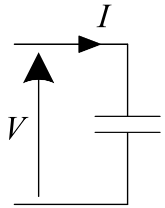
Resistor: $v = Ri$



Inductor: $v = L\frac{di}{dt}$



Capacitor: $i = C\frac{dv}{dt}$

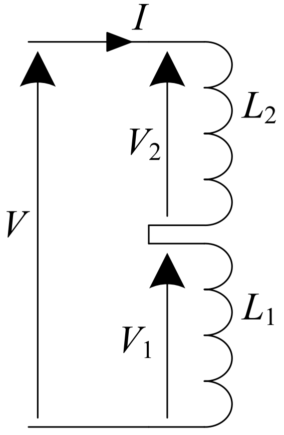


Notes: (1) There are no minus signs anywhere whatever you were taught at school.
(2) We use lower case, v, for time-varying voltages.

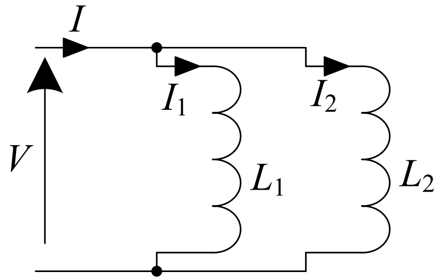
Series and Parallel Inductors

$$\begin{aligned} v = v_1 + v_2 &= L_1 \tfrac{di}{dt} + L_2 \tfrac{di}{dt} \\ &= (L_1 + L_2) \tfrac{di}{dt} \end{aligned}$$

Same equation as a single inductor of value $L_1 + L_2$



$$\begin{aligned} \tfrac{di}{dt} &= \tfrac{d(i_1+i_2)}{dt} = \tfrac{di_1}{dt} + \tfrac{di_2}{dt} \\ &= \tfrac{v}{L_1} + \tfrac{v}{L_2} = v\left(\tfrac{1}{L_1} + \tfrac{1}{L_2}\right) \\ v &= \tfrac{1}{\frac{1}{L_1}+\frac{1}{L_2}} \tfrac{di}{dt} \end{aligned}$$

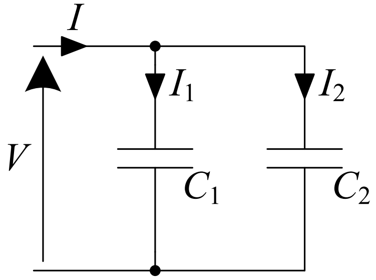


Same as a single inductor of value $\frac{1}{\frac{1}{L_1}+\frac{1}{L_2}} = \frac{L_1 L_2}{L_1+L_2}$

Inductors combine just like resistors.

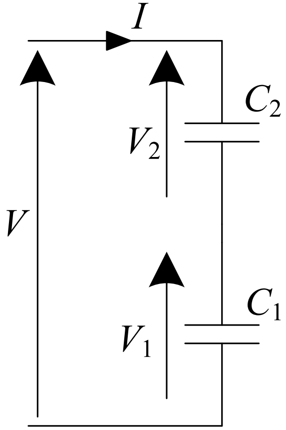
Series and Parallel Capacitors

$$i = i_1 + i_2 = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt}$$
$$= (C_1 + C_2)\frac{dv}{dt}$$



Same equation as a single capacitor of value $C_1 + C_2$

$$\frac{dv}{dt} = \frac{d(v_1+v_2)}{dt} = \frac{dv_1}{dt} + \frac{dv_2}{dt}$$
$$= \frac{i}{C_1} + \frac{i}{C_2} = i\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$
$$i = \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}} \frac{dv}{dt}$$

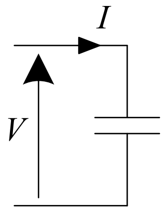


Same as a single capacitor of value $\frac{1}{\frac{1}{C_1}+\frac{1}{C_2}} = \frac{C_1 C_2}{C_1+C_2}$

Capacitors combine just like conductances (i.e. parallel capacitors add).

Current/Voltage Continuity

Capacitor: $i = C\frac{dv}{dt}$



For the voltage to change abruptly

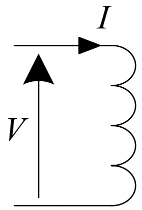
$$\frac{dv}{dt} = \infty \Rightarrow i = \infty.$$

This never happens so ...

The voltage across a capacitor never changes instantaneously.

Informal version: A capacitor "tries" to keep its voltage constant.

Inductor: $v = L\frac{di}{dt}$



For the current to change abruptly

$$\frac{di}{dt} = \infty \Rightarrow v = \infty.$$

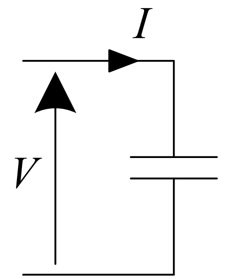
This never happens so ...

The current through an inductor never changes instantaneously.

Informal version: An inductor "tries" to keep its current constant.

