▷ 14: Power in AC Circuits
Average Power
Cosine Wave RMS
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Complex Power
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Average Power

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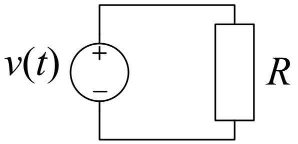

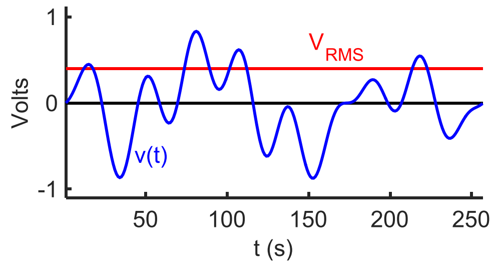

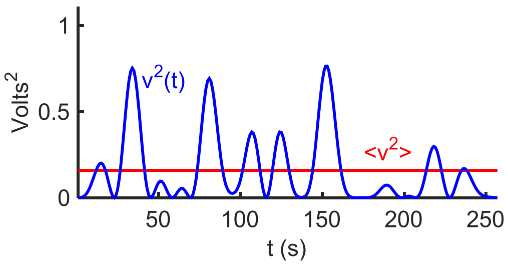

Intantaneous Power dissipated in R: $p(t) = \frac{v^2(t)}{R}$

Average Power dissipated in R:

$$P = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{R} \times \frac{1}{T}\int_0^T v^2(t)dt = \frac{\left\langle v^2(t)\right\rangle}{R}$$

$\left\langle v^2(t)\right\rangle$ is the value of $v^2(t)$ averaged over time

We define the *RMS Voltage* (Root Mean Square): $V_{rms} \triangleq \sqrt{\left\langle v^2(t)\right\rangle}$

The average power dissipated in R is $P = \frac{\left\langle v^2(t)\right\rangle}{R} = \frac{(V_{rms})^2}{R}$

V_{rms} is the DC voltage that would cause R to dissipate the same power.

We use *small letters* for time-varying voltages and *capital letters* for time-invariant values.

Cosine Wave RMS

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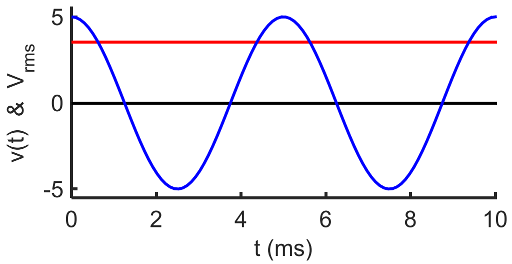

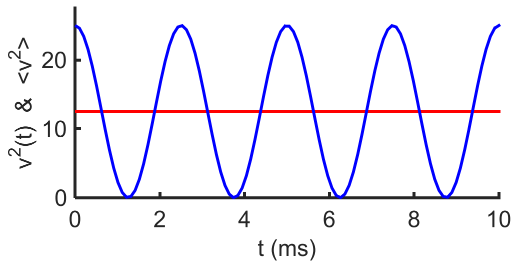

Cosine Wave: $v(t) = 5\cos\omega t$. Amplitude is $V = 5$ V.

Squared Voltage: $v^2(t) = V^2\cos^2\omega t = V^2\left(\frac{1}{2} + \frac{1}{2}\cos 2\omega t\right)$

Mean Square Voltage: $\langle v^2 \rangle = \frac{V^2}{2}$ since $\cos 2\omega t$ averages to zero.

RMS Voltage: $V_{rms} = \sqrt{\langle v^2 \rangle} = \frac{1}{\sqrt{2}}V = 3.54\,\text{V} = \widetilde{V}$

Note: Power engineers *always* use RMS voltages and currents exclusively and omit the "rms" subscript.
For example UK Mains voltage = 230 V rms = 325 V peak.

In this lecture course only, a ~ overbar means $\div\sqrt{2}$: thus $\widetilde{V} = \frac{1}{\sqrt{2}}V$.

