13: Filters

Filters
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Summary
A filter is a circuit whose gain varies with frequency. Often a filter aims to allow some frequencies to pass while blocking others.

- **Radio/TV:** a “tuning” filter blocks all frequencies except the wanted channel
- **Loudspeaker:** “crossover” filters send the right frequencies to different drive units
- **Sampling:** an “anti-aliasing filter” eliminates all frequencies above half the sampling rate
  - **Phones:** Sample rate = 8 kHz : filter eliminates frequencies above 3.4 kHz.
- **Computer cables:** filter eliminates interference
1st Order Low-Pass Filter

\[
\frac{Y}{X} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{j\omega RC + 1} = \frac{1}{j\frac{p}{R} + 1}
\]

Corner frequency: \( p = \left| \frac{b}{a} \right| = \frac{1}{RC} \)

Asymptotes: 1 and \( \frac{p}{j\omega} \)

Very low \( \omega \): Capacitor = open circuit

Very high \( \omega \): Capacitor short circuit

A **low-pass** filter because it allows low frequencies to pass but **attenuates** (makes smaller) high frequencies.

The **order** of a filter: highest power of \( j\omega \) in the denominator. Almost always equals the total number of \( L \) and/or \( C \).
Low-Pass with Gain Floor

\[
\frac{Y}{X} = \frac{R + \frac{1}{j\omega C}}{4R + \frac{1}{j\omega C}} = \frac{j\omega RC + 1}{j\omega 4RC + 1} = \frac{j\omega + 1}{q} \frac{1}{p} + 1
\]

Corner frequencies: \( p = \frac{1}{4RC} \), \( q = \frac{1}{RC} \)

Asymptotes: 1 and \( \frac{1}{4} \)

**Very low \( \omega \):**
- Capacitor = open circuit
- Resistor \( R \) unattached. Gain = 1

**Very high \( \omega \):**
- Capacitor short circuit
- Circuit is potential divider with gain \( 20 \log_{10} \frac{1}{4} = -12 \text{ dB} \).
Opamp filter

Inverting amplifier so

\[ \frac{Y}{X} = -\frac{3R||(R+1/j\omega C)}{R} = -\frac{3R(R+1/j\omega C)}{R \times (3R+R+1/j\omega C)} \]

\[ = -3 \times \frac{R+1/j\omega C}{4R+1/j\omega C} = -3 \times \frac{j\omega RC+1}{j\omega 4RC+1} \]

Same transfer function as before except \( \times -3 = +9.5 \) dB.

Advantages of op-amp circuit:

1. Can have gain > 1.
2. Low output impedance - loading does not affect filter
3. Resistive input impedance - does not vary with frequency

<table>
<thead>
<tr>
<th>Gain</th>
<th>(dB)</th>
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<tbody>
<tr>
<td>10</td>
<td>0.1q</td>
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<tr>
<td>5</td>
<td>p</td>
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<tr>
<td>0</td>
<td>q</td>
</tr>
<tr>
<td>10</td>
<td>10q</td>
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E1.1 Analysis of Circuits (2017-10116)
Integrator

$$\frac{Y}{X} = -\frac{1/j\omega C}{R} = -\frac{1}{j\omega RC}$$

Capacitor: \(i = C \frac{dv_C}{dt}\)

\(i = \frac{x}{R} = -C \frac{dy}{dt}\)

\(\frac{dy}{dt} = -\frac{1}{RC} x\)

\[\int_0^t \frac{dy}{dt} dt = -\frac{1}{RC} \int_0^t x dt\]

\(y(t) = -\frac{1}{RC} \int_0^t x dt + y(0)\)

Note: if \(x(t) = \cos \omega t\)

\[\int \cos(\omega t) dt = \frac{1}{\omega} \sin(\omega t) \Rightarrow \text{gain} \propto \frac{1}{\omega}.\]

