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A filter is a circuit whose gain varies with frequency. Often a filter aims to allow some frequencies to pass while blocking others.

- Radio/TV: a "tuning" filter blocks all frequencies except the wanted channel
- Loudspeaker: "crossover" filters send the right frequencies to different drive units
- Sampling: an "anti-aliasing filter" eliminates all frequencies above half the sampling rate
  - Phones: Sample rate $= 8$ kHz : filter eliminates frequencies above 3.4 kHz.
- Computer cables: filter eliminates interference

[Wikipedia]

1st Order Low-Pass Filter

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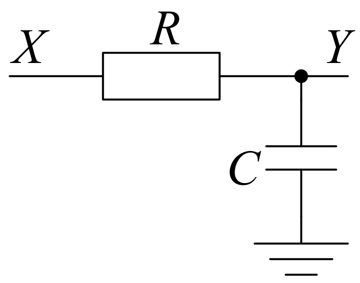
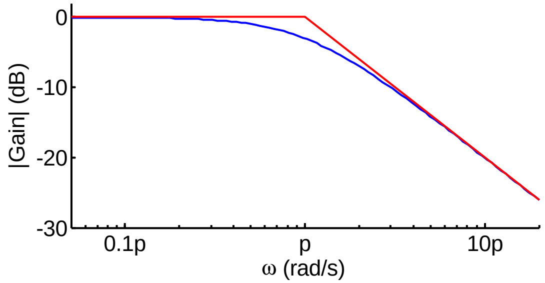
$$\frac{Y}{X} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{j\omega RC + 1} = \frac{1}{\frac{j\omega}{p} + 1}$$



Corner frequency: $p = \left|\frac{b}{a}\right| = \frac{1}{RC}$

Asymptotes: 1 and $\frac{p}{j\omega}$



Very low ω: Capacitor = open circuit
Very high ω: Capacitor short circuit

A *low-pass* filter because it allows low frequencies to pass but *attenuates* (makes smaller) high frequencies.

The *order* of a filter: highest power of $j\omega$ in the denominator.
Almost always equals the total number of L and/or C.

Low-Pass with Gain Floor

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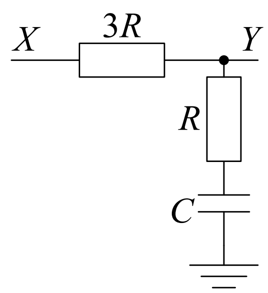
$$\frac{Y}{X} = \frac{R+1/j\omega C}{4R+1/j\omega C} = \frac{j\omega RC+1}{j\omega 4RC+1} = \frac{\frac{j\omega}{q}+1}{\frac{j\omega}{p}+1}$$



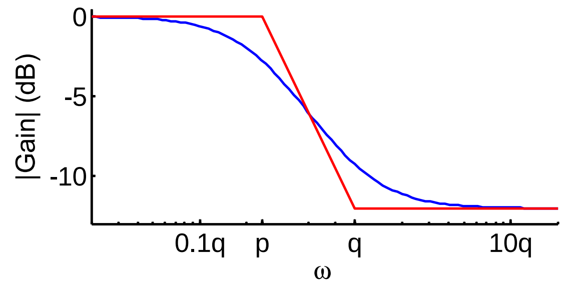
Corner frequencies: $p = \frac{1}{4RC}$, $q = \frac{1}{RC}$

Asymptotes: 1 and $\frac{1}{4}$

Very low ω:

Capacitor = open circuit

Resistor R unattached. Gain $= 1$



Very high ω:

Capacitor short circuit

Circuit is potential divider with gain $20\log_{10}\frac{1}{4} = -12\,\text{dB}$.

