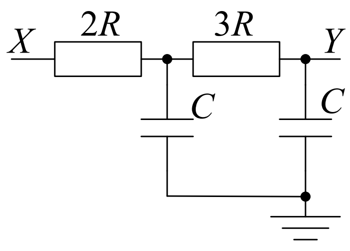
12: Resonance

Quadratic Factors

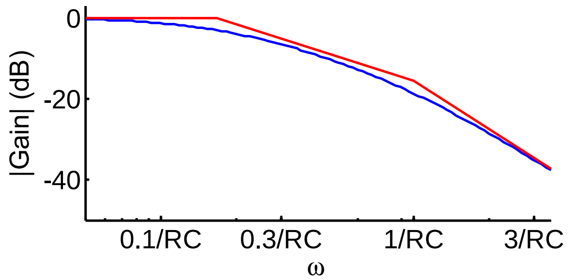
A quadratic factor in a transfer function is: $F(j\omega) = a\,(j\omega)^2 + b\,(j\omega) + c$.

Case 1: If $b^2 \geq 4ac$ then we can factorize it:



$$F(j\omega) = a(j\omega - p_1)(j\omega - p_2)$$

where $p_i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.



$$\frac{Y}{X}(j\omega) = \frac{1}{6R^2C^2(j\omega)^2 + 7RCj\omega + 1}$$

$$= \frac{1}{(6j\omega RC + 1)(j\omega RC + 1)}$$

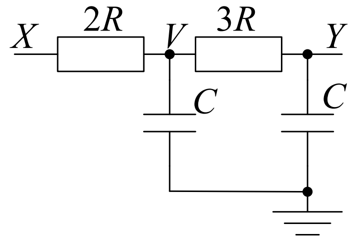
$$\omega_c = \frac{0.17}{RC},\ \frac{1}{RC} = |p_1|\,,\ |p_2|$$

Case 2: If $b^2 < 4ac$, we cannot factorize with real coefficients so we leave it as a quadratic. Sometimes called a *quadratic resonance*.

Any polynomial with real coefficients can be factored into linear and quadratic factors ⇒ a quadratic factor is as complicated as it gets.

[Derivation of Transfer Function]

KCL at V gives



$$\frac{V-X}{2R} + j\omega CV + \frac{V-Y}{3R} = 0 \Rightarrow 3(V-X) + 6j\omega RCV + 2(V-Y) = 0$$

$$\Rightarrow (5 + 6j\omega RC)V = 3X + 2Y.$$

KCL at Y gives

$$\frac{Y-V}{3R} + j\omega CY = 0 \quad \Rightarrow \quad (1 + 3j\omega RC)Y = V.$$

Eliminating V beween these two equations gives

$$(5 + 6j\omega RC)(1 + 3j\omega RC)Y = 3X + 2Y$$

$$\Rightarrow \left(5 + 21j\omega RC + 18(j\omega RC)^2 - 2\right)Y = 3X$$

$$\Rightarrow \frac{Y}{X} = \frac{3}{3+21j\omega RC+18(j\omega RC)^2} = \frac{1}{1+7j\omega RC+6(j\omega RC)^2} = \frac{1}{(1+6j\omega RC)(1+j\omega RC)}.$$

At high frequencies, the impedance of the capacitor is much less than $3R$ so we can think of the circuit as two potential dividers one after the other (i.e. the current through the $3R$ is negligible compared to the current throught the first C). The high frequency asymptote is therefore the product of the asymptotes for the two potential dividers which gives $\frac{Y}{X} \approx \frac{1}{2j\omega RC} \times \frac{1}{3j\omega RC} = \frac{1}{6(j\omega RC)^2}$.

Suppose $b^2 < 4ac$ in $F(j\omega) = a\,(j\omega)^2 + b\,(j\omega) + c$.