We can limit the LF gain to 20 dB:

$$\frac{Y}{X} = -\frac{10R \cdot 1/j\omega C}{R} = -\frac{10R \cdot 1/j\omega C}{R(10R + 1/j\omega C)}$$

\(= -\frac{10}{j\omega 10RC + 1} \quad (\omega_c = \frac{0.1}{RC})\)
High Pass Filter

\[ \frac{Y}{X} = \frac{R}{R + 1/j\omega C} = \frac{j\omega RC}{j\omega RC + 1} \]

Corner Freq: \[ p = \frac{1}{RC} \]

Asymptotes: \( j\omega RC \) and 1

Very low \( \omega \): \( C \) open circuit: gain = 0

Very high \( \omega \): \( C \) short circuit: gain = 1

We can add an op-amp to give a low-impedance output. Or add gain:

\[ \frac{Z}{X} = (1 + \frac{R_B}{R_A}) \times \frac{j\omega RC}{j\omega RC + 1} \]
2nd order filter

\[
\frac{Y}{X} = \frac{R_2 + j\omega L}{1/j\omega C + R_1 + R_2 + j\omega L}
\]

\[
= \frac{LC(j\omega)^2 + R_2 C j\omega}{LC(j\omega)^2 + (R_1 + R_2) C j\omega + 1}
\]

\[
= \frac{j\omega C(j\omega L + R_2)}{LC(j\omega)^2 + (R_1 + R_2) C j\omega + 1}
\]

Asymptotes: \(j\omega R_2 C\) and 1

Corner frequencies:

\(+20\ \text{dB/dec at } p = \frac{R_2}{L} = 100 \ \text{rad/s}\)

\(-40\ \text{dB/dec at }\)

\(q = \sqrt{\frac{c}{a}} = \frac{1}{\sqrt{LC}} = 1000 \ \text{rad/s}\)

Damping factor: \(\zeta = \frac{b \ \text{sgn}(a)}{\sqrt{4ac}} = \frac{q b}{2c} = \frac{q}{2} (R_1 + R_2) C = 0.6\).

Gain error at \(q\) is \(\frac{1}{2\zeta} = Q = 0.83 = -1.6 \ \text{dB} \ (0.04 \ \text{dB due to } p)\)

Compare with 1st order:

2nd order filter attenuates more rapidly than a 1st order filter.
Sallen-Key Filter

**Asymptotes:** \( \left( \frac{j\omega}{p} \right)^2 \) and 1

**KCL @ Y:** \( \frac{Y - X}{1/j\omega C} + \frac{Y - Z}{1/j\omega C} + \frac{Y - Z}{R} = 0 \)  
[assume \( V_+ = V_- = Z \)]

\( \Rightarrow Y \left( 1 + 2j\omega RC \right) - Z \left( 1 + j\omega RC \right) = X j\omega RC \)

**KCL @ V+:** \( \frac{Z}{mR} + \frac{Z - Y}{1/j\omega C} = 0 \Rightarrow Z \left( 1 + j\omega mRC \right) = Y j\omega mRC \)

**Sub Y:** \( Z \frac{(1+j\omega mRC)}{j\omega mRC} \left( 1 + 2j\omega RC \right) - Z \left( 1 + j\omega RC \right) = X j\omega RC \)

\( \Rightarrow \frac{Z}{X} = \frac{m(j\omega RC)^2}{m(j\omega RC)^2 + 2j\omega RC + 1} = \frac{(j\omega/p)^2}{(j\omega/p)^2 + 2\zeta(j\omega/p) + 1} \)

**Corner freq:** \( p = \frac{1}{\sqrt{mRC}} = 996 \text{ rad/s}, \; \zeta = \frac{1}{2Q} = pRC = \frac{1}{\sqrt{m}} = 0.6 \)