Opamp filter

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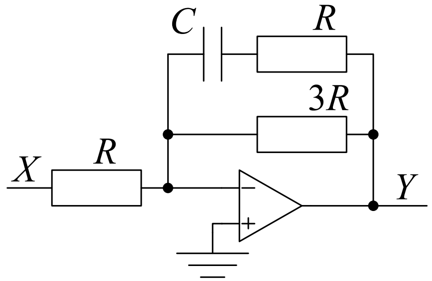
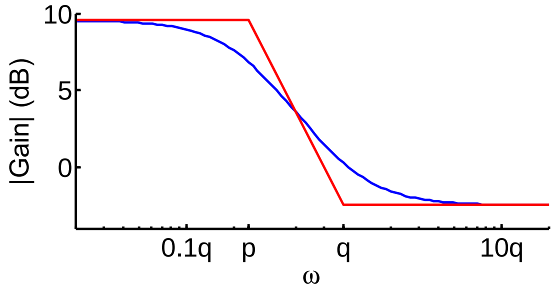
Inverting amplifier so



$$\frac{Y}{X} = -\frac{3R||(R+{}^1\!/_{j\omega C})}{R} = -\frac{3R(R+{}^1\!/_{j\omega C})}{R\times(3R+R+{}^1\!/_{j\omega C})}$$

$$= -3 \times \frac{R+{}^1\!/_{j\omega C}}{4R+{}^1\!/_{j\omega C}} = -3 \times \frac{j\omega RC+1}{j\omega 4RC+1}$$

Same transfer function as before except $\times -3 = +9.5\,\text{dB}$.



Advantages of op-amp crcuit:

1. Can have gain > 1.
2. Low output impedance - loading does not affect filter
3. Resistive input impedance - does not vary with frequency

Integrator

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$$\frac{Y}{X} = -\frac{1/j\omega C}{R} = -\frac{1}{j\omega RC}$$

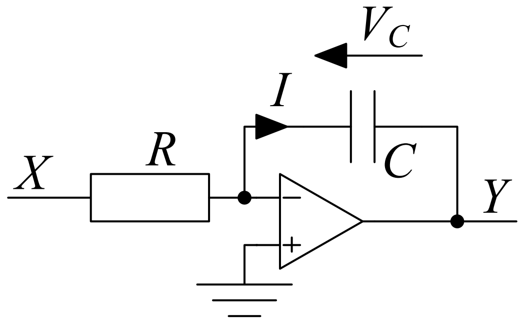
Capacitor: $i = C\frac{dv_C}{dt}$

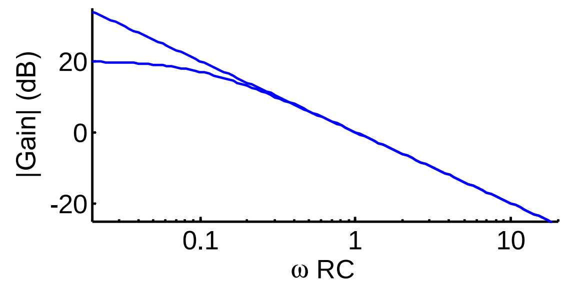
$i = \frac{x}{R} = -C\frac{dy}{dt}$

$\frac{dy}{dt} = \frac{-1}{RC}x$

$\int_0^t \frac{dy}{dt}dt = \frac{-1}{RC}\int_0^t x dt$

$y(t) = \frac{-1}{RC}\int_0^t x dt + y(0)$



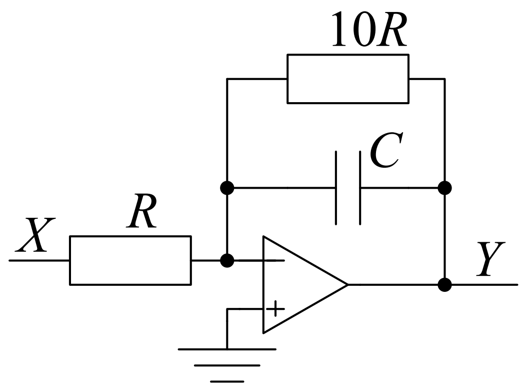


Note: if $x(t) = \cos\omega t$
$\int \cos(\omega t)dt = \frac{1}{\omega}\sin(\omega t) \Rightarrow$ gain $\propto \frac{1}{\omega}$.