Low/High freq asymptotes: $F_{\text{LF}}(j\omega) = c, \quad F_{\text{HF}}(j\omega) = a\,(j\omega)^2$

The asymptote magnitudes cross at the *corner frequency*:

$$\left|a\,(j\omega_c)^2\right| = |c| \Rightarrow \omega_c = \sqrt{\tfrac{c}{a}}.$$

We define the *damping factor*, "zeta", to be $\zeta = \frac{b}{2a\omega_c} = \frac{b\omega_c}{2c} = \frac{b\,\text{sgn}(a)}{\sqrt{4ac}}$

$$\Rightarrow F(j\omega) = c\left(\left(j\frac{\omega}{\omega_c}\right)^2 + 2\zeta\left(j\frac{\omega}{\omega_c}\right) + 1\right)$$

Properties to notice in this expression:

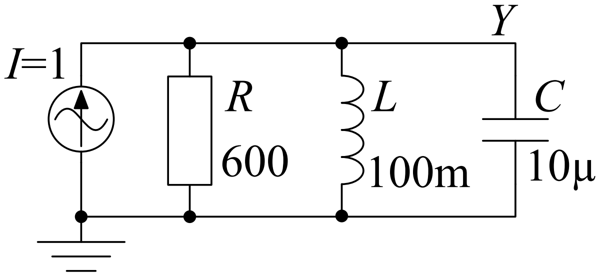
(a) c is just an overall scale factor.
(b) ω_c just scales the frequency axis since $F(j\omega)$ is a function of $\frac{\omega}{\omega_c}$.
(c) The shape of the $F(j\omega)$ graphs is determined entirely by ζ.
(d) The quadratic cannot be factorized $\Leftrightarrow b^2 < 4ac \Leftrightarrow |\zeta| < 1$.
(e) At $\omega = \omega_c$, asymptote gain $= c$ but $F(j\omega) = c \times 2j\zeta$.

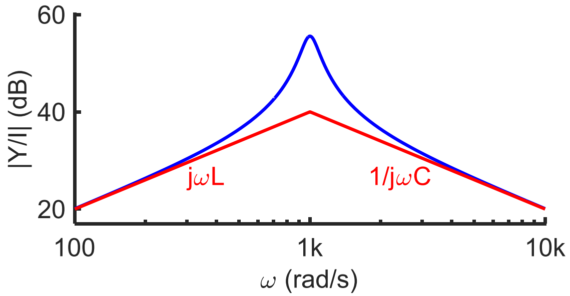
Alternatively, we sometimes use the *quality factor*, $Q \approx \frac{1}{2\zeta} = \frac{a\omega_c}{b}$.

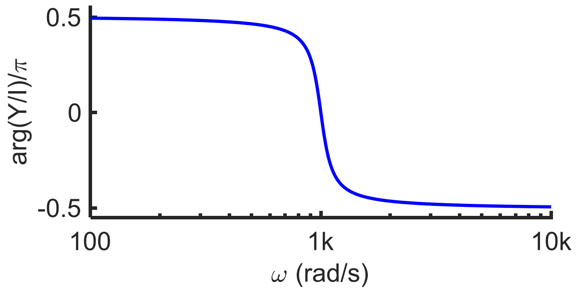
$\frac{Y}{I} = \frac{1}{\frac{1}{R}+\frac{1}{j\omega L}+j\omega C} = \frac{j\omega L}{LC(j\omega)^2+\frac{L}{R}j\omega+1}$

$\omega_c = \sqrt{\frac{c}{a}} = 1000, \ \zeta = \frac{b}{2a\omega_c} = 0.083$

Asymptotes: $j\omega L$ and $\frac{1}{j\omega C}$.