**Sallen-Key:** 2nd order filter without inductors. Can easily have gain > 1.

**Designing:** Choose \( m = \zeta^{-2} \); \( C \) any convenient value; \( R = \frac{\zeta}{pC} \).
After much algebra:

\[
\frac{Z}{X} = \frac{(1+m)((2j\omega RC)^2+1)}{(2j\omega RC)^2 + 4(1-m)j\omega RC + 1}
\]

\[
= \frac{(1+m)((j\omega/p)^2+1)}{(j\omega/p)^2 + 2\zeta(j\omega/p) + 1}
\]

\[
p = \frac{1}{2RC} = 314, \quad \zeta = 1 - m = 0.1
\]

**Very low \( \omega \):** \( C \) open circuit
Non-inverting amp, \( \frac{Z}{X} = 1 + m \)

**Very high \( \omega \):** \( C \) short circuit
Non-inverting amp, \( \frac{Z}{X} = 1 + m \)

At \( \omega = p \), \( \left( \frac{j\omega}{p} \right)^2 = -1 \): numerator = zero resulting in infinite attenuation.

The 3 dB notch width is approximately \( 2\zeta p = 2(1 - m)p \).

Used to remove one specific frequency (e.g. mains hum @ 50 Hz)

**Do not try to memorize this circuit**
Conformal Filter Transformations (A)

A dimensionless gain, \( \frac{V_Y}{V_X} \), can always be written using dimensionless impedance ratio terms: \( \frac{Z_R}{Z_C} = j\omega RC \), \( \frac{Z_L}{Z_R} = \frac{j\omega L}{R} \), \( \frac{Z_L}{Z_C} = -\omega^2 LC \).

**Impedance scaling:**

Scale all impedances by \( k \):

\( R' = kR \), \( C' = k^{-1}C \), \( L' = kL \)

Impedance ratios are unchanged so graph stays the same.

\( (k \text{ is arbitrary}) \)

**Frequency Shift:**

Scale reactive components by \( k \):

\( R' = R \), \( C' = kC \), \( L' = kL \)

\( \Rightarrow Z'(k^{-1}\omega) \equiv Z(\omega) \)

Graph shifts left by a factor of \( k \).

Must scale all reactive components in the circuit by the same factor.

\( k = 20 \)

\( k = 5 \)
Conformal Filter Transformations (B)

Change LR circuit to RC:

\[ R' = kL, \quad C' = \frac{1}{kR} \]
\[ \Rightarrow \frac{Z_{R'}}{Z_{C'}} = j\omega R'C' = \frac{j\omega L}{R} = \frac{Z_L}{Z_R} \]

Impedance ratios are unchanged at all \( \omega \) so graph stays the same. (\( k \) is arbitrary)

Reflect frequency axis around \( \omega_m \):

\[ R' = \frac{k}{\omega_m C'}, \quad C' = \frac{1}{\omega_m kR} \]
\[ \Rightarrow \frac{Z_{R'}}{Z_{C'}} \left( \frac{\omega_m^2}{\omega} \right) = \left( \frac{Z_C}{Z_R}(\omega) \right) \]

(a) Magnitude graph flips
(b) Phase graph flips and negates since \( \angle z^* = -\angle z \).

(\( k \) is arbitrary)
Summary

- The order of a filter is the highest power of $j\omega$ in the transfer function denominator.

- Active filters use op-amps and usually avoid the need for inductors.
  - Sallen-Key design for high-pass and low-pass.
  - Twin-T design for notch filter: gain = 0 at notch.

- For filters using $R$ and $C$ only:
  - Scale $R$ and $C$: Substituting $R' = kR$ and $C' = pC$ scales frequency by $(pk)^{-1}$.
  - Interchange $R$ and $C$: Substituting $R' = \frac{k}{\omega_0 C}$ and $C' = \frac{1}{k\omega_0 R}$ flips the frequency response around $\omega_0$ ($\forall k$). Changes a low-pass filter to high pass and vice-versa.

For further details see Hayt Ch 16 or Irwin Ch 12.