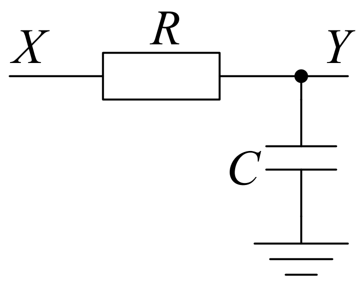
11: Frequency Responses

Frequency Response

If $x(t)$ is a sine wave, then $y(t)$ will also be a sine wave but with a different amplitude and phase shift. X is an input phasor and Y is the output phasor.

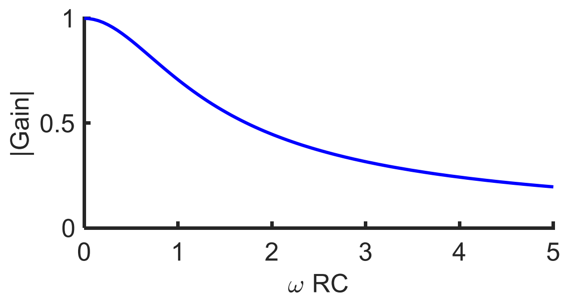


The *gain* of the circuit is $\frac{Y}{X} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{j\omega RC + 1}$

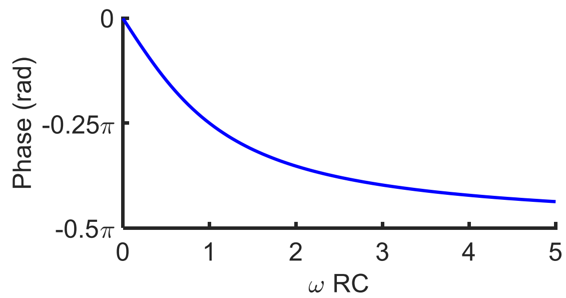
This is a complex function of ω so we plot separate graphs for:

Magnitude: $\left|\frac{Y}{X}\right| = \frac{1}{|j\omega RC+1|} = \frac{1}{\sqrt{1+(\omega RC)^2}}$

Phase Shift: $\angle\left(\frac{Y}{X}\right) = -\angle\left(j\omega RC + 1\right) = -\arctan\left(\frac{\omega RC}{1}\right)$