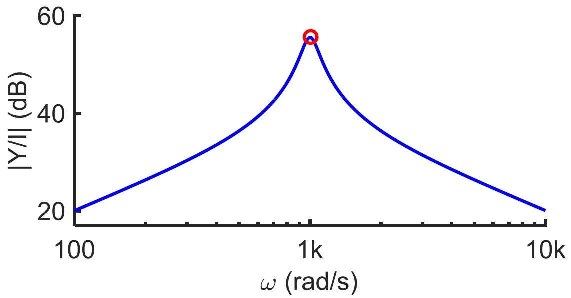


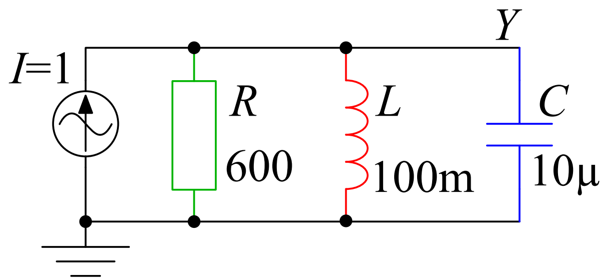




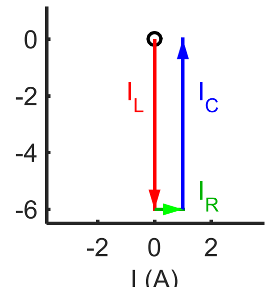
Power absorbed by resistor $\propto Y^2$. It peaks quite sharply at $\omega = 1000$. The resonant frequency, ω_r, is when the impedance is purely real:
at $\omega_r = 1000$, $Z_{RLC} = \frac{Y}{I} = R$.

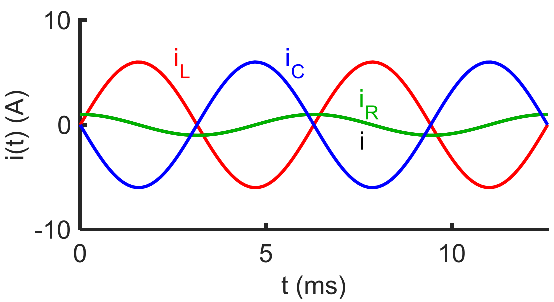
A system with a strong peak in power absorption is a *resonant* system.

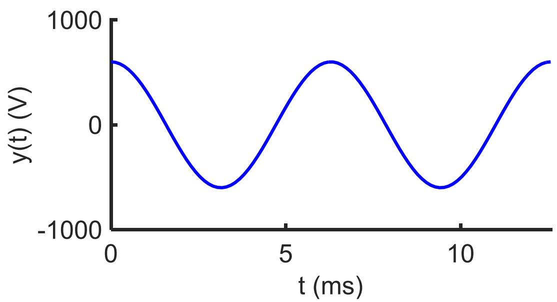




$\omega = 1000 \Rightarrow Z_L = 100j,\ Z_C = -100j.$
$Z_L = -Z_C \Rightarrow I_L = -I_C$
$\Rightarrow I = I_R + I_L + I_C = I_R = 1$
$\Rightarrow Y = I_R R = 600\angle 0° = 56\,\text{dBV}$
$\Rightarrow I_L = \frac{Y}{Z_L} = \frac{600}{100j} = -6j$

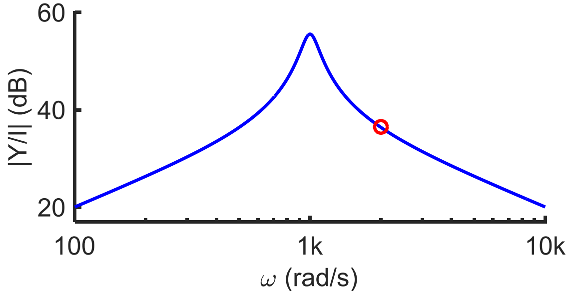


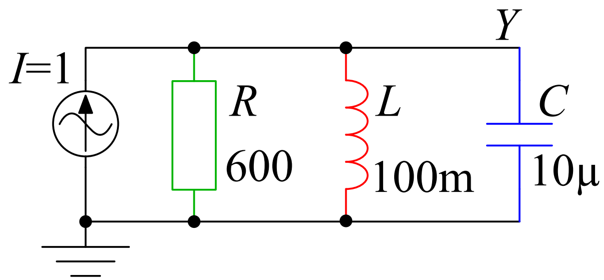




Large currents in L and C exactly cancel out $\Rightarrow I_R = I$ and $Z = R$ (real)

