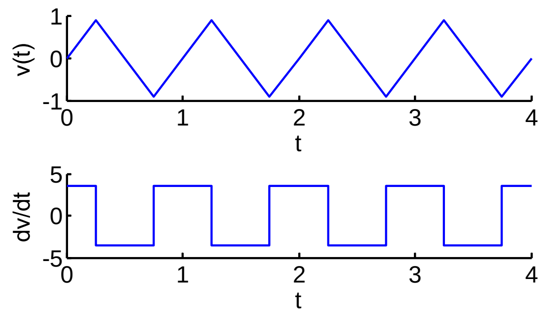
10: Sine waves and phasors

Sine Waves

For inductors and capacitors $i = C\frac{dv}{dt}$ and $v = L\frac{di}{dt}$ so we need to differentiate $i(t)$ and $v(t)$ when analysing circuits containing them.

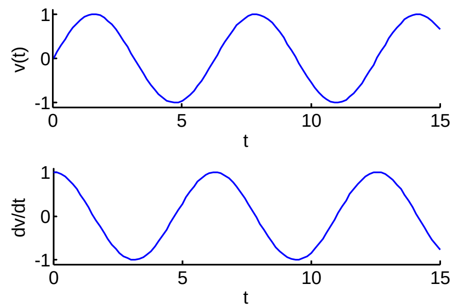
Usually differentiation changes the shape of a waveform.

For bounded waveforms there is only one exception:



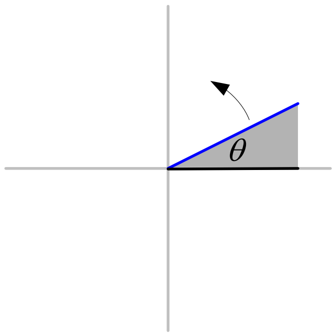
$v(t) = \sin t \Rightarrow \frac{dv}{dt} = \cos t$
same shape but with a time shift.

$\sin t$ completes one full period every time t increases by 2π.



$\sin 2\pi f t$ makes f complete repetitions every time t increases by 1; this gives a *frequency* of f cycles per second, or $f\,\mathrm{Hz}$.
We often use the *angular frequency*, $\omega = 2\pi f$ instead.
ω is measured in radians per second. E.g. $50\,\mathrm{Hz} \simeq 314\,\mathrm{rad.s}^{-1}$.

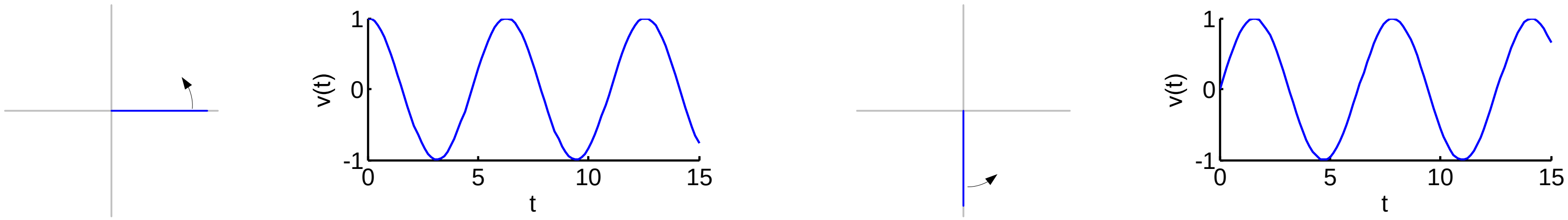
A useful way to think of a cosine wave is as the projection of a rotating rod onto the horizontal axis.



For a unit-length rod, the projection has length $\cos\theta$.

If the rod is rotating at a speed of f revolutions per second, then θ increases uniformly with time: $\theta = 2\pi f t$.