Magnitude Response

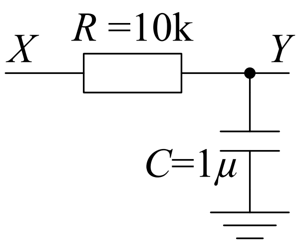


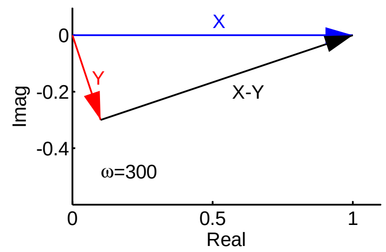
Phase Response

Sine Wave Response

$RC = 10\,\text{ms}$

$\frac{Y}{X} = \frac{1}{j\omega RC+1} = \frac{1}{0.01j\omega+1}$

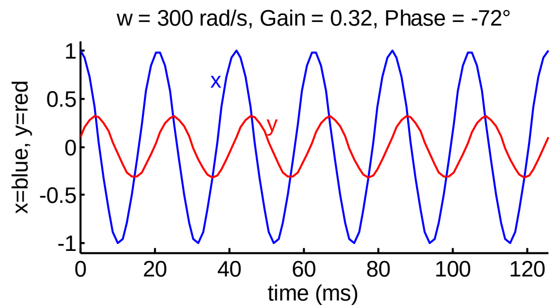


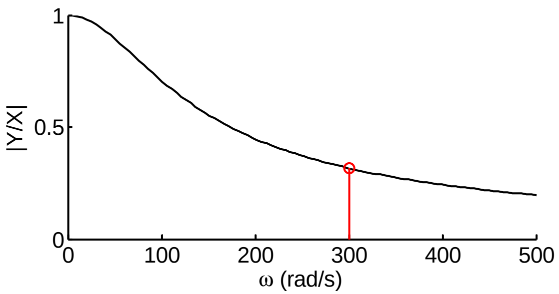


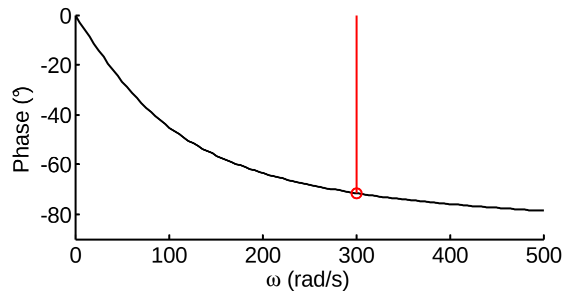
$\omega = 50 \Rightarrow \frac{Y}{X} = 0.89\angle - 27°$

$\omega = 100 \Rightarrow \frac{Y}{X} = 0.71\angle - 45°$

$\omega = 300 \Rightarrow \frac{Y}{X} = 0.32\angle - 72°$







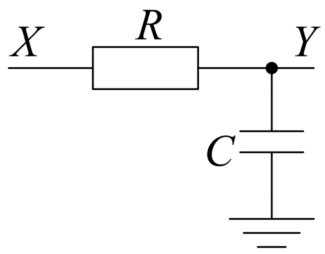
The output, $y(t)$, *lags* the input, $x(t)$, by up to $90°$.

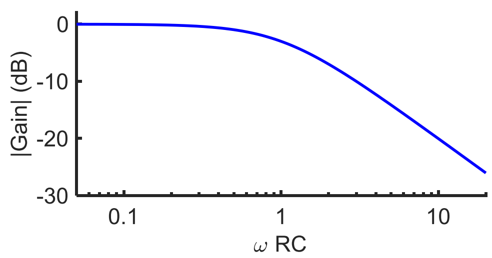
We usually use logarithmic axes for frequency and gain (but not phase) because % differences are more significant than absolute differences.
E.g. 5 kHz versus 5.005 kHz is less significant than 10 Hz versus 15 Hz even though both differences equal 5 Hz.

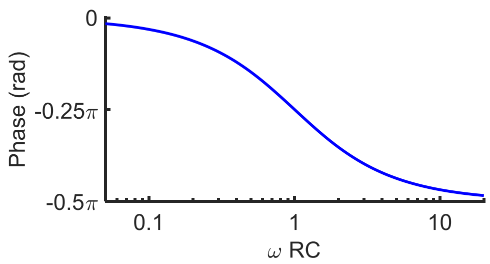
Logarithmic voltage ratios are specified in *decibels* (dB) $= 20\log_{10}\frac{|V_2|}{|V_1|}$.

Common voltage ratios:

$\frac{\lvert V_2\rvert}{\lvert V_1\rvert}$	0.1	0.5	$\sqrt{0.5}$	1	$\sqrt{2}$	2	10	100
dB	-20	-6	-3	0	3	6	20	40







Note that 0 does not exist on a log axis and so the starting point of the axis is arbitrary.

Note: $P \propto V^2 \Rightarrow$ decibel power ratios are given by $10\log_{10}\frac{P_2}{P_1}$

$H = c\,(j\omega)^r$ has a straight-line magnitude graph and a constant phase.

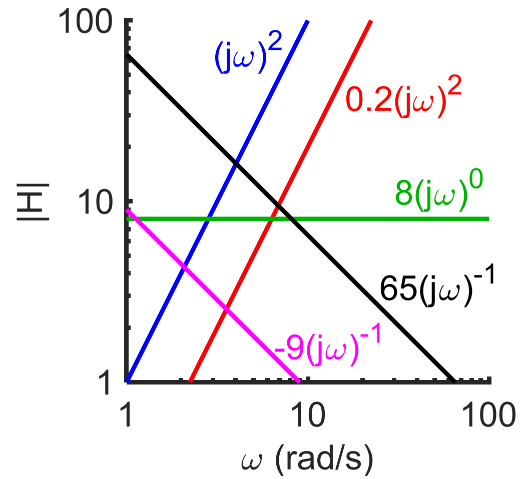
Magnitude (log-log graph):

$|H| = c\omega^r \Rightarrow \log|H| = \log|c| + r\log\omega$

This is a straight line with a slope of r.

c only affects the line's vertical position.