Away from resonance





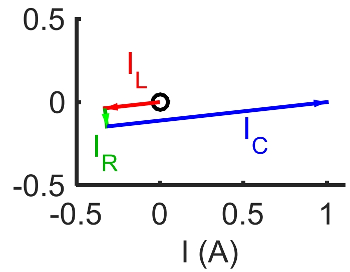
$$\omega = 2000 \Rightarrow Z_L = 200j,\ Z_C = -50j$$

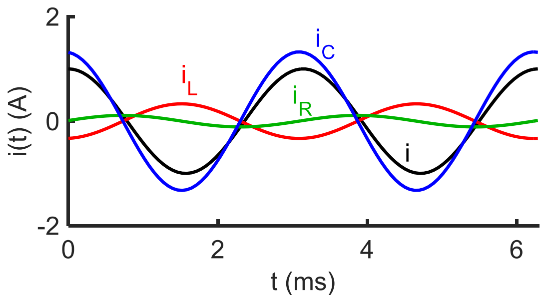
$$Z = \left(\frac{1}{R} + \frac{1}{Z_L} + \frac{1}{Z_C}\right)^{-1} = 66\angle -84^\circ$$

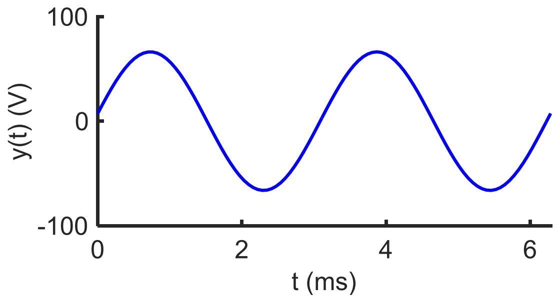
$$Y = I \times Z = 66\angle -84^\circ = 36\,\mathrm{dBV}$$

$$I_R = \frac{Y}{R} = 0.11\angle -84^\circ$$

$$I_L = \frac{Y}{Z_L} = 0.33\angle -174^\circ,\ I_C = 1.33\angle +6^\circ$$



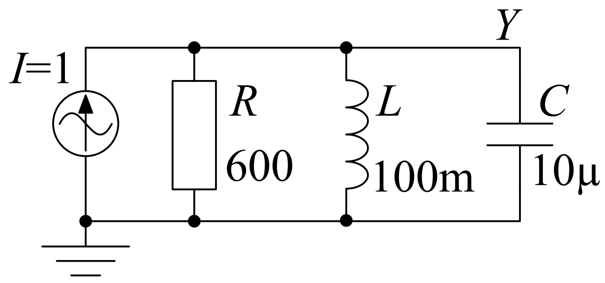




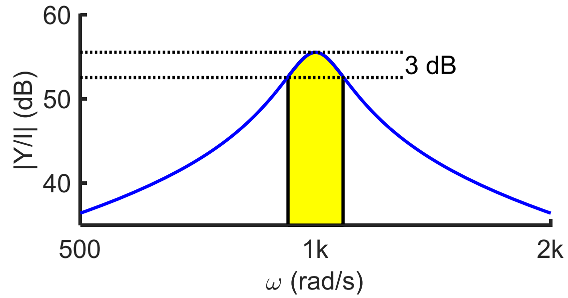
Most current now flows through C, only 0.11 through R.

Bandwidth and Q

$\frac{Y}{I} = \frac{1}{1/R + j(\omega C - 1/\omega L)}$



Bandwidth is the range of frequencies for which $\left|\frac{Y}{I}\right|^2$ is greater than half its peak. Also called *half-power bandwidth* or *3dB bandwidth*.



