13: Filters

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Filters

A filter is a circuit whose gain varies with frequency. Often a filter aims to allow some frequencies to pass while blocking others.

- Radio/TV: a “tuning” filter blocks all frequencies except the wanted channel
- Loudspeaker: “crossover” filters send the right frequencies to different drive units
- Sampling: an “anti-aliasing filter” eliminates all frequencies above half the sampling rate
  - Phones: Sample rate = 8 kHz; filter eliminates frequencies above 3.4 kHz.
- Computer cables: filter eliminates interference
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1st Order Low-Pass Filter

\[
\frac{Y}{X} = \frac{1/j \omega C}{R + 1/j \omega C} = \frac{1}{j \omega RC + 1}
\]
1st Order Low-Pass Filter

\[
\frac{Y}{X} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{j\omega RC + 1} = \frac{1}{j\omega p + 1}
\]

Corner frequency: \( p = \left| \frac{b}{a} \right| = \frac{1}{RC} \)
1st Order Low-Pass Filter

\[
\frac{Y}{X} = \frac{1/j\omega C}{R+1/j\omega C} = \frac{1}{j\omega RC+1} = \frac{1}{p} + 1
\]

Corner frequency: \( p = \left| \frac{b}{a} \right| = \frac{1}{RC} \)

Asymptotes: 1 and \( \frac{p}{j\omega} \)
1st Order Low-Pass Filter

\[
\frac{Y}{X} = \frac{1/jωC}{R+1/jωC} = \frac{1}{jωRC+1} = \frac{1}{jωp+1}
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Corner frequency: \( p = \left| \frac{b}{a} \right| = \frac{1}{RC} \)

Asymptotes: 1 and \( \frac{p}{jω} \)

Very low \( ω \): Capacitor = open circuit
1st Order Low-Pass Filter

\[
\frac{Y}{X} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{j\omega RC + 1} = \frac{1}{j\omega + 1}
\]

Corner frequency: \( p = \left| \frac{b}{a} \right| = \frac{1}{RC} \)

Asymptotes: 1 and \( \frac{j\omega}{p} \)

**Very low \( \omega \):** Capacitor = open circuit
**Very high \( \omega \):** Capacitor short circuit
1st Order Low-Pass Filter

\[
\frac{Y}{X} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{j\omega RC + 1} = \frac{1}{j\omega p + 1}
\]

Corner frequency: \( p = \left| \frac{b}{a} \right| = \frac{1}{RC} \)

Asymptotes: 1 and \( \frac{p}{j\omega} \)

Very low \( \omega \): Capacitor = open circuit

Very high \( \omega \): Capacitor short circuit

A *low-pass* filter because it allows low frequencies to pass but *attenuates* (makes smaller) high frequencies.
1st Order Low-Pass Filter

\[
\frac{Y}{X} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{j\omega RC + 1} = \frac{1}{j\omega/p + 1}
\]

Corner frequency: \( p = \left| \frac{b}{a} \right| = \frac{1}{RC} \)

Asymptotes: 1 and \( \frac{p}{j\omega} \)

Very low \( \omega \): Capacitor = open circuit
Very high \( \omega \): Capacitor short circuit

A low-pass filter because it allows low frequencies to pass but attenuates (makes smaller) high frequencies.

The order of a filter: highest power of \( j\omega \) in the denominator. Almost always equals the total number of \( L \) and/or \( C \).
Low-Pass with Gain Floor

\[ \frac{Y}{X} = \frac{R + \frac{1}{j\omega C}}{4R + \frac{1}{j\omega C}} = \frac{j\omega RC + 1}{j\omega 4RC + 1} \]
Low-Pass with Gain Floor

\[
\frac{Y}{X} = \frac{R + \frac{1}{j\omega C}}{4R + \frac{1}{j\omega C}} = \frac{j\omega RC + 1}{j\omega 4RC + 1} = \frac{j\omega}{q} + 1
\]

Corner frequencies: \(p = \frac{1}{4RC}, \quad q = \frac{1}{RC}\)
Low-Pass with Gain Floor

\[
\frac{Y}{X} = \frac{R + \frac{1}{j\omega C}}{4R + \frac{1}{j\omega C}} = \frac{j\omega RC + 1}{j\omega 4RC + 1} = \frac{j\omega}{q} + \frac{1}{p} + 1
\]

Corner frequencies: \( p = \frac{1}{4RC}, \ q = \frac{1}{RC} \)

Asymptotes: 1 and \( \frac{1}{4} \)
Low-Pass with Gain Floor

\[
\frac{Y}{X} = \frac{R + \frac{1}{j \omega C}}{4R + \frac{1}{j \omega C}} = \frac{j \omega RC + 1}{j \omega 4RC + 1} = \frac{j \omega}{q} + 1 \quad \frac{j \omega}{p} + 1
\]

Corner frequencies: \( p = \frac{1}{4RC}, \quad q = \frac{1}{RC} \)

Asymptotes: 1 and \( \frac{1}{4} \)

Very low \( \omega \):
- Capacitor = open circuit
- Resistor \( R \) unattached. Gain = 1
Low-Pass with Gain Floor

\[
\frac{Y}{X} = \frac{R + \frac{1}{j\omega C}}{4R + \frac{1}{j\omega C}} = \frac{j\omega RC + 1}{j\omega 4RC + 1} = \frac{\frac{j\omega}{q} + 1}{\frac{j\omega}{p} + 1}
\]

Corner frequencies: \( p = \frac{1}{4RC} \), \( q = \frac{1}{RC} \)

Asymptotes: 1 and \( \frac{1}{4} \)

**Very low \( \omega \):**
- Capacitor = open circuit
- Resistor \( R \) unattached. Gain = 1

**Very high \( \omega \):**
- Capacitor short circuit
- Circuit is potential divider with gain \( 20 \log_{10} \frac{1}{4} = -12 \text{ dB} \).
Inverting amplifier so

\[
\frac{Y}{X} = -\frac{3R||\left(R + \frac{1}{j\omega C}\right)}{R}
\]
Opamp filter

Inverting amplifier so

\[
\frac{Y}{X} = - \frac{3R\left|R \left|\frac{1}{j\omega C} + 1\right|\right.}{R} = - \frac{3R\left|R \left|\frac{1}{j\omega C} + 1\right|\right.}{R \times (3R + R + \frac{1}{j\omega C})}
\]
Inverting amplifier so

\[ \frac{Y}{X} = -\frac{3R||R+1/j\omega C}{R} = -\frac{3R(R+1/j\omega C)}{R \times (3R+R+1/j\omega C)} \]

\[ = -3 \times \frac{R+1/j\omega C}{4R+1/j\omega C} \]
Opamp filter

Inverting amplifier so

\[
\frac{Y}{X} = -\frac{3R|\left(\frac{R+1/j\omega C}{R}\right)}{R} = -\frac{3R\left(\frac{R+1/j\omega C}{R}\right)}{R \times \left(3R + \frac{R+1/j\omega C}{R}\right)}
\]

\[
= -3 \times \frac{R+1/j\omega C}{4R+1/j\omega C} = -3 \times \frac{j\omega RC+1}{j\omega 4RC+1}
\]
Inverting amplifier so

\[
\frac{Y}{X} = -\frac{3R\left|R + \frac{1}{j\omega C}\right|}{R} = -\frac{3R\left|R + \frac{1}{j\omega C}\right|}{R \times (3R + R + \frac{1}{j\omega C})} \\
= -3 \times \frac{R + \frac{1}{j\omega C}}{4R + \frac{1}{j\omega C}} = -3 \times \frac{j\omega RC + 1}{j\omega 4RC + 1}
\]