If $|H|$ is measured in decibels, a slope of r is called $6r$ dB/octave or $20r$ dB/decade.



Phase (log-lin graph):

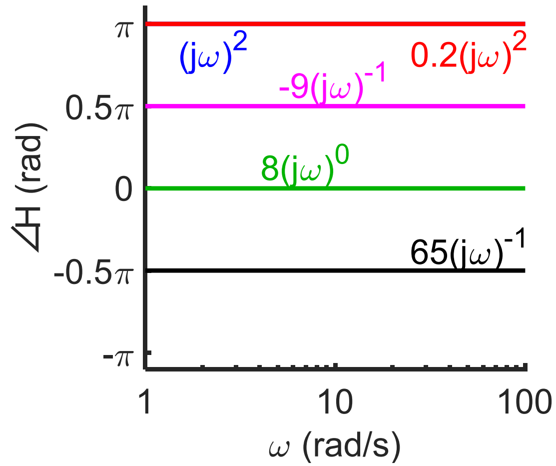
$\angle H = \angle j^r + \angle c = r \times \frac{\pi}{2}$ $(+\pi$ if $c < 0)$

The phase is constant $\forall\omega$.

If $c > 0$, phase $= 90° \times$ magnitude slope.

Negative c adds $\pm 180°$ to the phase.

Note: Phase angles are modulo $360°$, i.e. $+180° \equiv -180°$ and $450° \equiv 90°$.



[Octaves and Decades]

An "octave" is a factor of 2 in frequency; for example, $20\,\mathrm{Hz}$ is one octave greater than $10\,\mathrm{Hz}$. Similarly a "decade" is a factor of 10 in frequency; for example, $100\,\mathrm{Hz}$ is one decade greater than $10\,\mathrm{Hz}$.

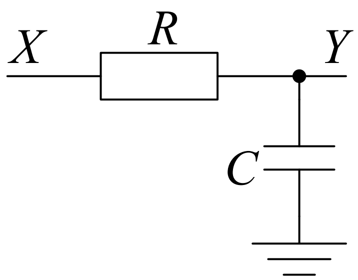
The number of decades between any two frequencies can be calculated by taking $\log_{10}$ of the frequency ratio. Thus, for the example given above, $\log_{10}\left(\frac{100\,\mathrm{Hz}}{10\,\mathrm{Hz}}\right) = \log_{10}(10) = 1\,\mathrm{decade}$. A slightly more complicated example is $\log_{10}\left(\frac{13\,\mathrm{kHz}}{25\,\mathrm{Hz}}\right) = \log_{10}\left(\frac{13000}{25}\right) = \log_{10}(520) = 2.716\,\mathrm{decades}$ so this means that $13\,\mathrm{kHz}$ is 2.716 decades greater than $25\,\mathrm{Hz}$.

As we shall discover in this lecture, frequency response graphs can be approximated as a series of straight lines whose gradients are easy to calculate. In particular magnitude response graphs can be approximated as a series of straight lines with gradients that are integer multiples of $20\,\mathrm{dB}$ per decade and phase response graphs can be approximated as a series of straight lines with gradients that are integer multiples of 0.25π radians per decade. This means that if you know the magnitude or phase at one frequency, you can calculate how much it has changed at any other frequency by multiplying the gradient of the line by the number of decades by which the frequency has changed.

Calculating the number of *octaves* between any two frequencies is done in the same way except that you must take a base-2 log. Thus between $10\,\mathrm{Hz}$ and $100\,\mathrm{Hz}$ is $\log_2\left(\frac{100\,\mathrm{Hz}}{10\,\mathrm{Hz}}\right) = \log_{10}\left(\frac{100\,\mathrm{Hz}}{10\,\mathrm{Hz}}\right) \div \log_{10} 2 = 3.322\,\log_{10}\left(\frac{100\,\mathrm{Hz}}{10\,\mathrm{Hz}}\right) = 3.322\,\mathrm{octaves}$. Thus one decade is equal to 3.322 octaves.