The only difference between $\cos$ and $\sin$ is the starting position of the rod:

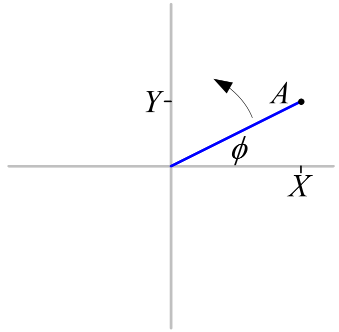


$v = \cos 2\pi f t$ $\quad$ $v = \sin 2\pi f t = \cos\left(2\pi f t - \frac{\pi}{2}\right)$

$\sin 2\pi f t$ *lags* $\cos 2\pi f t$ by 90° (or $\frac{\pi}{2}$ radians) because its peaks occurs $\frac{1}{4}$ of a cycle later (equivalently $\cos$ *leads* $\sin$) .

If the rod has length A and starts at an angle ϕ then the projection onto the horizontal axis is

$$\begin{aligned} &A\cos(2\pi ft+\phi) \\ &\quad = A\cos\phi\cos 2\pi ft - A\sin\phi\sin 2\pi ft \\ &\quad = X\cos 2\pi ft - Y\sin 2\pi ft \end{aligned}$$



At time $t=0$, the tip of the rod has coordinates $(X, Y) = (A\cos\phi, A\sin\phi)$.

If we think of the plane as an Argand Diagram (or complex plane), then the complex number $X + jY$ corresponding to the tip of the rod at $t = 0$ is called a *phasor*.

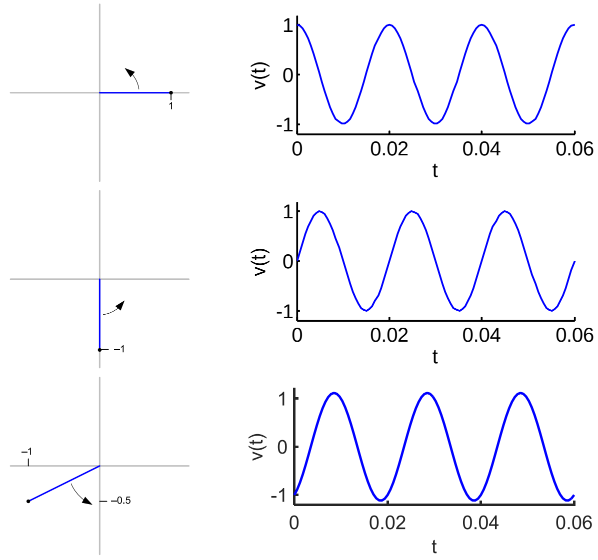
The *magnitude* of the phasor, $A = \sqrt{X^2+Y^2}$, gives the amplitude (peak value) of the sine wave.

The *argument* of the phasor, $\phi = \arctan\frac{Y}{X}$, gives the phase shift relative to $\cos 2\pi ft$.
If $\phi > 0$, it is *leading* and if $\phi < 0$, it is *lagging* relative to $\cos 2\pi ft$.

$$V = 1,\ f = 50\,\text{Hz}$$
$$v(t) = \cos 2\pi f t$$

$$V = -j$$
$$v(t) = \sin 2\pi f t$$

$$V = -1 - 0.5j = 1.12\angle -153^\circ$$
$$v(t) = -\cos 2\pi f t + 0.5 \sin 2\pi f t$$
$$= 1.12 \cos\left(2\pi f t - 2.68\right)$$



$$V = X + jY$$
$$v(t) = X \cos 2\pi f t - Y \sin 2\pi f t$$
Beware minus sign.

$$V = A\angle\phi = Ae^{j\phi}$$
$$v(t) = A\cos\left(2\pi f t + \phi\right)$$

A phasor represents an entire waveform (encompassing all time) as a single complex number. We assume the frequency, f, is known.

A phasor is not time-varying, so we use a capital letter: V.
A waveform is time-varying, so we use a small letter: $v(t)$.

Casio: $\mathrm{Pol}(X, Y) \to A, \phi$, $\mathrm{Rec}(A, \phi) \to X, Y$. Saved $\to X$ & Y mems.