Same transfer function as before except \( \times -3 = +9.5 \text{ dB.} \)
Opamp filter

Inverting amplifier so

\[
\frac{Y}{X} = -\frac{3R}{R} \frac{|(R+1/j\omega C)|}{(R+1/j\omega C)} = -\frac{3R(R+1/j\omega C)}{R(3R+R+1/j\omega C)}
\]

\[
= -3 \times \frac{R+1/j\omega C}{4R+1/j\omega C} = -3 \times \frac{j\omega RC+1}{j\omega 4RC+1}
\]

Same transfer function as before except \(-3 = +9.5\) dB.

Advantages of op-amp circuit:
Inverting amplifier so

\[
\frac{Y}{X} = -\frac{3R\left|R + \frac{1}{j\omega C}\right|}{R} = -\frac{3R\left|R + \frac{1}{j\omega C}\right|}{R \times (3R + R + \frac{1}{j\omega C})}
\]

\[
= -3 \times \frac{R + \frac{1}{j\omega C}}{4R + \frac{1}{j\omega C}} = -3 \times \frac{j\omega RC + 1}{j\omega 4RC + 1}
\]

Same transfer function as before except \(\times -3 = +9.5\) dB.

Advantages of op-amp circuit:

1. Can have gain \(> 1\).
**Opamp filter**

Inverting amplifier so

\[
\frac{Y}{X} = -\frac{3R\left|R + \frac{1}{j\omega C}\right|}{R} = -\frac{3R\left(R + \frac{1}{j\omega C}\right)}{R \times (3R + R + \frac{1}{j\omega C})}
\]

\[
= -3 \times \frac{R + \frac{1}{j\omega C}}{4R + \frac{1}{j\omega C}} = -3 \times \frac{j\omega RC + 1}{j\omega 4RC + 1}
\]

Same transfer function as before except \( \times -3 = +9.5 \text{ dB} \).

**Advantages of op-amp circuit:**

1. Can have gain > 1.
2. Low output impedance - loading does not affect filter.
Opamp filter

Inverting amplifier so

\[
\frac{Y}{X} = -\frac{3R |(R+1/j\omega C)|}{R} = -\frac{3R(R+1/j\omega C)}{R \times (3R+R+1/j\omega C)}
\]

\[
= -3 \times \frac{R+1/j\omega C}{4R+1/j\omega C} = -3 \times \frac{j\omega RC+1}{j\omega 4RC+1}
\]

Same transfer function as before except \( \times -3 = +9.5 \text{ dB}. \)

Advantages of op-amp circuit:

1. Can have gain \( > 1 \).
2. Low output impedance - loading does not affect filter.
3. Resistive input impedance - does not vary with frequency.
Integrator

\[ \frac{Y}{X} = -\frac{1/j \omega C}{R} \]

[Diagram of integrator circuit]
Integrator

\[ \frac{Y}{X} = -\frac{1}{j\omega C}R = -\frac{1}{j\omega RC} \]

\[ Y = -\frac{1}{j\omega RC}X \]

\[ Y = \frac{I}{j\omega C}V_C \]

\[ X \rightarrow R \rightarrow C \rightarrow Y \]

\[ |\text{Gain} (\text{dB})| \]

\[ 0 \quad 20 \]

\[ 0.1 \quad 1 \quad 10 \]

\[ \omega RC \]
### Integrator

\[
\frac{Y}{X} = -\frac{1/j\omega C}{R} = -\frac{1}{j\omega RC}
\]

Capacitor: \( i = C \frac{dv_C}{dt} \)
Integrator

\[
\frac{Y}{X} = -\frac{1/j\omega C}{R} = -\frac{1}{j\omega RC}
\]

Capacitor: \( i = C\frac{dv_C}{dt} \)

\( i = \frac{x}{R} = -C\frac{dy}{dt} \)
Integrator

\[
\frac{Y}{X} = -\frac{1}{j\omega C} R = -\frac{1}{j\omega RC}
\]

**Capacitor:**

\[
i = C \frac{dv_C}{dt}
\]

\[
i = \frac{x}{R} = -C \frac{dy}{dt}
\]

\[
\frac{dy}{dt} = -\frac{1}{RC} x
\]
Integrator

\[
\frac{Y}{X} = -\frac{1/j\omega C}{R} = -\frac{1}{j\omega RC}
\]

Capacitor: \( i = C \frac{dv_C}{dt} \)

\[
i = \frac{x}{R} = -C \frac{dy}{dt}
\]

\[
\frac{dy}{dt} = -\frac{1}{RC} x
\]

\[
\int_0^t \frac{dy}{dt} \, dt = -\frac{1}{RC} \int_0^t x \, dt
\]
Integrator

\[ \frac{Y}{X} = -\frac{1/j\omega C}{R} = -\frac{1}{j\omega RC} \]

Capacitor: \( i = C \frac{dv_C}{dt} \)
\[
i = \frac{x}{R} = -C \frac{dy}{dt}
\]
\[
\frac{dy}{dt} = -\frac{1}{RC} x
\]
\[
\int_0^t \frac{dy}{dt} dt = -\frac{1}{RC} \int_0^t x dt
\]
\[
y(t) = -\frac{1}{RC} \int_0^t x dt + y(0)
\]
### Integrator

\[
\frac{Y}{X} = -\frac{1}{j\omega C R} = -\frac{1}{j\omega RC}
\]

**Capacitor:**

\[
i = \frac{x}{R} = -C \frac{dy}{dt}
\]

\[
\frac{dy}{dt} = -\frac{1}{RC} x
\]

\[
\int_0^t \frac{dy}{dt} dt = -\frac{1}{RC} \int_0^t x dt
\]

\[
y(t) = -\frac{1}{RC} \int_0^t x dt + y(0)
\]

**Note:** if \( x(t) = \cos \omega t \)

\[
\int \cos(\omega t) dt = \frac{1}{\omega} \sin(\omega t) \Rightarrow \text{gain} \propto \frac{1}{\omega}.
\]
Integrator

\[
\frac{Y}{X} = -\frac{1/j\omega C}{R} = -\frac{1}{j\omega RC}
\]

**Capacitor:** \( i = C \frac{dv_C}{dt} \)

\[
i = \frac{x}{R} = -C \frac{dy}{dt}
\]

\[
\frac{dy}{dt} = \frac{-1}{RC} x
\]

\[
\int_{0}^{t} \frac{dy}{dt} dt = \frac{-1}{RC} \int_{0}^{t} x dt
\]

\[
y(t) = \frac{-1}{RC} \int_{0}^{t} x dt + y(0)
\]