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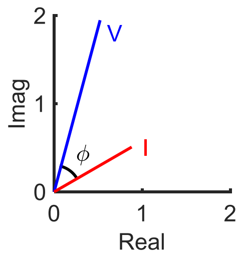
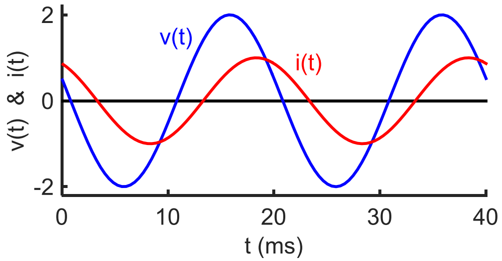
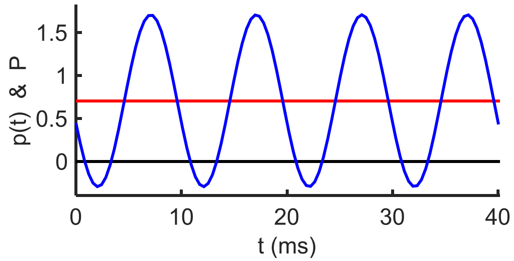


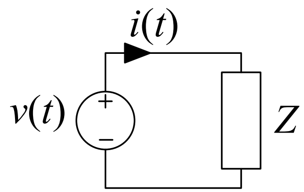




Suppose voltage and current phasors are:

$$V = |V|\, e^{j\theta_V} \quad \Leftrightarrow \quad v(t) = |V| \cos(\omega t + \theta_V)$$

$$I = |I|\, e^{j\theta_I} \quad \Leftrightarrow \quad i(t) = |I| \cos(\omega t + \theta_I)$$



Power dissipated in load Z is

$$\begin{aligned} p(t) = v(t)i(t) &= |V|\,|I| \cos(\omega t + \theta_V)\cos(\omega t + \theta_I) \\ &= |V|\,|I| \left(\tfrac{1}{2}\cos(2\omega t + \theta_V + \theta_I) + \tfrac{1}{2}\cos(\theta_V - \theta_I)\right) \\ &= \tfrac{1}{2}|V|\,|I|\cos(\theta_V - \theta_I) + \tfrac{1}{2}|V|\,|I|\cos(2\omega t + \theta_V + \theta_I) \end{aligned}$$

Average power: $P = \frac{1}{2}|V|\,|I|\cos(\phi)$ where $\phi = \theta_V - \theta_I$

$= \left|\widetilde{V}\right|\left|\widetilde{I}\right|\cos(\phi)$ $\cos\phi$ is the *power factor*

$\phi > 0 \Leftrightarrow$ a *lagging power factor* (normal case: Current lags Voltage)
$\phi < 0 \Leftrightarrow$ a *leading power factor* (rare case: Current leads Voltage)

[Multiplying Phasors]

From the previous slide, if the phasor voltage and current are $V = |V|e^{j\theta_V}$ and $I = |I|e^{j\theta_I}$, then the corresponding waveforms are $v(t) = |V|\cos(\omega t + \theta_V)$ and $i(t) = |I|\cos(\omega t + \theta_I)$. When you multiply these two wavefoms together you get $p(t) = \frac{1}{2}|V|\,|I|\,\cos(\theta_V - \theta_I) + \frac{1}{2}|V|\,|I|\,\cos(2\omega t + \theta_V + \theta_I)$. This product contains two components: a constant, or DC, term that doesn't involve t and a second term that is a cosine wave of frequency 2ω.

The time-average of the second term is zero (because a cosine wave of any non-zero frequency goes symmetrically positive and negative and so averages to zero) and so the average power is just equal to the first term: $\frac{1}{2}|V|\,|I|\,\cos(\theta_V - \theta_I)$. It is easy to see that $V \times I^* = |V|e^{j\theta_V} \times |I|e^{-j\theta_I} = |V|\,|I|\,e^{j(\theta_V - \theta_I)} = |V|\,|I|\,\cos(\theta_V - \theta_I) + j|V|\,|I|\,\sin(\theta_V - \theta_I)$ and so the average power is the real part of $\frac{1}{2}V \times I^*$.

The second term is a cosine wave at a frequency of $2\omega$and so it is possible to represent this waveform, $\frac{1}{2}|V|\,|I|\,\cos(2\omega t + \theta_V + \theta_I)$, as a phasor whose value is $\frac{1}{2}V \times I = \frac{1}{2}|V|\,|I|\,e^{j(\theta_V + \theta_I)}$.

So to sum up, if you multiply together the two sinusoidal waveforms corresponding to phasors V and I, you get two components: (a) a DC component of value $\Re\left(\frac{1}{2}V \times I^*\right)$ and (b) a sinusoidal component of twice the frequency which corresponds to the phasor $\frac{1}{2}V \times I$.