We can limit the LF gain to 20 dB:

$$\frac{Y}{X} = -\frac{10R||1/j\omega C}{R} = -\frac{10R\times 1/j\omega C}{R(10R+1/j\omega C)}$$

$$= -\frac{10}{j\omega 10RC+1} \qquad \left(\omega_c = \frac{0.1}{RC}\right)$$



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$\frac{Y}{X} = \frac{R}{R+1/j\omega C} = \frac{j\omega RC}{j\omega RC+1}$

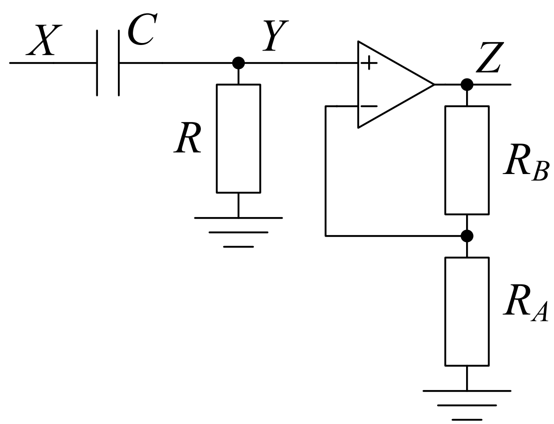
Corner Freq: $p = \frac{1}{RC}$

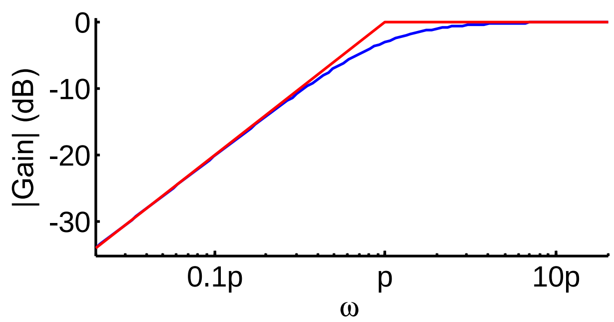
Asymptotes: $j\omega RC$ and 1

Very low ω: C open circuit: gain $= 0$
Very high ω: C short circuit: gain $= 1$

We can add an op-amp to give a low-impedance output. Or add gain:

$\frac{Z}{X} = \left(1 + \frac{R_B}{R_A}\right) \times \frac{j\omega RC}{j\omega RC+1}$





2nd order filter

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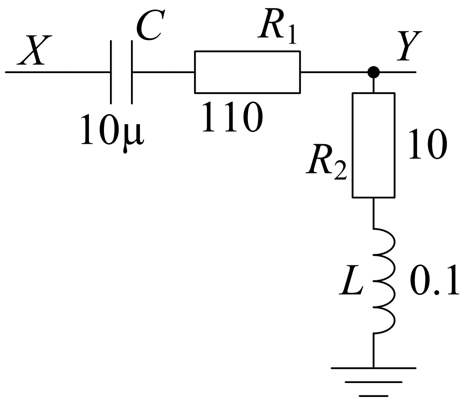
Summary

$$\frac{Y}{X} = \frac{R_2 + j\omega L}{1/j\omega C + R_1 + R_2 + j\omega L}$$

$$= \frac{LC(j\omega)^2 + R_2 C j\omega}{LC(j\omega)^2 + (R_1 + R_2)Cj\omega + 1}$$

$$= \frac{j\omega C(j\omega L + R_2)}{LC(j\omega)^2 + (R_1 + R_2)Cj\omega + 1}$$



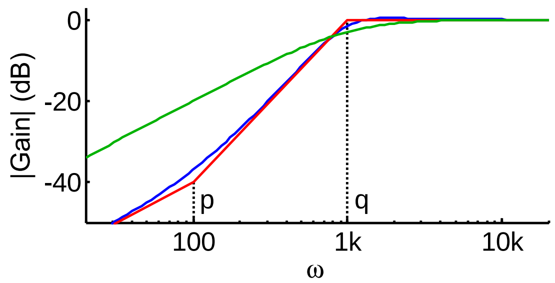
Asymptotes: $j\omega R_2 C$ and 1

Corner frequencies:

$+20\,\text{dB/dec}$ at $p = \frac{R_2}{L} = 100\,\text{rad/s}$

$-40\,\text{dB/dec}$ at

$$q = \sqrt{\frac{c}{a}} = \frac{1}{\sqrt{LC}} = 1000\,\text{rad/s}$$



Damping factor: $\zeta = \frac{b\,\text{sgn}(a)}{\sqrt{4ac}} = \frac{qb}{2c} = \frac{q}{2}\left(R_1 + R_2\right)C = 0.6.$

Gain error at q is $\frac{1}{2\zeta} = Q = 0.83 = -1.6\,\text{dB}$ $(+0.04\,\text{dB}$ due to $p)$

Compare with 1st order:

2nd order filter attenuates more rapidly than a 1st order filter.

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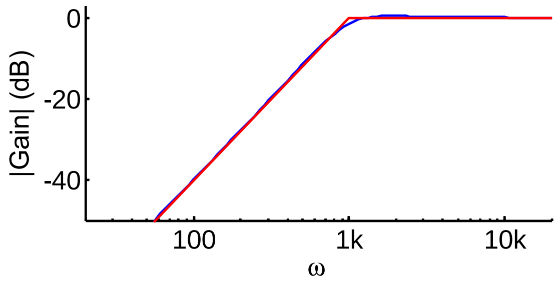

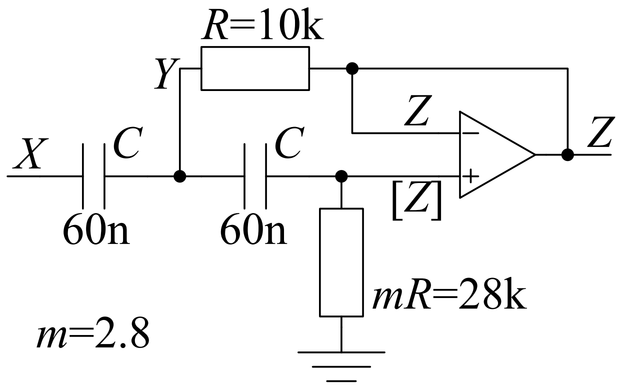

Asymptotes: $\left(\frac{j\omega}{p}\right)^2$ and 1

KCL @ Y: $\frac{Y-X}{1/j\omega C} + \frac{Y-Z}{1/j\omega C} + \frac{Y-Z}{R} = 0$ [assume $V_+ = V_- = Z$]

$$\Rightarrow Y(1+2j\omega RC) - Z(1+j\omega RC) = Xj\omega RC$$

KCL @ V_+: $\frac{Z}{mR} + \frac{Z-Y}{1/j\omega C} = 0 \Rightarrow Z(1+j\omega mRC) = Yj\omega mRC$

Sub Y: $Z\frac{(1+j\omega mRC)}{j\omega mRC}(1+2j\omega RC) - Z(1+j\omega RC) = Xj\omega RC$

$$\Rightarrow \frac{Z}{X} = \frac{m(j\omega RC)^2}{m(j\omega RC)^2 + 2j\omega RC + 1} = \frac{(j\omega/p)^2}{(j\omega/p)^2 + 2\zeta(j\omega/p) + 1}$$

Corner freq: $p = \frac{1}{\sqrt{m}RC} = 996$ rad/s, $\zeta = \frac{1}{2Q} = pRC = \frac{1}{\sqrt{m}} = 0.6$

Sallen-Key: 2nd order filter without inductors. Can easily have gain >1.

Designing: Choose $m = \zeta^{-2}$; C any convenient value; $R = \frac{\zeta}{pC}$.