**Note:** if \( x(t) = \cos \omega t \)

\[
\int \cos(\omega t) dt = \frac{1}{\omega} \sin(\omega t) \Rightarrow \text{gain} \propto \frac{1}{\omega}.
\]

We can limit the LF gain to 20 dB:
**Integrator**

\[ \frac{Y}{X} = -\frac{1/j\omega C}{R} = -\frac{1}{j\omega RC} \]

**Capacitor:**

\[ i = \frac{x}{R} = -C \frac{dy}{dt} \]

\[ \frac{dy}{dt} = -\frac{1}{RC} x \]

\[ \int_0^t \frac{dy}{dt} dt = -\frac{1}{RC} \int_0^t x dt \]

\[ y(t) = -\frac{1}{RC} \int_0^t x dt + y(0) \]

**Note:** if \( x(t) = \cos \omega t \)

\[ \int \cos(\omega t) dt = \frac{1}{\omega} \sin(\omega t) \Rightarrow \text{gain} \propto \frac{1}{\omega}. \]

We can limit the LF gain to 20 dB:

\[ \frac{Y}{X} = -10R\left|\frac{1}{j\omega C}\right| \frac{1}{R} \]
### Integrator

\[
\frac{Y}{X} = -\frac{1/j\omega C}{R} = -\frac{1}{j\omega RC}
\]

**Capacitor:**
\[
i = C\frac{dv_C}{dt} \\
i = \frac{x}{R} = -C\frac{dy}{dt}
\]

\[
\frac{dy}{dt} = -\frac{1}{RC} x
\]

\[
\int_0^t \frac{dy}{dt} dt = -\frac{1}{RC} \int_0^t x dt
\]

\[
y(t) = -\frac{1}{RC} \int_0^t x dt + y(0)
\]

**Note:** if \( x(t) = \cos \omega t \)
\[
\int \cos(\omega t) dt = \frac{1}{\omega} \sin(\omega t) \Rightarrow \text{gain} \propto \frac{1}{\omega}.
\]

We can limit the LF gain to 20 dB:
\[
\frac{Y}{X} = -\frac{10R}{R} || \frac{1/j\omega C}{R} = -\frac{10R \times \frac{1}{j\omega C}}{R(10R + \frac{1}{j\omega C})}
\]
Integrator

\[ \frac{Y}{X} = -\frac{1/j \omega C}{R} = -\frac{1}{j \omega RC} \]

**Capacitor:**
\[ i = C \frac{dv_C}{dt} \]
\[ i = \frac{x}{R} = -C \frac{dy}{dt} \]
\[ \frac{dy}{dt} = -\frac{1}{RC} x \]
\[ \int_0^t \frac{dy}{dt} dt = -\frac{1}{RC} \int_0^t x dt \]
\[ y(t) = -\frac{1}{RC} \int_0^t x dt + y(0) \]

**Note:** if \( x(t) = \cos \omega t \)
\[ \int \cos(\omega t) dt = \frac{1}{\omega} \sin(\omega t) \Rightarrow \text{gain} \propto \frac{1}{\omega}. \]

We can limit the LF gain to 20 dB:
\[ \frac{Y}{X} = -10R \left| \frac{1/j \omega C}{R} \right| = -\frac{10R}{R(10R + 1/j \omega C)} \]
\[ = -\frac{10}{j \omega 10RC + 1} \]
Integrator

\[
\frac{Y}{X} = -\frac{1/j\omega C}{R} = -\frac{1}{j\omega RC}
\]

Capacitor: \[i = C \frac{dv_C}{dt}\]
\[i = \frac{x}{R} = -C \frac{dy}{dt}\]
\[\frac{dy}{dt} = \frac{-1}{RC} x\]
\[\int_0^t \frac{dy}{dt} dt = \frac{-1}{RC} \int_0^t x dt\]
\[y(t) = \frac{-1}{RC} \int_0^t x dt + y(0)\]

Note: if \[x(t) = \cos \omega t\]
\[\int \cos(\omega t) dt = \frac{1}{\omega} \sin(\omega t) \Rightarrow \text{gain} \propto \frac{1}{\omega}.\]

We can limit the LF gain to 20 dB:

\[
\frac{Y}{X} = -\frac{10R}{10RC+1} = -\frac{10R \times 1/j\omega C}{R(10R+1/j\omega C)}
\]
\[= -\frac{10}{j\omega 10RC+1} \quad (\omega_c = \frac{0.1}{RC})\]
High Pass Filter

\[
\frac{Y}{X} = \frac{R}{R + \frac{1}{j\omega C}}
\]
High Pass Filter

\[
\frac{Y}{X} = \frac{R}{R+1/j\omega C} = \frac{j\omega RC}{j\omega RC+1}
\]
High Pass Filter

\[
\frac{Y}{X} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{j\omega RC + 1}
\]

Corner Freq: \( p = \frac{1}{RC} \)
High Pass Filter

\[ \frac{Y}{X} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{j\omega RC + 1} \]

Corner Freq: \( p = \frac{1}{RC} \)

Asymptotes: \( j\omega RC \) and 1

\[ X \rightarrow C \rightarrow Y \]

\[ R \]
High Pass Filter

\[
\frac{Y}{X} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{j\omega RC + 1}
\]

Corner Freq: \( p = \frac{1}{RC} \)

Asymptotes: \( j\omega RC \) and 1
High Pass Filter

\[
\frac{Y}{X} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{j\omega RC + 1}
\]

Corner Freq: \( p = \frac{1}{RC} \)

Asymptotes: \( j\omega RC \) and 1

Very low \( \omega \): \( C \) open circuit: gain = 0

Very high \( \omega \): \( C \) short circuit: gain = 1
High Pass Filter

\[
\frac{Y}{X} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{j\omega RC + 1}
\]

Corner Freq: \( p = \frac{1}{RC} \)

Asymptotes: \( j\omega RC \) and 1

Very low \( \omega \): \( C \) open circuit: gain = 0

Very high \( \omega \): \( C \) short circuit: gain = 1

We can add an op-amp to give a low-impedance output.
High Pass Filter

\[
\frac{Y}{X} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{j\omega RC + 1}
\]

Corner Freq: \( p = \frac{1}{RC} \)