Average Current/Voltage

For a capacitor $i = C\frac{dv}{dt}$. Take the average of both sides:

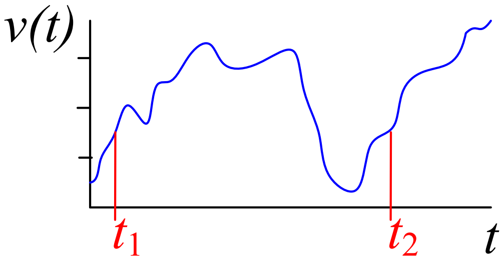


$$\frac{1}{t_2-t_1}\int_{t_1}^{t_2} i dt = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} C\frac{dv}{dt}dt = \frac{C}{t_2-t_1}\int_{v(t_1)}^{v(t_2)} dv$$

$$= \frac{C}{t_2-t_1}\left[v\right]_{v(t_1)}^{v(t_2)} = \frac{C}{t_2-t_1}\left(v(t_2) - v(t_1)\right)$$

(1) If $v(t_1) = v(t_2)$ then the average current exactly equals zero.

(2) If v is bounded then the average current $\to 0$ as $(t_2 - t_1) \to \infty$.



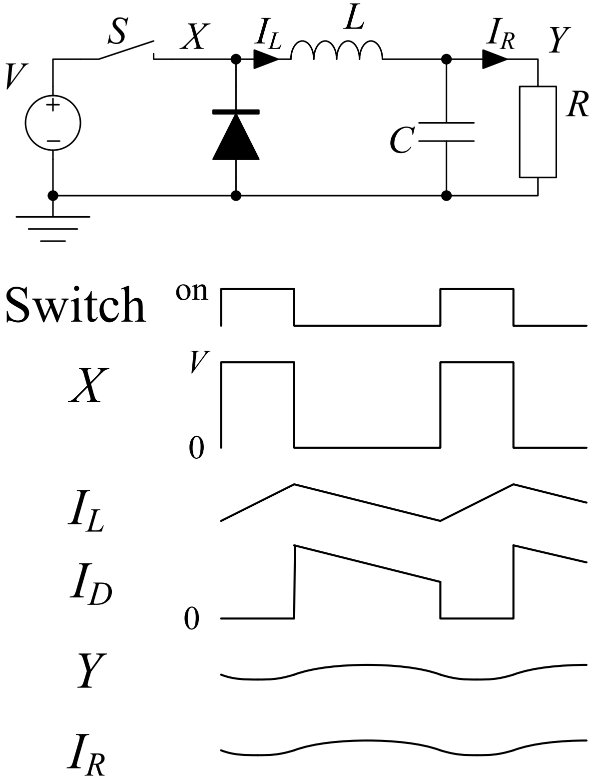
The average current through a capacitor is zero and, likewise, the average voltage across an inductor is zero. The circuit symbols remind you of this.

"Average" can either be over an exact number of periods of a repetitive waveform or else the long-term average (provided v and i remain bounded).

"v is bounded" means $|v|$ always stays less than a predefined maximum value.

Buck Converter

[Do not memorize this circuit]



A buck converter converts a high voltage, V, into a lower one, Y.

The switch, S, closes for a fraction a of the time. a is the *duty cycle* and is $\frac{1}{3}$ in this example.

When S is closed, $x = v$, and a current i_L flows.

When S opens, the current i_L cannot change instantly and so it must flow through the diode (we assume the diode is ideal).

The average value of x is aV $\Rightarrow$ the average value of y must also be aV.

The average current through R is $\frac{aV}{R}$ so, since the average current through C must be zero, the average current i_L must also be $\frac{aV}{R}$.

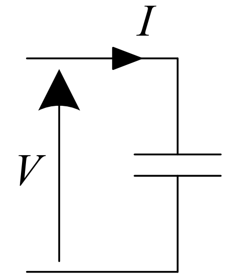
$C\frac{dy}{dt} = i_L - i_R$ $\Rightarrow$ if C is large, then the variations in y will be very small.

Power and Energy

Electrical power absorbed by any component at the instant t is $v(t) \times i(t)$.

So total energy absorbed between times t_1 and t_2 is $W = \int_{t=t_1}^{t_2} vi\, dt$.

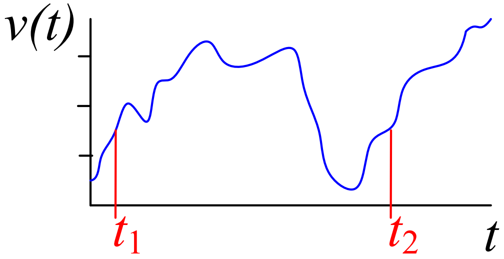
For a capacitor $i = C\frac{dv}{dt}$, so



$$W = C\int_{t=t_1}^{t_2} v\frac{dv}{dt}dt = C\int_{v=v(t_1)}^{v(t_2)} vdv$$

$$= C\left[\tfrac{1}{2}v^2\right]_{v(t_1)}^{v(t_2)} = \tfrac{1}{2}C\left(v^2(t_2) - v^2(t_1)\right)$$

If $v(t_1) = v(t_2)$ then there has been no nett energy absorbed: all the energy absorbed when the voltage rises is returned to the circuit when it falls.



The energy stored in a capacitor is $\frac{1}{2}Cv^2$ and likewise in an inductor $\frac{1}{2}Li^2$.

If v and i remain bounded, then the average power absorbed by a capacitor or inductor is always zero.

Summary

- Capacitor:
  - $i = C\frac{dv}{dt}$
  - parallel capacitors add in value
  - average i is zero, v never changes instantaneously.
  - average power absorbed is zero
- Inductor:
  - $v = L\frac{di}{dt}$
  - series inductors add in value (like resistors)
  - average v is zero, i never changes instantaneously.
  - average power absorbed is zero

For further details see Hayt Ch 7 or Irwin Ch 6.