$\left|\frac{Y}{I}\right|^2 = \frac{1}{(1/R)^2 + (\omega C - 1/\omega L)^2}$

Peak is $\left|\frac{Y}{I}(\omega_0)\right|^2 = R^2$ @ $\omega_0 = 1000$

At $\omega_{3\text{dB}}$: $\left|\frac{Y}{I}(\omega_{3\text{dB}})\right|^2 = \frac{1}{2}\left|\frac{Y}{I}(\omega_0)\right|^2$

$$\frac{1}{(1/R)^2 + (\omega_{3\text{dB}} C - 1/\omega_{3\text{dB}} L)^2} = \frac{R^2}{2} \Rightarrow 1 + \left(\omega_{3\text{dB}} RC - \frac{R}{\omega_{3\text{dB}} L}\right)^2 = 2$$

$$\omega_{3\text{dB}} RC - R/\omega_{3\text{dB}} L = \pm 1 \quad \Rightarrow \quad \omega_{3\text{dB}}^2 RLC \pm \omega_{3\text{dB}} L - R = 0$$

Positive roots: $\omega_{3\text{dB}} = \frac{\pm L + \sqrt{L^2 + 4R^2 LC}}{2RLC} = \{920,\ 1086\}$ rad/s

Bandwidth: $B = 1086 - 920 = 167\,\text{rad/s}.$

Q factor $\approx \frac{\omega_0}{B} = \frac{1}{2\zeta} = 6.$ (Q = "Quality")

Absorbed Power $= v(t)i(t)$:

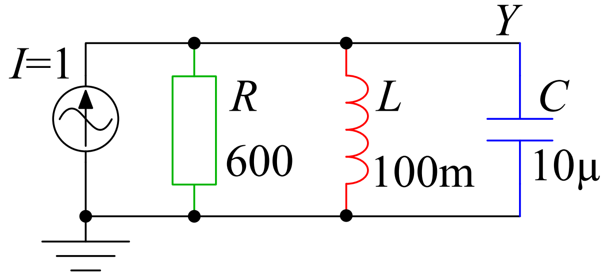
P_L and P_C opposite and $\gg P_R$.

Stored Energy $= \frac{1}{2}Li_L^2 + \frac{1}{2}Cy^2$:

sloshes between L and C.

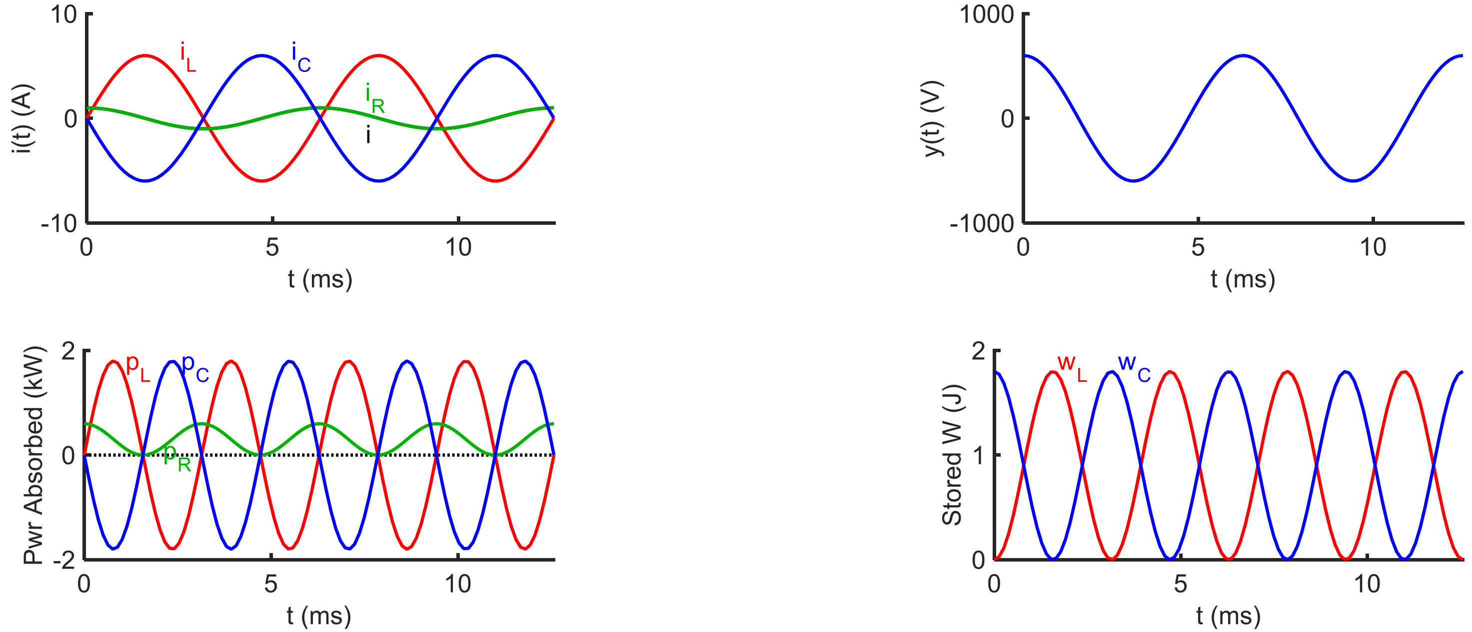
$Q \triangleq \omega \times W_{\text{stored}} \div \overline{P}_R$