Asymptotes: \( j\omega RC \) and 1

Very low \( \omega \): \( C \) open circuit: gain = 0

Very high \( \omega \): \( C \) short circuit: gain = 1

We can add an op-amp to give a low-impedance output. Or add gain:

\[
\frac{Z}{X} = \left(1 + \frac{R_B}{R_A}\right) \times \frac{j\omega RC}{j\omega RC + 1}
\]
2nd order filter

\[
\frac{Y}{X} = \frac{R_2 + j\omega L}{1/j\omega C + R_1 + R_2 + j\omega L}
\]
2nd order filter

\[
\frac{Y}{X} = \frac{R_2 + j\omega L}{1/j\omega C + R_1 + R_2 + j\omega L} = \frac{LC(j\omega)^2 + R_2 C j\omega}{LC(j\omega)^2 + (R_1 + R_2) C j\omega + 1}
\]
2nd order filter

\[
\frac{Y}{X} = \frac{R_2 + j\omega L}{1/j\omega C + R_1 + R_2 + j\omega L}
\]

\[
= \frac{LC(j\omega)^2 + R_2 C j\omega}{LC(j\omega)^2 + (R_1 + R_2) C j\omega + 1}
\]

\[
= \frac{j\omega C(j\omega L + R_2)}{LC(j\omega)^2 + (R_1 + R_2) C j\omega + 1}
\]
**2nd order filter**

\[
\frac{Y}{X} = \frac{R_2 + j\omega L}{1/j\omega C + R_1 + R_2 + j\omega L}
\]

\[
= \frac{LC(j\omega)^2 + R_2 C j\omega}{LC(j\omega)^2 + (R_1 + R_2) C j\omega + 1}
\]

\[
= \frac{j\omega C (j\omega L + R_2)}{LC(j\omega)^2 + (R_1 + R_2) C j\omega + 1}
\]

**Asymptotes:** \(j\omega R_2 C\) and 1
2nd order filter

\[
\frac{Y}{X} = \frac{R_2 + j\omega L}{1/j\omega C + R_1 + R_2 + j\omega L}
\]

\[
= \frac{LC(j\omega)^2 + R_2 C j\omega}{LC(j\omega)^2 + (R_1 + R_2) C j\omega + 1}
\]

\[
= \frac{j\omega C(j\omega L + R_2)}{LC(j\omega)^2 + (R_1 + R_2) C j\omega + 1}
\]

Asymptotes: \(j\omega R_2 C\) and 1

Corner frequencies:

\[+20 \text{ dB/dec at } p = \frac{R_2}{L} = 100 \text{ rad/s}\]

\[-40 \text{ dB/dec at } q = \sqrt{\frac{C}{a}} = \frac{1}{\sqrt{LC}} = 1000 \text{ rad/s}\]
2nd order filter

\[
\frac{Y}{X} = \frac{R_2 + j\omega L}{1/j\omega C + R_1 + R_2 + j\omega L}
\]

\[
= \frac{LC(j\omega)^2 + R_2Cj\omega}{LC(j\omega)^2 + (R_1 + R_2)Cj\omega + 1}
\]

Asymptotes: \(j\omega R_2 C\) and 1

Corner frequencies:

\[+20 \text{ dB/dec at } p = \frac{R_2}{L} = 100 \text{ rad/s}\]

\[-40 \text{ dB/dec at } q = \sqrt{\frac{C}{a}} = \frac{1}{\sqrt{LC}} = 1000 \text{ rad/s}\]
2nd order filter

\[
\frac{Y}{X} = \frac{R_2 + j\omega L}{1/j\omega C + R_1 + R_2 + j\omega L}
\]

\[
= \frac{LC(j\omega)^2 + R_2 C j\omega}{LC(j\omega)^2 + (R_1 + R_2) C j\omega + 1}
\]

Asymptotes: \(j\omega R_2 C\) and 1

Corner frequencies:
- +20 dB/dec at \(p = \frac{R_2}{L} = 100\) rad/s
- −40 dB/dec at \(q = \sqrt{\frac{C}{a}} = \frac{1}{\sqrt{LC}} = 1000\) rad/s

Damping factor: \(\zeta = \frac{b \text{sgn}(a)}{\sqrt{4ac}} = \frac{g_b}{2c} = \frac{q^2}{2} (R_1 + R_2) C = 0.6\).

Gain error at \(q\) is \(\frac{1}{2\zeta} = Q = 0.83 = -1.6\) dB (+0.04 dB due to \(p\))
2nd order filter

\[
\frac{Y}{X} = \frac{R_2 + j\omega L}{1/j\omega C + R_1 + R_2 + j\omega L} \\
= \frac{LC(j\omega)^2 + R_2 Cj\omega}{LC(j\omega)^2 + (R_1 + R_2)Cj\omega + 1} \\
= \frac{j\omega C(j\omega L + R_2)}{LC(j\omega)^2 + (R_1 + R_2)Cj\omega + 1}
\]

Asymptotes: \( j\omega R_2 C \) and 1

Corner frequencies:
\[ +20 \text{ dB/dec at } p = \frac{R_2}{L} = 100 \text{ rad/s} \]
\[ -40 \text{ dB/dec at } q = \sqrt{\frac{C}{a}} = \frac{1}{\sqrt{LC}} = 1000 \text{ rad/s} \]

Damping factor: \( \zeta = \frac{b \text{sgn}(a)}{\sqrt{4ac}} = \frac{qb}{2c} = \frac{q}{2} (R_1 + R_2) C = 0.6 \).

Gain error at \( q \) is \( \frac{1}{2\zeta} = Q = 0.83 = -1.6 \text{ dB} (+0.04 \text{ dB due to } p) \)

Compare with 1st order:
2nd order filter attenuates more rapidly than a 1st order filter.
Sallen-Key Filter

13: Filters
- Filters
- 1st Order Low-Pass Filter
- Low-Pass with Gain Floor
- Opamp filter
- Integrator
- High Pass Filter
- 2nd order filter
- Sallen-Key Filter
- Twin-T Notch Filter
- Conformal Filter
- Transformations (A)
- Conformal Filter
- Transformations (B)
- Summary

KCL @ Y: \[ \frac{Y-X}{j\omega C} + \frac{Y-Z}{j\omega C} + \frac{Y-Z}{R} = 0 \] [assume \( V_+ = V_- = Z \)]
Sallen-Key Filter

\[ Y \left(1 + 2j\omega RC \right) - Z \left(1 + j\omega RC \right) = Xj\omega RC \]

\[ Y = \frac{Y-X}{1/j\omega C} + \frac{Y-Z}{1/j\omega C} + \frac{Y-Z}{R} = 0 \]  

[assume \( V_+ = V_- \equiv Z \)]
### Sallen-Key Filter

**KCL @ Y:** \[ \frac{Y-X}{j\omega C} + \frac{Y-Z}{j\omega C} + \frac{Y-Z}{R} = 0 \]

[assume \( V_+ = V_- = Z \)]

\[ \Rightarrow Y \left(1 + 2j\omega RC\right) - Z \left(1 + j\omega RC\right) = Xj\omega RC \]

**KCL @ \( V_+ \):** \[ \frac{Z}{mR} + \frac{Z-Y}{j\omega C} = 0 \]
Sallen-Key Filter

KCL @ $Y$: $\frac{Y-X}{1/j\omega C} + \frac{Y-Z}{1/j\omega C} + \frac{Y-Z}{R} = 0$  
[assume $V_+ = V_- = Z$]

$\Rightarrow Y \left( 1 + 2j\omega RC \right) - Z \left( 1 + j\omega RC \right) = X j\omega RC$