[Algebraic Phasor↔Waveform Mapping]

A phasor is a complex number, V, that uniquely defines a waveform, $v(t)$, via the mapping $V = Ae^{j\phi} \longleftrightarrow v(t) = A\cos(2\pi ft + \phi)$. It is sometimes convenient to give an algebraic formula for this.

For the direction $V \longrightarrow v(t)$ the mapping is easy:

$$v(t) = \Re\left(Ve^{j2\pi ft}\right) = \tfrac{1}{2}\left(V + V^*\right)\cos 2\pi ft + \tfrac{1}{2}j\left(V - V^*\right)\sin 2\pi ft.$$

The reverse mapping, $V \longleftarrow v(t)$ is a bit more complicated and we use a technique that you will also use in the Maths of Fourier transforms. The mapping is given by

$$V = 2f\int_0^{\frac{1}{f}} v(t)e^{-j2\pi ft}dt.$$

To confrm that this is true, we can substitute $v(t) = A\cos(2\pi ft + \phi)$ and do the integration:

$$\begin{aligned}
2f\int_0^{\frac{1}{f}} v(t)e^{-j2\pi ft}dt &= Af\int_0^{\frac{1}{f}} \left(e^{j(2\pi ft + j\phi} + e^{-j2\pi ft - j\phi}\right)e^{-j2\pi ft}dt \\
&= Af\int_0^{\frac{1}{f}} \left(e^{j\phi} + e^{-j4\pi ft - j\phi}\right)dt = Ae^{j\phi} + Afe^{-j\phi}\int_0^{\frac{1}{f}} e^{-j4\pi ft}dt \\
&= Ae^{j\phi} + \frac{Afe^{-j\phi}}{-j4\pi f}\left[e^{-j4\pi ft}\right]_0^{\frac{1}{f}} = Ae^{j\phi} + \frac{Afe^{-j\phi}}{-j4\pi f}\left(e^{-j4\pi} - 1\right) = Ae^{j\phi}
\end{aligned}$$

Phasors	Waveforms
$V = P + jQ$	$v(t) = P\cos\omega t - Q\sin\omega t$ where $\omega = 2\pi f$.
aV	$a \times v(t) = aP\cos\omega t - aQ\sin\omega t$
$V_1 + V_2$	$v_1(t) + v_2(t)$

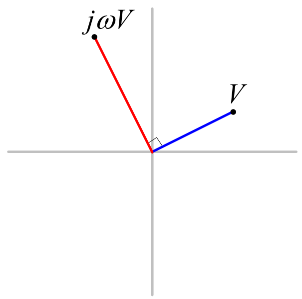
Adding or scaling is the same for waveforms and phasors.

$$\begin{aligned}\dot{V} &= (-\omega Q) + j(\omega P)\\ &= j\omega(P + jQ)\\ &= j\omega V\end{aligned}$$

$$\begin{aligned}\frac{dv}{dt} &= -\omega P\sin\omega t - \omega Q\cos\omega t\\ &= (-\omega Q)\cos\omega t - (\omega P)\sin\omega t\end{aligned}$$

Differentiating waveforms corresponds to multiplying phasors by $j\omega$.

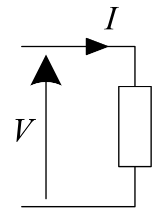
Rotate anti-clockwise 90° and scale by $\omega = 2\pi f$.



Complex Impedances

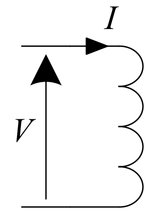
Resistor:

$$v(t) = Ri(t) \Rightarrow V = RI \quad \Rightarrow \frac{V}{I} = R$$



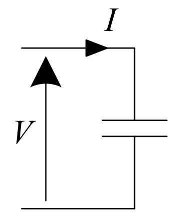
Inductor:

$$v(t) = L\frac{di}{dt} \Rightarrow V = j\omega LI \quad \Rightarrow \frac{V}{I} = j\omega L$$



Capacitor:

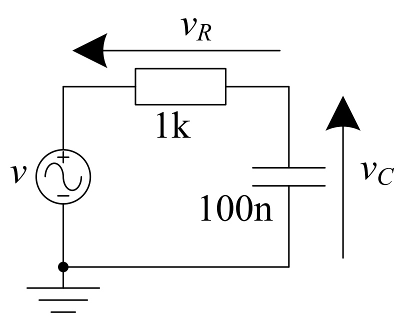
$$i(t) = C\frac{dv}{dt} \Rightarrow I = j\omega CV \quad \Rightarrow \frac{V}{I} = \frac{1}{j\omega C}$$



For all three components, phasors obey Ohm's law if we use the *complex impedances* $j\omega L$ and $\frac{1}{j\omega C}$ as the "resistance" of an inductor or capacitor.

If all sources in a circuit are sine waves having the same frequency, we can do circuit analysis exactly as before by using complex impedances.

Given $v = 10\sin\omega t$ where $\omega = 2\pi \times 1000$, find $v_C(t)$.

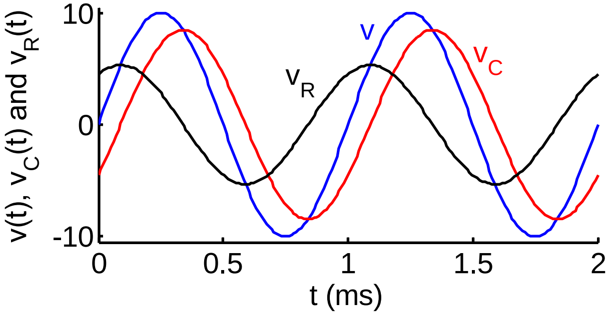


(1) Find capacitor complex impedance

$$Z = \frac{1}{j\omega C} = \frac{1}{6.28j\times 10^{-4}} = -1592j$$

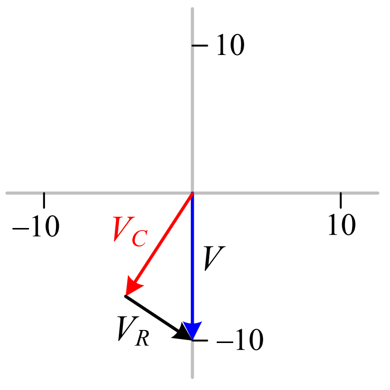
(2) Solve circuit with phasors

$$V_C = V \times \frac{Z}{R+Z}$$
$$= -10j \times \frac{-1592j}{1000-1592j}$$
$$= -4.5 - 7.2j = 8.47\angle -122^\circ$$
$$v_C = 8.47\cos(\omega t - 122^\circ)$$



(3) Draw a *phasor diagram* showing KVL:

$$V = -10j$$
$$V_C = -4.5 - 7.2j$$
$$V_R = V - V_C = 4.5 - 2.8j = 5.3\angle -32^\circ$$



Phasors add like vectors

[Differential Equation Analysis]

To solve the problem form the previous slide without using phasors, we define i to be the current flowing clockwise and use the capacitor equation $i = C\frac{dv_C}{dt}$.

From KVL, we have $v = v_R + v_C = iR + v_C$.
Differentiating and applying the capacitor equation gives $\frac{dv}{dt} = 10\omega\cos\omega t = R\frac{di}{dt} + \frac{1}{C}i$.

We need to find the particular integral for the above equation. To do so, we guess that the answer will be of the form $i = A\cos\omega t + B\sin\omega t$ and substitute it into the equation (multiplied by C).

$$\begin{aligned} 10C\omega\cos\omega t &= RC\left(-A\omega\sin\omega t + B\omega\cos\omega t\right) + \left(A\cos\omega t + B\sin\omega t\right) \\ &= \left(A + RCB\omega\right)\cos\omega t + \left(B - RCA\omega\right)\sin\omega t \end{aligned}$$

which gives two siultaneous equations: $A + RC\omega B = 10C\omega$ and $-RC\omega A + B = 0$. Substituting values for R, C and ω gives $A + 0.628B = 0.00628$ and $-0.628A + B = 0$. Solving these simultaneous equations gives $A = 4.5\,\mathrm{mA}$ and $B = 2.8\,\mathrm{mA}$.