Straight Line Approximations

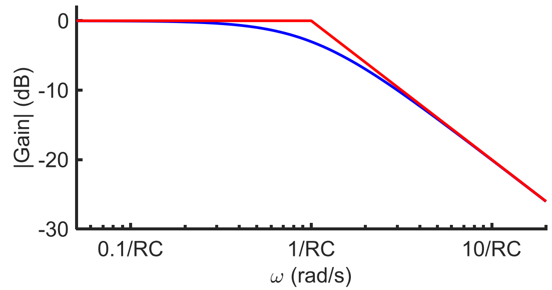
Key idea: $(aj\omega + b) \approx \begin{cases} aj\omega & \text{for } |a\omega| \gg |b| \\ b & \text{for } |a\omega| \ll |b| \end{cases}$



Gain: $H(j\omega) = \frac{1}{j\omega RC+1}$

Low frequencies ($\omega \ll \frac{1}{RC}$): $H(j\omega) \approx 1 \Rightarrow |H(j\omega)| \approx 1$
High frequencies ($\omega \gg \frac{1}{RC}$): $H(j\omega) \approx \frac{1}{j\omega RC} \Rightarrow |H(j\omega)| \approx \frac{1}{RC}\omega^{-1}$

Approximate the magnitude response as two straight lines intersecting at the *corner frequency*, $\omega_c = \frac{1}{RC}$.



At the corner frequency:

(a) the gradient changes by -1 ($= -6$ dB/octave $= -20$ dB/decade).
(b) $|H(j\omega_c)| = \left|\frac{1}{1+j}\right| = \frac{1}{\sqrt{2}} = -3$ dB (worst-case error).

A linear factor $(aj\omega + b)$ has a corner frequency of $\omega_c = \left|\frac{b}{a}\right|$.

Plot Magnitude Response

The gain of a linear circuit is always a *rational polynomial* in $j\omega$ and is called the *transfer function* of the circuit. For example:

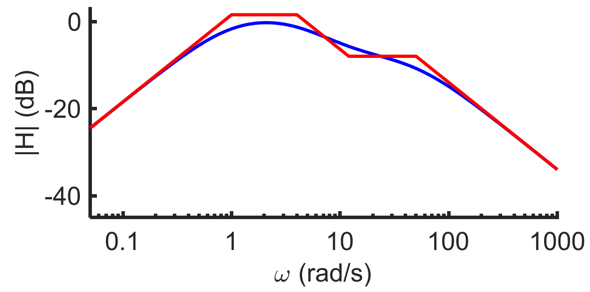
$$H(j\omega) = \frac{60(j\omega)^2+720(j\omega)}{3(j\omega)^3+165(j\omega)^2+762(j\omega)+600} = \frac{20j\omega(j\omega+12)}{(j\omega+1)(j\omega+4)(j\omega+50)}$$

Step 1: Factorize the polynomials

Step 2: Sort corner freqs: $1, 4, 12, 50$

Step 3: For $\omega < 1$ all linear factors equal their constant terms:

$$|H| \approx \frac{20\omega\times12}{1\times4\times50} = 1.2\omega^1.$$



Step 4: For $1 < \omega < 4$, the factor $(j\omega+1) \approx j\omega$ so

$$|H| \approx \frac{20\omega\times12}{\omega\times4\times50} = 1.2\omega^0 = +1.58\,\text{dB}.$$

Step 5: For $4 < \omega < 12$, $|H| \approx \frac{20\omega\times12}{\omega\times\omega\times50} = 4.8\omega^{-1}$.

Step 6: For $12 < \omega < 50$, $|H| \approx \frac{20\omega\times\omega}{\omega\times\omega\times50} = 0.4\omega^0 = -7.96\,\text{dB}$.

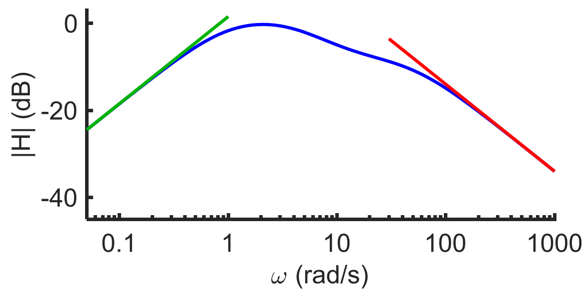
Step 7: For $\omega > 50$, $|H| \approx \frac{20\omega\times\omega}{\omega\times\omega\times\omega} = 20\omega^{-1}$.