KCL @ $V_+$: $\frac{Z}{mR} + \frac{Z-Y}{1/j\omega C} = 0 \Rightarrow Z \left( 1 + j\omega mRC \right) = Y j\omega mRC$
Sallen-Key Filter

\[ R = 10k \]

\[ m = 2.8 \]

\[ mR = 28k \]

\[
\begin{align*}
KCL @ Y: & \quad \frac{Y-X}{1/j\omega C} + \frac{Y-Z}{1/j\omega C} + \frac{Y-Z}{R} = 0 \quad \text{[assume } V_+ = V_- = Z] \\
& \Rightarrow Y \left( 1 + 2j\omega RC \right) - Z \left( 1 + j\omega RC \right) = Xj\omega RC
\end{align*}
\]

\[
\begin{align*}
KCL @ V+: & \quad \frac{Z}{mR} + \frac{Z-Y}{1/j\omega C} = 0 \Rightarrow Z \left( 1 + j\omega mRC \right) = Y j\omega mRC
\end{align*}
\]

\[
\begin{align*}
\text{Sub } Y: & \quad Z \left( 1 + j\omega mRC \right) \left( 1 + 2j\omega RC \right) - Z \left( 1 + j\omega RC \right) = Xj\omega RC
\end{align*}
\]
Sallen-Key Filter

KCL @ Y:
\[
\frac{Y-X}{j\omega C} + \frac{Y-Z}{j\omega C} + \frac{Y-Z}{R} = 0 \quad \text{[assume } V_+ = V_- = Z]\]
\[\Rightarrow Y (1 + 2j\omega RC') - Z (1 + j\omega RC') = X j\omega RC\]

KCL @ V_+:
\[\frac{Z}{mR} + \frac{Z-Y}{1/j\omega C} = 0 \Rightarrow Z(1 + j\omega mRC) = Y j\omega mRC\]

Sub Y:
\[Z \frac{(1+j\omega mRC)}{j\omega mRC} (1 + 2j\omega RC') - Z (1 + j\omega RC') = X j\omega RC\]
\[\Rightarrow \frac{Z}{X} = \frac{m(j\omega RC)^2}{m(j\omega RC)^2 + 2j\omega RC + 1}\]
Sallen-Key Filter

KCL @ Y: \( \frac{Y-X}{j\omega C} + \frac{Y-Z}{j\omega C} + \frac{Y-Z}{R} = 0 \)  [assume \( V_+ = V_- = Z \)]
\( \Rightarrow Y (1 + 2j\omega RC') - Z (1 + j\omega RC') = X j\omega RC \)

KCL @ \( V_+ \): \( \frac{Z}{mR} + \frac{Z-Y}{j\omega C} = 0 \) \( \Rightarrow Z(1 + j\omega mRC) = Y j\omega mRC \)

Sub \( Y \): \( Z \frac{(1+j\omega mRC)}{mR} (1 + 2j\omega RC') - Z (1 + j\omega RC') = X j\omega RC \)
\( \Rightarrow \frac{Z}{X} = \frac{m(j\omega RC)^2}{(j\omega RC)^2 + 2j\omega RC + 1} = \frac{(j\omega/p)^2}{(j\omega/p)^2 + 2\zeta(j\omega/p) + 1} \)

Corner freq: \( p = \frac{1}{\sqrt{mRC}} = 996 \text{ rad/s} \)
**Sallen-Key Filter**

13: Filters
- Filters
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- Transformations (A)
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- Summary

\[
\begin{align*}
\text{KCL @ } Y: & \quad \frac{Y-X}{j\omega C} + \frac{Y-Z}{j\omega C} + \frac{Y-Z}{R} = 0 \quad [\text{assume } V_+ = V_- = Z] \\
& \Rightarrow Y (1 + 2j\omega RC) - Z (1 + j\omega RC) = X j\omega RC
\\
\text{KCL @ } V+: & \quad \frac{Z}{mR} + \frac{Z-Y}{j\omega C} = 0 \Rightarrow Z(1 + j\omega mRC) = Y j\omega mRC
\\
\text{Sub } Y: & \quad Z \frac{(1+j\omega mRC)}{j\omega mRC} (1 + 2j\omega RC) - Z (1 + j\omega RC) = X j\omega RC
\\
& \Rightarrow \frac{Z}{X} = \frac{m(j\omega RC)^2}{m(j\omega RC)^2 + 2j\omega RC + 1} = \frac{(j\omega/p)^2}{(j\omega/p)^2 + 2\zeta(j\omega/p) + 1}
\\
\text{Corner freq: } & \quad p = \frac{1}{\sqrt{mRC}} = 996 \text{ rad/s}, \quad \zeta = \frac{1}{2Q} = pRC = \frac{1}{\sqrt{m}} = 0.6
\end{align*}
\]


**Sallen-Key Filter**

KCL @ Y: \( \frac{Y-X}{j\omega C} + \frac{Y-Z}{j\omega C} + \frac{Y-Z}{R} = 0 \)  
[assume \( V_+ = V_- = Z \)]

\( \Rightarrow Y (1 + 2j\omega RC) - Z (1 + j\omega RC) = X j\omega RC \)

KCL @ \( V_+ \): \( \frac{Z}{mR} + \frac{Z-Y}{j\omega C} = 0 \) \( \Rightarrow Z(1 + j\omega mRC) = Y j\omega mRC \)

Sub \( Y \): \( \frac{Z(1+j\omega mRC)}{j\omega mRC} (1 + 2j\omega RC) - Z (1 + j\omega RC) = X j\omega RC \)

\( \Rightarrow \frac{Z}{X} = \frac{m(j\omega RC)^2}{m(j\omega RC)^2 + 2j\omega RC + 1} = \frac{(j\omega/p)^2}{(j\omega/p)^2 + 2\zeta(j\omega/p) + 1} \)

Corner freq: \( p = \frac{1}{\sqrt{mRC}} = 996 \text{ rad/s}, \quad \zeta = \frac{1}{2Q} = pRC = \frac{1}{\sqrt{m}} = 0.6 \)
Sallen-Key Filter

Asymptotes: \( \left( \frac{j\omega}{p} \right)^2 \) and 1

KCL @ Y: \[ \frac{Y - X}{j\omega C} + \frac{Y - Z}{j\omega C} + \frac{Y - Z}{R} = 0 \] [assume \( V_+ = V_- = Z \)]
\[ \Rightarrow Y \left( 1 + 2j\omega RC \right) - Z \left( 1 + j\omega RC \right) = X j\omega RC \]

KCL @ V+: \[ \frac{Z}{mR} + \frac{Z - Y}{j\omega C} = 0 \Rightarrow Z \left( 1 + j\omega mRC \right) = Y j\omega mRC \]

Sub Y: \[ \frac{Z \left( 1 + j\omega mRC \right)}{j\omega mRC} \left( 1 + 2j\omega RC \right) - Z \left( 1 + j\omega RC \right) = X j\omega RC \]
\[ \Rightarrow \frac{Z}{X} = \frac{m\left( j\omega RC \right)^2}{m\left( j\omega RC \right)^2 + 2j\omega RC + 1} = \frac{(j\omega/p)^2}{(j\omega/p)^2 + 2\zeta(j\omega/p) + 1} \]