$= \omega \times \frac{1}{2}C\,|IR|^2 \div \frac{1}{2}\,|I|^2\,R = \omega RC$



@$\omega = 1000$: $Y = 600$,

$I_R = 1$, $I_L = -6j$, $I_C = +6j$



$Q \triangleq \omega \times$ peak stored energy $\div$ average power loss.

[Derivation of Power and Energy Waveforms]

The input current is a phasor $I = 1$ (i.e. $i(t) = \cos\omega t$ where $\omega = 1000\,\text{rad/s}$).

The complex impedances are $Z_L = j\omega L = 100j\,\Omega$ and $Z_C = \frac{1}{j\omega C} = -100j\,\Omega$. Using the formula for parallel impedances, the total impedance satisfies $\frac{1}{Z} = \frac{1}{600} + \frac{1}{100j} + \frac{1}{-100j} = \frac{1}{600}$. So, at the resonant frequency, the impedances of L and C cancel out and the total impedance is just $Z = 600\,\Omega$.

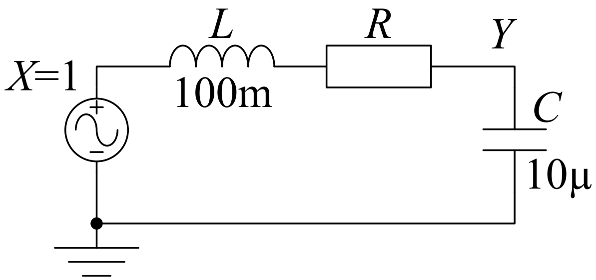
The voltage phasor across the three passive components is $V = IZ = 1 \times 600 = 600\,\text{V}$. The waveform corresponding to this phasor is $v(t) = 600\cos\omega t$ and is plotted in the upper right graph. From knowing V, we can use Ohm's law to work out the individual current phasors in the three components as $I_R = \frac{V}{R} = \frac{600}{600} = 1$, $I_C = \frac{V}{Z_C} = \frac{600}{-100j} = 6j$ and $I_L = \frac{V}{Z_L} = \frac{600}{100j} = -6j$. The waveforms corresponding to these three phasors are plotted in the upper left graph.