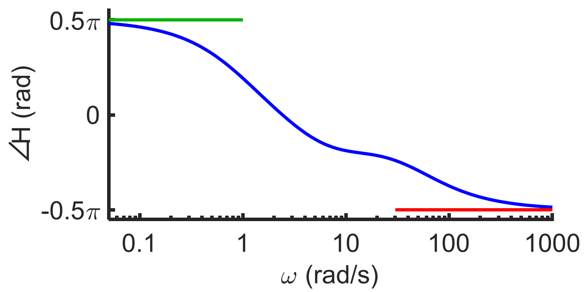
At each corner frequency, the graph is continuous but its gradient changes abruptly by $+1$ (numerator factor) or -1 (denominator factor).

Low and High Frequency Asymptotes

You can find the low and high frequency asymptotes without factorizing:

$$H(j\omega) = \frac{60(j\omega)^2+720(j\omega)}{3(j\omega)^3+165(j\omega)^2+762(j\omega)+600} = \frac{20j\omega(j\omega+12)}{(j\omega+1)(j\omega+4)(j\omega+50)}$$





Low Frequency Asymptote:

From factors: $H_{\text{LF}}(j\omega) = \frac{20j\omega(12)}{(1)(4)(50)} = 1.2j\omega$

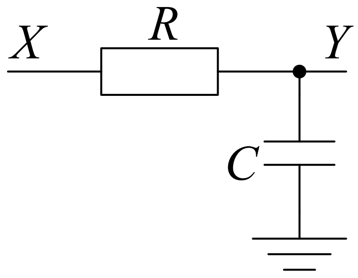
Lowest power of $j\omega$ on top and bottom: $H(j\omega) \simeq \frac{720(j\omega)}{600} = 1.2j\omega$

High Frequency Asymptote:

From factors: $H_{\text{HF}}(j\omega) = \frac{20j\omega(j\omega)}{(j\omega)(j\omega)(j\omega)} = 20(j\omega)^{-1}$

Highest power of $j\omega$ on top and bottom: $H(j\omega) \simeq \frac{60(j\omega)^2}{3(j\omega)^3} = 20(j\omega)^{-1}$

Gain: $H(j\omega) = \frac{1}{j\omega RC+1}$



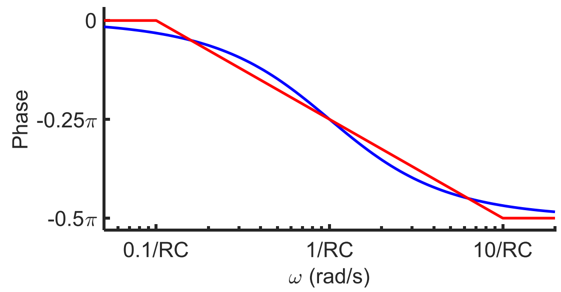
Low frequencies ($\omega \ll \frac{1}{RC}$):

$$H(j\omega) \approx 1 \Rightarrow \angle 1 = 0$$

High frequencies ($\omega \gg \frac{1}{RC}$): $H(j\omega) \approx \frac{1}{j\omega RC} \Rightarrow \angle j^{-1} = -\frac{\pi}{2}$

Approximate the phase response as three straight lines.

By chance, they intersect close to $0.1\omega_c$ and $10\omega_c$ where $\omega_c = \frac{1}{RC}$.



Between $0.1\omega_c$ and $10\omega_c$ the phase changes by $-\frac{\pi}{2}$ over two decades. This gives a gradient $= -\frac{\pi}{4}$ radians/decade.