Corner freq: \( p = \frac{1}{\sqrt{mRC}} = 996 \text{ rad/s}, \quad \zeta = \frac{1}{2Q} = pRC = \frac{1}{\sqrt{m}} = 0.6 \)
Sallen-Key Filter

Asymptotes: \( \left( \frac{j\omega}{p} \right)^2 \) and 1

KCL @ \( Y \): \( \frac{Y-X}{j\omega C} + \frac{Y-Z}{j\omega C} + \frac{Y-Z}{R} = 0 \)  
[assume \( V_+ = V_- = Z \)]
\[ \Rightarrow Y \left( 1 + 2j\omega RC' \right) - Z \left( 1 + j\omega RC' \right) = Xj\omega RC \]

KCL @ \( V_+ \): \( \frac{Z}{mR} + \frac{Z-Y}{j\omega C} = 0 \Rightarrow Z \left( 1 + j\omega mRC \right) = Yj\omega mRC \)

Sub \( Y \): \( \frac{Z}{j\omega mRC} \left( 1 + 2j\omega RC' \right) - Z \left( 1 + j\omega RC' \right) = Xj\omega RC \)
\[ \Rightarrow \frac{Z}{X} = \frac{m(j\omega RC)^2}{m(j\omega RC)^2 + 2j\omega RC + 1} = \frac{(j\omega/p)^2}{(j\omega/p)^2 + 2\zeta(j\omega/p) + 1} \]

Corner freq: \( p = \frac{1}{\sqrt{mRC}} = 996 \text{ rad/s}, \ \zeta = \frac{1}{2Q} = pRC = \frac{1}{\sqrt{m}} = 0.6 \)

Sallen-Key: 2nd order filter without inductors. Can easily have gain >1.
Sallen-Key Filter

Asymptotes: \( \left( \frac{j \omega}{p} \right)^2 \) and 1

KCL @ Y: \( \frac{Y-X}{1/j \omega C} + \frac{Y-Z}{1/j \omega C} + \frac{Y-Z}{R} = 0 \)

⇒ \( Y \left( 1 + 2j \omega RC \right) - Z \left( 1 + j \omega RC \right) = X j \omega RC \)

KCL @ V+: \( \frac{Z}{mR} + \frac{Z-Y}{1/j \omega C} = 0 \) ⇒ \( Z \left( 1 + j \omega mRC \right) = Y j \omega mRC \)

Sub Y: \( \frac{Z (1+j \omega mRC)}{j \omega mRC} \left( 1 + 2j \omega RC \right) - Z \left( 1 + j \omega RC \right) = X j \omega RC \)

⇒ \( \frac{Z}{X} = \frac{m(j \omega RC)^2}{m(j \omega RC)^2 + 2j \omega RC + 1} = \frac{(j \omega /p)^2}{(j \omega /p)^2 + 2 \zeta (j \omega /p) + 1} \)

Corner freq: \( p = \frac{1}{\sqrt{mRC}} = 996 \text{ rad/s}, \; \zeta = \frac{1}{2Q} = pRC = \frac{1}{\sqrt{m}} = 0.6 \)

Sallen-Key: 2nd order filter without inductors. Can easily have gain >1.

Designing: Choose \( m = \zeta^{-2}; \; C \) any convenient value; \( R = \frac{\zeta}{p \omega} \).
Twin-T Notch Filter

After much algebra:

\[ \frac{Z}{X} = \frac{(1+m)(2j\omega RC)^2 + 1}{(2j\omega RC)^2 + 4(1-m)j\omega RC + 1} \]

Do not try to memorize this circuit
Twin-T Notch Filter

After much algebra:

\[
\frac{Z}{X} = \frac{(1+m)((2j\omega RC)^2 + 1)}{(2j\omega RC)^2 + 4(1-m)j\omega RC + 1}
\]

\[
= \frac{(1+m)((j\omega/p)^2 + 1)}{(j\omega/p)^2 + 2\zeta(j\omega/p) + 1}
\]

Do not try to memorize this circuit
**Twin-T Notch Filter**

After much algebra:

\[
\frac{Z}{X} = \frac{(1+m)((2j\omega RC)^2+1)}{(2j\omega RC)^2+4(1-m)j\omega RC+1}
\]

\[
= \frac{(1+m)((j\omega/p)^2+1)}{(j\omega/p)^2+2\zeta(j\omega/p)+1}
\]

\[
p = \frac{1}{2RC} = 314, \quad \zeta = 1 - m = 0.1
\]

Do not try to memorize this circuit
After much algebra:

\[
\frac{Z}{X} = \frac{(1+m)((2j\omega RC)^2+1)}{(2j\omega RC)^2+4(1-m)j\omega RC+1}
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\[
= \frac{(1+m)((j\omega/p)^2+1)}{(j\omega/p)^2+2\zeta(j\omega/p)+1}
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\[
p = \frac{1}{2RC} = 314, \quad \zeta = 1 - m = 0.1
\]

**Very low** \(\omega\): \(C\) open circuit

Do not try to memorize this circuit
Twin-T Notch Filter

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\[
= \frac{(1+m)((j\omega/p)^2+1)}{(j\omega/p)^2+2\zeta(j\omega/p)+1}
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p = \frac{1}{2RC} = 314, \quad \zeta = 1 - m = 0.1
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**Very low \(\omega\):** \(C\) open circuit

Non-inverting amp, \(\frac{Z}{X} = 1 + m\)

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Twin-T Notch Filter

After much algebra:

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\frac{Z}{X} = \frac{(1+m)((2j\omega RC)^2+1)}{(2j\omega RC)^2+4(1-m)j\omega RC+1}
\]

\[
= \frac{(1+m)((j\omega/p)^2+1)}{(j\omega/p)^2+2\zeta(j\omega/p)+1}
\]

\[p = \frac{1}{2RC} = 314, \quad \zeta = 1 - m = 0.1\]

**Very low \(\omega\):** \(C\) open circuit

**Non-inverting amp**, \(\frac{Z}{X} = 1 + m\)

**Very high \(\omega\):** \(C\) short circuit

Do not try to memorize this circuit
**Twin-T Notch Filter**

After much algebra:

\[
\frac{Z}{X} = \frac{(1+m)((2j\omega RC)^2+1)}{(2j\omega RC)^2 + 4(1-m)j\omega RC + 1}
\]

\[
= \frac{(1+m)((j\omega/p)^2+1)}{(j\omega/p)^2 + 2\zeta(j\omega/p) + 1}
\]

\[p = \frac{1}{2RC} = 314, \quad \zeta = 1 - m = 0.1\]

**Very low \(\omega\):** \(C\) open circuit
Non-inverting amp, \(\frac{Z}{X} = 1 + m\)

**Very high \(\omega\):** \(C\) short circuit
Non-inverting amp, \(\frac{Z}{X} = 1 + m\)

Do not try to memorize this circuit
Twin-T Notch Filter

After much algebra:

\[
\frac{Z}{X} = \frac{(1+m)((2j\omega RC)^2+1)}{(2j\omega RC)^2+4(1-m)j\omega RC+1}
= \frac{(1+m)((j\omega/p)^2+1)}{(j\omega/p)^2+2\zeta(j\omega/p)+1}
\]

\[
p = \frac{1}{2RC} = 314, \quad \zeta = 1 - m = 0.1
\]

**Very low \(\omega\):**  \(C\) open circuit
Non-inverting amp, \(\frac{Z}{X} = 1 + m\)

**Very high \(\omega\):**  \(C\) short circuit
Non-inverting amp, \(\frac{Z}{X} = 1 + m\)

At \(\omega = p\), \(\left(\frac{j\omega}{p}\right)^2 = -1\): numerator = zero resulting in infinite attenuation.