Complex Power

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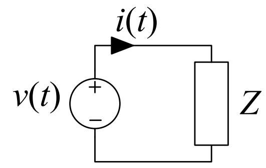
If $\widetilde{V} = \frac{1}{\sqrt{2}} |V| e^{j\theta_V}$ and $\widetilde{I} = \frac{1}{\sqrt{2}} |I| e^{j\theta_I}$

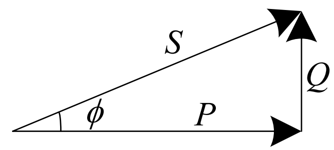


The *complex power* absorbed by Z is $S \triangleq \widetilde{V} \times \widetilde{I}^*$ where * means complex conjugate.

$$\widetilde{V} \times \widetilde{I}^* = \left|\widetilde{V}\right| e^{j\theta_V} \times \left|\widetilde{I}\right| e^{-j\theta_I} = \left|\widetilde{V}\right| \left|\widetilde{I}\right| e^{j(\theta_V - \theta_I)}$$

$$= \left|\widetilde{V}\right| \left|\widetilde{I}\right| e^{j\phi} = \left|\widetilde{V}\right| \left|\widetilde{I}\right| \cos\phi + j \left|\widetilde{V}\right| \left|\widetilde{I}\right| \sin\phi$$

$$= P + jQ$$



Complex Power: $S \triangleq \widetilde{V}\widetilde{I}^* = P + jQ$ measured in Volt-Amps (VA)

Apparent Power: $|S| \triangleq \left|\widetilde{V}\right| \left|\widetilde{I}\right|$ measured in Volt-Amps (VA)

Average Power: $P \triangleq \Re(S)$ measured in Watts (W)

Reactive Power: $Q \triangleq \Im(S)$ Measured in Volt-Amps Reactive (VAR)

Power Factor: $\cos\phi \triangleq \cos\left(\angle\widetilde{V} - \angle\widetilde{I}\right) = \frac{P}{|S|}$

Machines and transformers have capacity limits and power losses that are independent of $\cos\phi$; their ratings are always given in apparent power.

Power Company: Costs $\propto$ apparent power, Revenue $\propto$ average power.

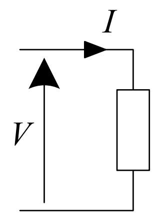
Power in R, L, C

For any impedance, Z, complex power absorbed: $S = \widetilde{V}\widetilde{I}^* = P + jQ$

Using (a) $\widetilde{V} = \widetilde{I}Z$ (b) $\widetilde{I} \times \widetilde{I}^* = \left|\widetilde{I}\right|^2$ we get $S = \left|\widetilde{I}\right|^2 Z = \frac{\left|\widetilde{V}\right|^2}{Z^*}$

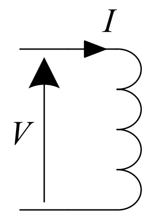
Resistor: $S = \left|\widetilde{I}\right|^2 R = \frac{\left|\widetilde{V}\right|^2}{R}$ $\quad \phi = 0$

Absorbs average power, no VARs $(Q = 0)$



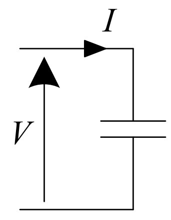
Inductor: $S = j\left|\widetilde{I}\right|^2 \omega L = j\frac{\left|\widetilde{V}\right|^2}{\omega L}$ $\quad \phi = +90^\circ$

No average power, Absorbs VARs $(Q > 0)$



Capacitor: $S = -j\frac{\left|\widetilde{I}\right|^2}{\omega C} = -j\left|\widetilde{V}\right|^2 \omega C$ $\quad \phi = -90^\circ$

No average power, Generates VARs $(Q < 0)$



VARs are generated by capacitors and absorbed by inductors
The phase, ϕ, of the absorbed power, S, equals the phase of Z

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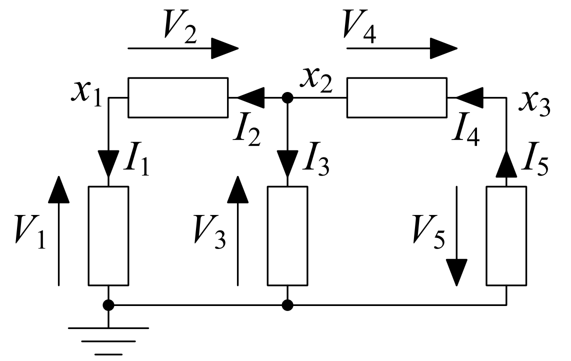
Tellegen's Theorem: The complex power, S, dissipated in any circuit's components sums to zero.