$(aj\omega + b)$ in denominator

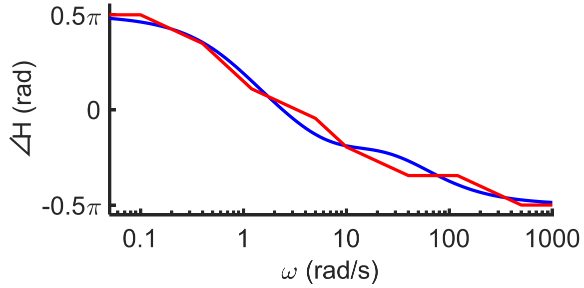
$$\Rightarrow \Delta\text{gradient} = \mp\frac{\pi}{4}/\text{decade at } \omega = 10^{\mp 1}\left|\frac{b}{a}\right|.$$

The sign of Δgradient is reversed for (a) numerator factors and (b) $\frac{b}{a} < 0$.

[Phase Approximation ++]

Like the magnitude response, the phase response can be approximated by a graph that consists of a sequence of straight line segments that are joined at "corners". For this to be true, we need to plot the phase response using a *linear* axis for the phase but a *logarithmic* axis for the frequency.
The previous slide showed the phase response of a filter whose frequency response, $H(z)$, has a single linear factor in the denominator. On the next slide this is extended to a more complicated frequency response.
Recall that the argument of a complex number is $\angle(a+jb) = \tan^{-1}\frac{b}{a}$ and $\angle\frac{1}{a+jb} = -\tan^{-1}\frac{b}{a}$.
Therefore if the frequency response is $H(j\omega) = \frac{1}{j\omega RC+1}$, then the phase is given by $\angle H(j\omega) = -\tan^{-1}\omega RC$ which is plotted as the blue curve. At low frequencies, this tends to zero (since $\tan^{-1} 0 = 0$) and at high frequencies it tends to $-\frac{\pi}{2}$ (since $\tan^{-1}\infty = \frac{\pi}{2}$). The magnitude response graph has a corner frequency at $\omega_c = \frac{1}{RC}$ and at this frequency, $\angle H(j\omega_c) = -\tan^{-1} 1 = -\frac{\pi}{4}$.
It turns out that we can approximate this curve with three straight lines which meet at two "phase response corner frequencies" of $0.1\omega_c$ and $10\omega_c$. Since the frequency range $0.1\omega_c$ to $10\omega_c$ is two decades (a factor of 100), the gradient of the central segment of the approximation must be $-\frac{\pi}{4}$ radians/decade. This approximation is not actually the best possible approximation using 3 straight lines but it is very close and much easier to remember that the optimum approximation.
To summarise: A linear factor of $(aj\omega + b)$ in the denominator will result in two corner frequencies in the phase response at $\omega = 10^{-1}\left|\frac{b}{a}\right|$ and $10^{+1}\left|\frac{b}{a}\right|$. At these frequencies, the gradient of the graph will change by $-\frac{\pi}{4}$ and $+\frac{\pi}{4}$ radians/decade respectively. The signs of the gradient changes will be reversed for numerator factors and reversed again if $\frac{b}{a}$ is negative (which is rare and can only happen in the numerator).

$$H(j\omega) = \frac{60(j\omega)^2+720(j\omega)}{3(j\omega)^3+165(j\omega)^2+762(j\omega)+600} = \frac{20j\omega(j\omega+12)}{(j\omega+1)(j\omega+4)(j\omega+50)}$$



Step 1: Factorize the polynomials

Step 2: List corner freqs: $\pm =$ num/den

$\omega_c = \{1^-, 4^-, 12^+, 50^-\}$

Step 3: Gradient changes at $10^{\mp 1}\omega_c$.
Sign depends on num/den and sgn $\left(\frac{b}{a}\right)$:
$.1^-, 10^+; .4^-, 40^+; 1.2^+, 120^-; 5^-, 500^+$

Step 4: Put in ascending order and calculate gaps as $\log_{10}\frac{\omega_2}{\omega_1}$ decades:
$.1^-\ (.6)\ .4^-\ (.48)\ 1.2^+\ (.62)\ 5^-\ (.3)\ 10^+\ (.6)\ 40^+\ (.48)\ 120^-\ (.62)\ 500^+$.

Step 5: Find phase of LF asymptote: $\angle 1.2j\omega = +\frac{\pi}{2}$.

Step 6: At $\omega = 0.1$ gradient becomes $-\frac{\pi}{4}$ rad/decade. ϕ is still $\frac{\pi}{2}$.