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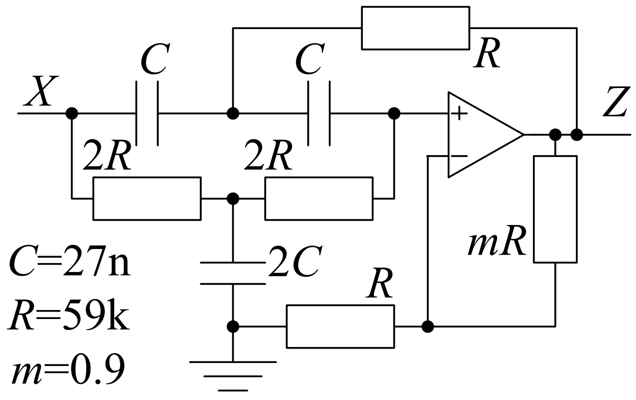
Twin-T Notch Filter

After much algebra:

$$\frac{Z}{X} = \frac{(1+m)\left((2j\omega RC)^2+1\right)}{(2j\omega RC)^2+4(1-m)j\omega RC+1}$$

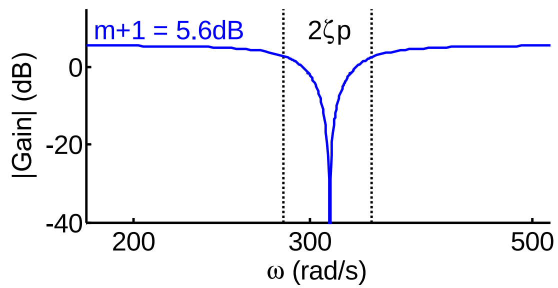
$$= \frac{(1+m)\left((j\omega/p)^2+1\right)}{(j\omega/p)^2+2\zeta(j\omega/p)+1}$$

$p = \frac{1}{2RC} = 314,\ \zeta = 1 - m = 0.1$



Very low ω: C open circuit
Non-inverting amp, $\frac{Z}{X} = 1 + m$

Very high ω: C short circuit
Non-inverting amp, $\frac{Z}{X} = 1 + m$



At $\omega = p$, $\left(\frac{j\omega}{p}\right)^2 = -1$: numerator = zero resulting in infinite attenuation.
The $3\,\text{dB}$ notch width is approximately $2\zeta p = 2(1-m)p$.

Used to remove one specific frequency (e.g. mains hum @ 50 Hz)

Do not try to memorize this circuit

Conformal Filter Transformations (A)

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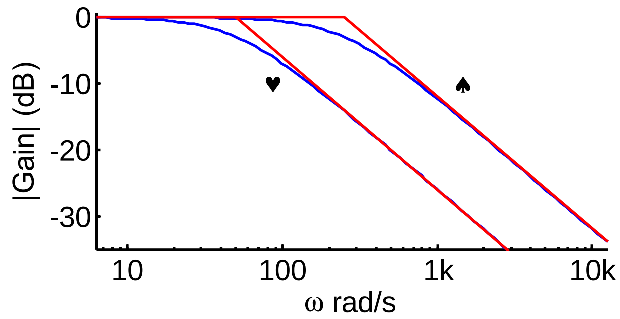
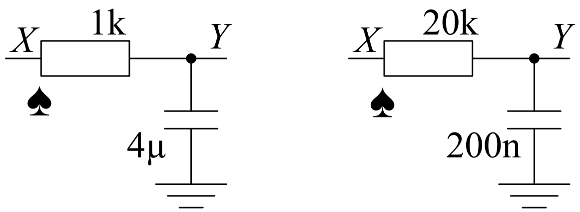
A dimensionless gain, $\frac{V_Y}{V_X}$, can always be written using dimensionless impedance ratio terms: $\frac{Z_R}{Z_C} = j\omega RC$, $\frac{Z_L}{Z_R} = \frac{j\omega L}{R}$, $\frac{Z_L}{Z_C} = -\omega^2 LC$.