Multiplying phasors together doesn't directly give the correct result and so we calculate the power waveforms directly by multiplying $v(t) \times i(t)$. For the resistor, $V = 600$ and $I_R = 1$, so $p_R(t) = 600\cos\omega t \times \cos\omega t = 600\cos^2\omega t = 300 + 300\cos 2\omega t$. For the inductor, $V = 600$ and $I_L = -6j$, so $p_R(t) = 600\cos\omega t \times 6\sin\omega t = 3600\sin\omega t\cos\omega t = 1800\sin 2\omega t$. Finally, for the capacitor, $V = 600$ and $I_L = +6j$, so $p_R(t) = 600\cos\omega t \times -6\sin\omega t = -3600\sin\omega t\cos\omega t = -1800\sin 2\omega t$. These are plotted in the lower left graph.

The energy stored in an inductor is $w_L(t) = \frac{1}{2}Li^2(t) = \frac{1}{2} \times 0.1 \times 36\sin^2\omega t = 1.8\sin^2\omega t = 0.9\,(1 - \cos 2\omega t)$. The energy stored in a capacitor is $w_C(t) = \frac{1}{2}Cv^2(t) = \frac{1}{2} \times 10^{-5} \times 600^2\cos^2\omega t = 1.8\cos^2\omega t = 0.9\,(1 + \cos 2\omega t)$. These are plotted in the lower right graph. The total stored energy in the circuit is $w_L(t) + w_C(t) = 1.8\,\text{J}$ which does not vary with time.

Low Pass Filter

$$\frac{Y}{X} = \frac{1/j\omega C}{R + j\omega L + \frac{1}{j\omega C}} = \frac{1}{LC(j\omega)^2 + RCj\omega + 1}$$



Asymptotes: 1 and $\frac{1}{LC}(j\omega)^{-2}$.

$\omega_c = \sqrt{\frac{c}{a}} = 1000,\ \zeta = \frac{b}{2a\omega_c} = \frac{R}{200}$

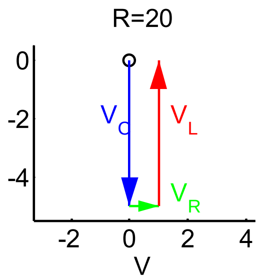
$@\omega_c : Z_L = -Z_C = 100j,\ I = \frac{X}{R},\ \left|\frac{Y}{X}\right| = \frac{1}{RC\omega} = \frac{1}{2\zeta},\ \angle\frac{Y}{X} = -\frac{\pi}{2}$

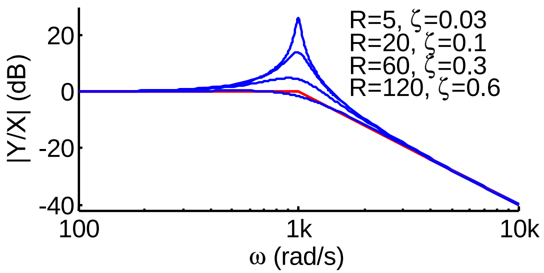
Magntitude Plot:

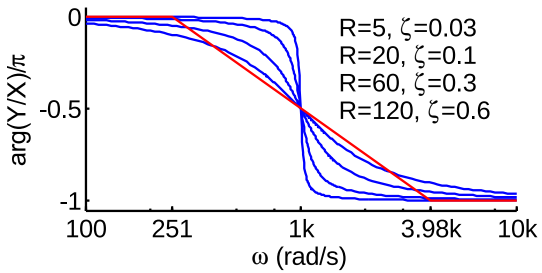
Small $\zeta \Rightarrow$ less loss, higher peak, smaller bandwidth.
Large ζ more loss, smaller peak at a lower ω, larger bandwidth.