Do not try to memorize this circuit
### Twin-T Notch Filter

After much algebra:

\[
\frac{Z}{X} = \frac{(1+m)((2j\omega RC)^2+1)}{(2j\omega RC)^2+4(1-m)j\omega RC+1} = \frac{(1+m)((\frac{j\omega}{p})^2+1)}{(\frac{j\omega}{p})^2+2\zeta(\frac{j\omega}{p})+1}
\]

\[
p = \frac{1}{2RC} = 314, \ \zeta = 1 - m = 0.1
\]

**Very low \(\omega\):** \(C\) open circuit  
Non-inverting amp, \(\frac{Z}{X} = 1 + m\)

**Very high \(\omega\):** \(C\) short circuit  
Non-inverting amp, \(\frac{Z}{X} = 1 + m\)

At \(\omega = p\), \((\frac{j\omega}{p})^2 = -1\): numerator = zero resulting in infinite attenuation.

---

Do not try to memorize this circuit
Twin-T Notch Filter

After much algebra:

\[
\frac{Z}{X} = \frac{(1+m)((2j\omega RC)^2+1)}{(2j\omega RC)^2+4(1-m)j\omega RC+1}
\]

\[
= \frac{(1+m)((j\omega/p)^2+1)}{(j\omega/p)^2+2\zeta(j\omega/p)+1}
\]

\[
p = \frac{1}{2RC} = 314, \quad \zeta = 1 - m = 0.1
\]

Very low \(\omega\): \(C\) open circuit
Non-inverting amp, \(\frac{Z}{X} = 1 + m\)

Very high \(\omega\): \(C\) short circuit
Non-inverting amp, \(\frac{Z}{X} = 1 + m\)

At \(\omega = p\), \(\left(\frac{j\omega}{p}\right)^2 = -1\): numerator = zero resulting in infinite attenuation.

The 3 dB notch width is approximately \(2\zeta p = 2(1 - m)p\).

Do not try to memorize this circuit
Twin-T Notch Filter

After much algebra:

\[
\frac{Z}{X} = \frac{(1+m)((2j\omega RC)^2 + 1)}{(2j\omega RC)^2 + 4(1-m)j\omega RC + 1}
\]

\[
= \frac{(1+m)((j\omega/p)^2 + 1)}{(j\omega/p)^2 + 2\zeta(j\omega/p) + 1}
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\[p = \frac{1}{2RC} = 314, \ \zeta = 1 - m = 0.1\]

**Very low \(\omega\):** \(C\) open circuit
Non-inverting amp, \(\frac{Z}{X} = 1 + m\)

**Very high \(\omega\):** \(C\) short circuit
Non-inverting amp, \(\frac{Z}{X} = 1 + m\)

At \(\omega = p\), \((\frac{j\omega}{p})^2 = -1\): numerator = zero resulting in infinite attenuation.
The 3 dB notch width is approximately \(2\zeta p = 2(1 - m)p\).

Used to remove one specific frequency (e.g. mains hum @ 50 Hz)

Do not try to memorize this circuit
Conformal Filter Transformations (A)

A dimensionless gain, $\frac{V_Y}{V_X}$, can always be written using dimensionless impedance ratio terms: $\frac{Z_R}{Z_C} = j\omega RC$, $\frac{Z_L}{Z_R} = \frac{j\omega L}{R}$, $\frac{Z_L}{Z_C} = -\omega^2 LC$. 
A dimensionless gain, $\frac{V_Y}{V_X}$, can always be written using dimensionless impedance ratio terms: $\frac{Z_R}{Z_C} = j\omega RC$, $\frac{Z_L}{Z_R} = \frac{j\omega L}{R}$, $\frac{Z_L}{Z_C} = -\omega^2 LC$. 

![Graph showing the magnitude of the gain vs. angular frequency](image)
Conformal Filter Transformations (A)

A dimensionless gain, $\frac{V_Y}{V_X}$, can always be written using dimensionless impedance ratio terms: $\frac{Z_R}{Z_C} = j\omega RC$, $\frac{Z_L}{Z_R} = \frac{j\omega L}{R}$, $\frac{Z_L}{Z_C} = -\omega^2 LC$.

Impedance scaling:

Scale all impedances by $k$:

![Impedance scaling diagram]
Conformal Filter Transformations (A)

A dimensionless gain, \( \frac{V_Y}{V_X} \), can always be written using dimensionless impedance ratio terms:
\[
\frac{Z_R}{Z_C} = j\omega RC, \quad \frac{Z_L}{Z_R} = \frac{j\omega L}{R}, \quad Z_L = -\omega^2 LC.
\]

Impedance scaling:

Scale all impedances by \( k \):
\[
R' = kR, \quad C' = k^{-1}C, \quad L' = kL
\]

\[k = 20\]
Conformal Filter Transformations (A)

A dimensionless gain, \( \frac{V_Y}{V_X} \), can always be written using dimensionless impedance ratio terms:

\[
\frac{Z_R}{Z_C} = j\omega RC, \quad \frac{Z_L}{Z_R} = \frac{j\omega L}{R}, \quad \frac{Z_L}{Z_C} = -\omega^2 LC.
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Impedance scaling:

Scale all impedances by \( k \):

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Impedance ratios are unchanged so graph stays the same.

\[ k = 20 \]
Conformal Filter Transformations (A)

A dimensionless gain, \( \frac{V_Y}{V_X} \), can always be written using dimensionless impedance ratio terms: 
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Impedance scaling:

Scale all impedances by \( k \):
\[ R' = kR, \quad C' = k^{-1}C, \quad L' = kL \]
Impedance ratios are unchanged so graph stays the same. \((k\) is arbitrary)
A dimensionless gain, \( \frac{V_Y}{V_X} \), can always be written using dimensionless impedance ratio terms:

\[
\frac{Z_R}{Z_C} = j\omega RC, \quad \frac{Z_L}{Z_R} = \frac{j\omega L}{R}, \quad \frac{Z_L}{Z_C} = -\omega^2 LC.
\]

**Impedance scaling:**

Scale all impedances by \( k \):

\[
R' = kR, \quad C' = k^{-1}C, \quad L' = kL
\]