Impedance scaling:

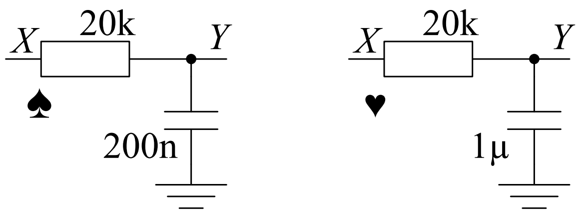
Scale all impedances by k:
$R' = kR,\ C' = k^{-1}C,\ L' = kL$
Impedance ratios are unchanged so graph stays the same.
(k is arbitrary)



$k = 20$

Frequency Shift:

Scale reactive components by k:
$R' = R,\ C' = kC,\ L' = kL$
$\Rightarrow Z'(k^{-1}\omega) \equiv Z(\omega)$
Graph shifts left by a factor of k.



$k = 5$

Must scale all reactive components in the circuit by the same factor.

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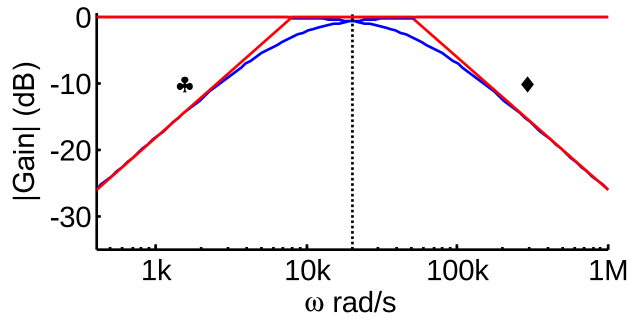
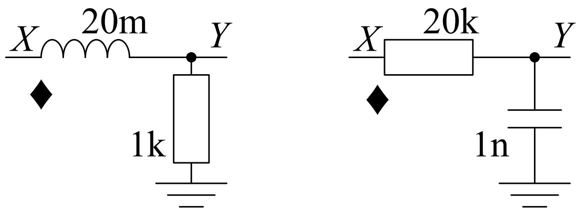
Change LR circuit to RC:

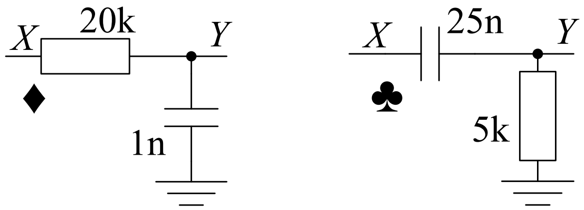
Change $R' = kL$, $C' = \frac{1}{kR}$
$\Rightarrow \frac{Z_{R'}}{Z_{C'}} = j\omega R'C' = \frac{j\omega L}{R} = \frac{Z_L}{Z_R}$

Impedance ratios are unchanged at all ω so graph stays the same. (k is arbitrary)





$k = 10^6$



$k = 0.1,\ \omega_m = 20\,\text{k}$

Reflect frequency axis around ω_m:

Change $R' = \frac{k}{\omega_m C}$, $C' = \frac{1}{\omega_m kR}$
$\Rightarrow \frac{Z_{R'}}{Z_{C'}}\left(\frac{\omega_m^2}{\omega}\right) = \left(\frac{Z_C}{Z_R}(\omega)\right)^*$

(a) Magnitude graph flips
(b) Phase graph flips and negates since $\angle z^* = -\angle z$.
(k is arbitrary)

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▷ Summary

- The *order* of a filter is the highest power of $j\omega$ in the transfer function denominator.
- *Active filters* use op-amps and usually avoid the need for inductors.
  - Sallen-Key design for high-pass and low-pass.
  - Twin-T design for notch filter: gain $= 0$ at notch.
- For filters using R and C only:
  - Scale R and C: Substituting $R' = kR$ and $C' = pC$ scales frequency by $(pk)^{-1}$.
  - Interchange R and C: Substituting $R' = \frac{k}{\omega_0 C}$ and $C' = \frac{1}{k\omega_0 R}$ flips the frequency response around ω_0 $(\forall k)$.
    Changes a low-pass filter to high pass and vice-versa.

For further details see Hayt Ch 16 or Irwin Ch 12.