Step 7: At $\omega = 0.4$, $\phi = \frac{\pi}{2} - 0.6\frac{\pi}{4} = 0.35\pi$. New gradient is $-\frac{\pi}{2}$.

Step 8: At $\omega = 1.2$, $\phi = 0.35\pi - 0.48\frac{\pi}{2} = 0.11\pi$. New gradient is $-\frac{\pi}{4}$.

Steps 9-13: Repeat for each gradient change. Final gradient is always 0.

At 0.1 and 10 times each corner frequency, the graph is continuous but its gradient changes abruptly by $\pm\frac{\pi}{4}$ rad/decade.

[Plot Phase Response ++]

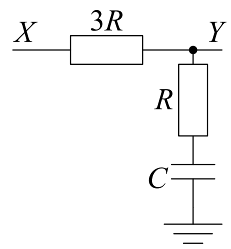
Like the magnitude response, the phase response can be approximated by a graph that consists of a sequence of straight line segments that are joined at "corners". For this to be true, we need to plot the phase response using a *linear* axis for the phase but a *logarithmic* axis for the frequency. As we saw on the previous slide, each linear factor in either the numerator or the denominator gives rise to two corners in the phase response graph. At each of these corners, the gradient of the graph changes abruptly by $\pm\frac{\pi}{4}$ radians/decade; it follows that the gradient will always be an integer multiple of $\frac{\pi}{4}$ radians/decade.
In order to plot the phase response graph, we need to determine three things: (a) the frequencies of all the corners, (b) the sign of the gradient change at each one and (c) the phase at low frequencies (i.e. frequencies less than the first corner). The example response on the slide, $H(j\omega) = \frac{20j\omega(j\omega+12)}{(j\omega+1)(j\omega+4)(j\omega+50)}$ has four linear factors: one in the numerator and three in the denominator. This means we will have a total of eight corners (two from each linear factor). Since all the factors have $\frac{b}{a} > 0$ the signs of the gradient changes will be $+$ followed by $-$for the numerator factor and $-$ followed by $+$ for the denominator factors. The two corner frequencies corresponding to a factor $(aj\omega + b)$ are at $\omega = 0.1\left|\frac{b}{a}\right|$ and $10\left|\frac{b}{a}\right|$. So, using a superscript for the sign of the gradient change, we get corners at 1.2^+ and 120^- for the numerator factor and at 0.1^-, 0.4^-, 10^+, 40^+, 5^- and 500^+ from the three denominator factors. Sorting these into ascending order of ω gives corners at 0.1^-, 0.4^-, 1.2^+, 5^-, 10^+, 40^+, 120^- and 500^+.

[Plot Phase Response ++]

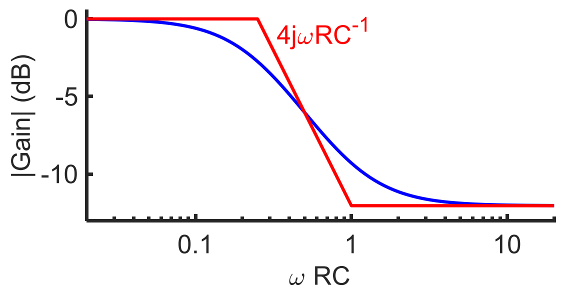
To plot the phase response, we calculate the low frequency asymptote by taking the terms with the lowest power of $j\omega$ in numerator and denominator; this gives $1.2j\omega$ which has a phase of $+\frac{\pi}{2} = 1.57$ radians. So we begin with a horizontal line at 1.57 radians until the first corner frequency at $\omega = 0.1^-$ where the gradient becomes $-\frac{\pi}{4}$. The graph will continue with this gradient until the next corner frequency which is at $\omega = 0.4^-$ where the gradient will decrease by another $\frac{\pi}{4}$ to become $-\frac{\pi}{2}$.
To work out the phase at the second corner frequency ($\omega = 0.4$) we calculate how much the phase has changed between $\omega = 0.1$ and 0.4 by multiplying the gradient of the graph ($-\frac{\pi}{4}$ radians/decade) by the separation of these two corner frequencies in decades ($\log_{10} \frac{0.4}{0.1} = 0.602$ decades). This product gives gives a phase change of -0.473 radians. So the phase is 1.571 radians at $\omega = 0.1$ and decreases by -0.473 to become 1.098 radians at $\omega = 0.4$.
The next corner is at $\omega = 1.2^+$ which is $\log_{10} \frac{1.2}{0.4} = 0.477$ decades away from $\omega = 0.4$. Since the gradient in this segment is $-\frac{\pi}{2} = -1.571$ rads/decade, the phase change between these two frequencies is $-1.571 \times 0.477 = -0.749$ radians. So the phase at $\omega = 1.2$ is $1.098 - 0.749 = 0.349$ radians.
You continue like this hopping from each corner frequency to the next. At each corner frequency, you know the new gradient (measured in radians/decade) and so you multiply this by the distance to the next corner frequency (measured in decades) to get the phase change between the two corner frequencies.
As a check, the gradient after the final corner frequency should be zero and the phase should match the phase of the high frequency asymptote. In this example, the high frequency asymptote is $20\,(j\omega)^{-1}$ which has a phase of $-\frac{\pi}{2}$. (Remember that j^r has a phase of $\left(\frac{\pi}{2}\right)^r$).