Impedance ratios are unchanged so graph stays the same.  
\( (k \text{ is arbitrary}) \)

**Frequency Shift:**

Scale reactive components by \( k \):

\( k = 20 \)
A dimensionless gain, \( \frac{V_Y}{V_X} \), can always be written using dimensionless impedance ratio terms: 
\[
\frac{Z_R}{Z_C} = j\omega RC, \quad \frac{Z_L}{Z_R} = \frac{j\omega L}{R}, \quad \frac{Z_L}{Z_C} = -\omega^2 LC.
\]

**Impedance scaling:**

Scale all impedances by \( k \):
\[
R' = kR, \quad C' = k^{-1}C, \quad L' = kL
\]
Impedance ratios are unchanged so graph stays the same. \((k \text{ is arbitrary})\)

**Frequency Shift:**

Scale reactive components by \( k \):
\[
R' = R, \quad C' = kC, \quad L' = kL
\]
Conformal Filter Transformations (A)

A dimensionless gain, \( \frac{V_Y}{V_X} \), can always be written using dimensionless impedance ratio terms: \( \frac{Z_R}{Z_C} = j\omega RC \), \( \frac{Z_L}{Z_R} = \frac{j\omega L}{R} \), \( \frac{Z_L}{Z_C} = -\omega^2 LC \).

Impedance scaling:

Scale all impedances by \( k \):
\[
R' = kR, \quad C' = k^{-1}C, \quad L' = kL
\]
Impedance ratios are unchanged so graph stays the same. (\( k \) is arbitrary)

Frequency Shift:

Scale reactive components by \( k \):
\[
R' = R, \quad C' = kC, \quad L' = kL
\]
\[
\Rightarrow Z'(k^{-1}\omega) \equiv Z(\omega)
\]
A dimensionless gain, $\frac{V_Y}{V_X}$, can always be written using dimensionless impedance ratio terms:

$$\frac{Z_R}{Z_C} = j\omega RC, \quad \frac{Z_L}{Z_R} = \frac{j\omega L}{R}, \quad \frac{Z_L}{Z_C} = -\omega^2 LC.$$ 

**Impedance scaling:**

Scale all impedances by $k$:

$$R' = kR, \quad C' = k^{-1}C, \quad L' = kL$$

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**Frequency Shift:**

Scale reactive components by $k$:

$$R' = R, \quad C' = kC, \quad L' = kL$$

$$\Rightarrow Z'(k^{-1}\omega) \equiv Z(\omega)$$

Graph shifts left by a factor of $k$. 

$k = 20$

$k = 5$
Conformal Filter Transformations (A)

A dimensionless gain, \( \frac{V_Y}{V_X} \), can always be written using dimensionless impedance ratio terms: 
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Scale reactive components by \( k \):
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R' = R, \quad C' = kC, \quad L' = kL
\]
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\Rightarrow Z'(k^{-1}\omega) \equiv Z(\omega)
\]
Graph shifts left by a factor of \( k \).

Must scale all reactive components in the circuit by the same factor.
Conformal Filter Transformations (B)

Change LR circuit to RC:

![Diagram of LR circuit to RC conversion](image)
Conformal Filter Transformations (B)

Change LR circuit to RC:

\[ R' = kL, \quad C' = \frac{1}{kR} \]

\[ k = 10^6 \]
Conformal Filter Transformations (B)

Change LR circuit to RC:

\[ R' = kL, \quad C' = \frac{1}{kR} \]

\[ \Rightarrow \frac{Z_{R'}}{Z_{C'}} = j\omega R'C' = \frac{j\omega L}{R} = \frac{Z_L}{Z_R} \]

\( k = 10^6 \)
Conformal Filter Transformations (B)

Change LR circuit to RC:

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\text{Change } R' = kL, \quad C' = \frac{1}{kR} \\
\Rightarrow \frac{Z_{R'}}{Z_{C'}} = j\omega R' C' = \frac{j\omega L}{R} = \frac{Z_L}{Z_R}
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Impedance ratios are unchanged at all \( \omega \) so graph stays the same.

\[ k = 10^6 \]
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Impedance ratios are unchanged at all \( \omega \) so graph stays the same. \( (k: \text{is arbitrary}) \)

Reflect frequency axis around \( \omega_m \):

Change \( R' = \frac{k}{\omega_mC} \), \( C' = \frac{1}{\omega_mkR} \)
Conformal Filter Transformations (B)

Change LR circuit to RC:

Change \( R' = kL, \ C' = \frac{1}{kR} \)

\[ \Rightarrow \frac{Z_{R'}}{Z_{C'}} = j\omega R'C' = \frac{j\omega L}{R} = \frac{Z_L}{Z_R} \]

Impedance ratios are unchanged at all \( \omega \) so graph stays the same. (\( k \) is arbitrary)

Reflect frequency axis around \( \omega_m \):

Change \( R' = \frac{k}{\omega_m C}, \ C' = \frac{1}{\omega_m kR} \)

\[ \Rightarrow \frac{Z_{R'}}{Z_{C'}} \left( \frac{\omega_m^2}{\omega} \right) = \left( \frac{Z_C}{Z_R} (\omega) \right) \]

\[ k = 10^6 \]

\[ k = 0.1, \ \omega_m = 20 k \]
Conformal Filter Transformations (B)

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\]

(a) Magnitude graph flips

\( k = 10^6 \)

\( k = 0.1, \omega_m = 20k \)
Conformal Filter Transformations (B)

Change LR circuit to RC:

\[ R' = kL, \quad C' = \frac{1}{kR} \]

\[ \Rightarrow \frac{Z_{R'}}{Z_{C'}} = j\omega R'C' = \frac{j\omega L}{R} = \frac{Z_L}{Z_R} \]

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(a) Magnitude graph flips
(b) Phase graph flips and negates since \( \angle z^* = -\angle z \).

(\( k \) is arbitrary)
Summary

- The order of a filter is the highest power of $j\omega$ in the transfer function denominator.
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- Active filters use op-amps and usually avoid the need for inductors.
  - Sallen-Key design for high-pass and low-pass.
  - Twin-T design for notch filter: gain = 0 at notch.
Summary

- The **order** of a filter is the highest power of $j\omega$ in the transfer function denominator.

- **Active filters** use op-amps and usually avoid the need for inductors.
  - Sallen-Key design for high-pass and low-pass.
  - Twin-T design for notch filter: gain = 0 at notch.

- For filters using $R$ and $C$ only:
  - **Scale $R$ and $C$:** Substituting $R' = kR$ and $C' = pC$ scales frequency by $(pk)^{-1}$. 
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- Scale $R$ and $C$: Substituting $R' = kR$ and $C' = pC$ scales frequency by $(pk)^{-1}$.
- Interchange $R$ and $C$: Substituting $R' = \frac{k}{\omega_0 C}$ and $C' = \frac{1}{k\omega_0 R}$ flips the frequency response around $\omega_0$ ($\forall k$).
  Changes a low-pass filter to high pass and vice-versa.
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For further details see Hayt Ch 16 or Irwin Ch 12.