The resistor voltage is therefore $v_R = iR = 4.5\cos\omega t + 2.8\sin\omega t$ and therefore, from KVL, the capacitor votage is $v_C = v - v_R = -4.5\cos\omega t + 7.2\sin\omega t$.

Thus we get the same answer as using phasors but with more work even for a simple circuit like this. For more complicated circuits the difference is much much bigger.

Capacitors: $i = C\frac{dv}{dt} \quad \Rightarrow I$ leads V

Inductors: $v = L\frac{di}{dt} \quad \Rightarrow V$ leads I

Mnemonic: CIVIL = "In a capacitor I lead V but V leads I in an inductor".

COMPLEX ARITHMETIC TRICKS:

(1) $j \times j = -j \times -j = -1$
(2) $\frac{1}{j} = -j$
(3) $a + jb = r\angle\theta = re^{j\theta}$
where $r = \sqrt{a^2 + b^2}$ and $\theta = \arctan\frac{b}{a}$ ($\pm 180°$ if $a < 0$)
(4) $r\angle\theta = re^{j\theta} = (r\cos\theta) + j\,(r\sin\theta)$
(5) $a\angle\theta \times b\angle\phi = ab\angle\,(\theta + \phi)$ and $\frac{a\angle\theta}{b\angle\phi} = \frac{a}{b}\angle\,(\theta - \phi)$.
Multiplication and division are much easier in polar form.
(6) All scientific calculators will convert rectangular to/from polar form.

Casio fx-991 (available in all exams except Maths) will do complex arithmetic $(+, -, \times, \div, x^2, \frac{1}{x}, |x|, x^*)$ in CMPLX mode.

Learn how to use this: it will save lots of time and errors.

Impedance and Admittance

For any network (resistors+capacitors+inductors):

(1) Impedance = Resistance + $j\times$ Reactance

$$Z = R + jX \ (\Omega)$$

$$|Z|^2 = R^2 + X^2 \qquad \angle Z = \arctan\frac{X}{R}$$

(2) Admittance = $\frac{1}{\text{Impedance}}$ = Conductance + $j\times$ Susceptance

$$Y = \frac{1}{Z} = G + jB \text{ Siemens (S)}$$

$$|Y|^2 = \frac{1}{|Z|^2} = G^2 + B^2 \qquad \angle Y = -\angle Z = \arctan\frac{B}{G}$$

Note:

$$Y = G + jB = \frac{1}{Z} = \frac{1}{R+jX} = \frac{R}{R^2+X^2} + j\frac{-X}{R^2+X^2}$$

So $G = \frac{R}{R^2+X^2} = \frac{R}{|Z|^2}$

$$B = \frac{-X}{R^2+X^2} = \frac{-X}{|Z|^2}$$

Beware: $G \neq \frac{1}{R}$ unless $X = 0$.

Summary

- Sine waves are the only bounded signals whose shape is unchanged by differentiation.
- Think of a sine wave as the projection of a rotating rod onto the horizontal (or real) axis.
  - A *phasor* is a complex number representing the length and position of the rod at time $t = 0$.
  - If $V = a + jb = r\angle\theta = re^{j\theta}$, then
    $$v(t) = a\cos\omega t - b\sin\omega t = r\cos(\omega t + \theta) = \Re\left(Ve^{j\omega t}\right)$$
  - The *angular frequency* $\omega = 2\pi f$ is assumed known.
- If all sources in a linear circuit are sine waves having the same frequency, we can use phasors for circuit analysis:
  - Use complex impedances: $j\omega L$ and $\frac{1}{j\omega C}$
  - Mnemonic: CIVIL tells you whether I leads V or vice versa ("leads" means "reaches its peak before").
  - Phasors eliminate time from equations ☺, converts simultaneous differential equations into simultaneous linear equations ☺☺☺.
  - Needs complex numbers ☹ but worth it.

See Hayt Ch 10 or Irwin Ch 8