Phase Plot:

Small $\zeta \Rightarrow$ fast phase change: π over 2ζ decades.
$\angle\frac{Y}{X} \approx \frac{-\pi}{2}\left(1 + \frac{1}{\zeta}\log_{10}\frac{\omega}{\omega_c}\right)$ for $10^{-\zeta} < \frac{\omega}{\omega_c} < 10^{+\zeta}$

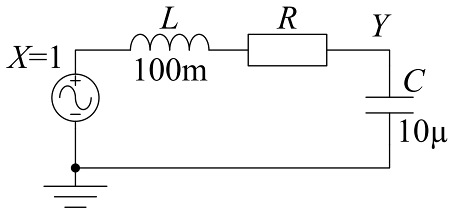






Resonance Peak for LP filter

$$\frac{Y}{X} = \frac{1}{LC(j\omega)^2+RCj\omega+1} = \frac{1}{\left(j\frac{\omega}{\omega_c}\right)^2+2\zeta j\frac{\omega}{\omega_c}+1}$$



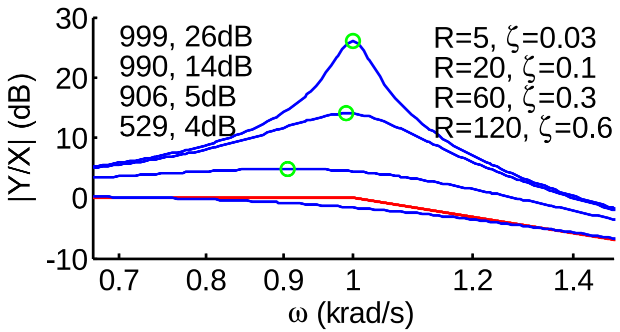
$$\omega_c = \sqrt{\frac{c}{a}} = 1000,\ \zeta = \frac{b}{2a\omega_c} = \frac{b\omega_c}{2c} = \frac{R}{200}$$

$\frac{Y}{X}$ is a function of $\frac{\omega}{\omega_c}$ so ω_c just scales frequency axis (= shift on log axis).
The *damping factor*, ζ, ("zeta") determines the shape of the peak.

Peak frequency:

$$\omega_p = \omega_c\sqrt{1-2\zeta^2}$$

$\zeta \geq 0.5 \Rightarrow$ passes under corner,
$\zeta \geq 0.71 \Rightarrow$ no peak,
$\zeta \geq 1 \Rightarrow$ can factorize



Gain relative to asymptote: @ ω_p: $\frac{1}{2\zeta\sqrt{1-\zeta^2}}$ @ ω_c: $\frac{1}{2\zeta} \approx Q$

Three frequencies: ω_p= peak, ω_c= asymptotes cross, ω_r= real impedance
For $\zeta < 0.3$, $\omega_p \approx \omega_c \approx \omega_r$. All get called the resonant frequency.

The exact relationship between ω_p, ω_c and ω_r and the gain at these frequencies is affected by any other corner frequencies in the response.

Summary

- Resonance is a peak in energy absorption
  - Parallel or series circuit has a real impedance at ω_r
    - ▷ peak response may be at a slightly different frequency
  - The quality factor, Q, of the resonance is

$$Q \triangleq \frac{\omega_0 \times \text{stored energy}}{\text{power in } R} \approx \frac{\omega_0}{3 \text{ dB bandwidth}} \approx \frac{1}{2\zeta}$$

  - 3 dB bandwidth is where power falls by $\frac{1}{2}$ or voltage by $\frac{1}{\sqrt{2}}$
  - The stored energy sloshes between L and C
- Quadratic factor: $\left(\frac{j\omega}{\omega_c}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_c}\right) + 1$
  - $a\,(j\omega)^2 + b\,(j\omega) + c \quad \Rightarrow \quad \omega_c = \sqrt{\frac{c}{a}}$ and $\zeta = \frac{b}{2a\omega_c} = \frac{b\,\mathrm{sgn}(a)}{\sqrt{4ac}}$
  - ± 40 dB/decade slope change in magnitude response
  - phase changes rapidly by 180° over $\omega = 10^{\mp\zeta}\omega_c$
  - Gain error in asymptote is $\frac{1}{2\zeta} \approx Q$ at ω_0

For further details see Hayt Ch 16 or Irwin Ch 12.