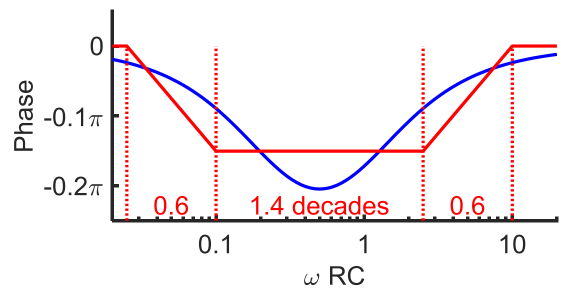
RCR Circuit

$$\frac{Y}{X} = \frac{R+\frac{1}{j\omega C}}{3R+R+\frac{1}{j\omega C}} = \frac{j\omega RC+1}{4j\omega RC+1}$$



Corner freqs: $\frac{0.25}{RC}^{-}, \frac{1}{RC}^{+}$ LF Asymptote: $H(j\omega) = 1$





Magnitude Response:

Gradient Changes: $-20\,\text{dB/dec}$ at $\omega = \frac{0.25}{RC}$ and $+20$ at $\omega = \frac{1}{RC}$

Line equations: $H(j\omega) =$ (a) 1, (b) $\frac{1}{4j\omega RC}$, (c) $\frac{j\omega RC}{4j\omega RC} = 0.25$

Phase Response:

LF asymptote: $\phi = \angle 1 = 0$

Gradient changes of $\pm\frac{\pi}{4}$/decade at: $\omega = \frac{0.025}{RC}^{-}, \frac{0.1}{RC}^{+}, \frac{2.5}{RC}^{+}, \frac{10}{RC}^{-}$.

At $\omega = \frac{0.1}{RC}$, $\phi = 0 - \frac{\pi}{4}\log_{10}\frac{0.1}{0.025} = -\frac{\pi}{4} \times 0.602 = -0.15\pi$

Summary

- Frequency response: magnitude and phase of $\frac{Y}{X}$ as a function of ω
  - Only applies to sine waves
  - Use log axes for frequency and gain but linear for phase
    - Decibels $= 20\log_{10}\frac{V_2}{V_1} = 10\log_{10}\frac{P_2}{P_1}$
- Linear factor $(aj\omega + b)$ gives corner frequency at $\omega = \left|\frac{b}{a}\right|$.
  - Magnitude plot gradient changes by ± 20 dB/decade @ $\omega = \left|\frac{b}{a}\right|$.
  - Phase gradient changes in two places by:
    - $\pm\frac{\pi}{4}$ rad/decade @ $\omega = 0.1 \times \left|\frac{b}{a}\right|$
    - $\mp\frac{\pi}{4}$ rad/decade @ $\omega = 10 \times \left|\frac{b}{a}\right|$
- LF/HF asymptotes: keep only the terms with the lowest/highest power of $j\omega$ in numerator and denominator polynomials

For further details see Hayt Ch 16 or Irwin Ch 12.