$x_n =$ voltage at node n

$V_b, I_b =$ voltage/current in branch b
(obeying passive sign convention)



$$a_{bn} \triangleq \begin{cases} -1 & \text{if } V_b \text{ starts from node } n \\ +1 & \text{if } V_b \text{ ends at node } n \\ 0 & \text{else} \end{cases}$$

e.g. branch 4 goes from 2 to 3 $\Rightarrow a_{4*} = [0, -1, 1]$

branches (b) \ a_{bn}, nodes (n)	1	2	3
1	1	0	0
2	–1	1	0
3	0	1	0
4	0	–1	1
5	0	0	–1

Branch voltages: $V_b = \sum_n a_{bn} x_n$ (e.g. $V_4 = x_3 - x_2$)

KCL @ node n: $\sum_b a_{bn} I_b = 0 \quad \Rightarrow \quad \sum_b a_{bn} I_b^* = 0$

Tellegen: $\sum_b V_b I_b^* = \sum_b \sum_n a_{bn} x_n I_b^*$

$$= \sum_n \sum_b a_{bn} I_b^* x_n = \sum_n x_n \sum_b a_{bn} I_b^* = \sum_n x_n \times 0$$

Note: $\sum_b S_b = 0 \quad \Rightarrow \quad \sum_b P_b = 0$ and also $\sum_b Q_b = 0$.

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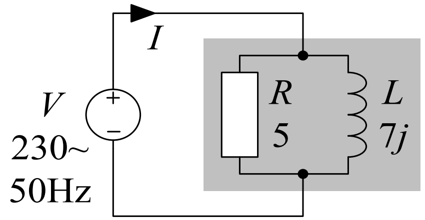
$\widetilde{V} = 230$. Motor modelled as $5||7j\,\Omega$.

$\widetilde{I} = \frac{\widetilde{V}}{R} + \frac{\widetilde{V}}{Z_L} = 46 - j32.9\,\text{A} = 56.5\angle - 36°$

$S = \widetilde{V}\widetilde{I}^* = 10.6 + j7.6\,\text{kVA} = 13\angle 36°\,\text{kVA}$

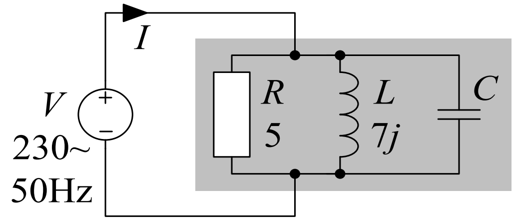
$\cos\phi = \frac{P}{|S|} = \cos 36° = 0.81$



Add parallel capacitor of $300\,\mu$F:

$Z_C = \frac{1}{j\omega C} = -10.6j\,\Omega \Rightarrow \widetilde{I}_C = 21.7j\,\text{A}$

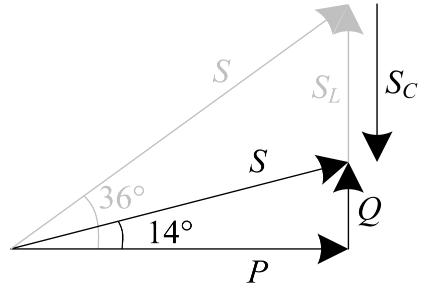
$\widetilde{I} = 46 - j11.2\,\text{A} = 47\angle - 14°\,\text{A}$



$S_C = \widetilde{V}\widetilde{I}_C^* = -j5\,\text{kVA}$

$S = \widetilde{V}\widetilde{I}^* = 10.6 + j2.6\,\text{kVA} = 10.9\angle 14°\,\text{kVA}$

$\cos\phi = \frac{P}{|S|} = \cos 14° = 0.97$



Average power to motor, P, is $10.6\,\text{kW}$ in both cases.

$\left|\widetilde{I}\right|$, reduced from $56.5 \searrow 47\,\text{A}$ (-16%) $\Rightarrow$ lower losses.

Effect of C: VARs $= 7.6 \searrow 2.6\,\text{kVAR}$, power factor $= 0.81 \nearrow 0.97$.

Ideal Transformer

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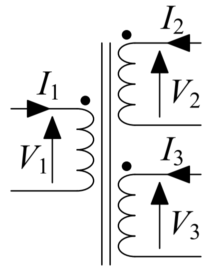
A transformer has ≥ 2 windings on the same magnetic core.

Ampère's law: $\sum N_r I_r = \frac{l\Phi}{\mu A}$; Faraday's law: $\frac{V_r}{N_r} = \frac{d\Phi}{dt}$.
$N_1 : N_2 + N_3$ shows the turns ratio between the windings.
The • indicates the voltage polarity of each winding.

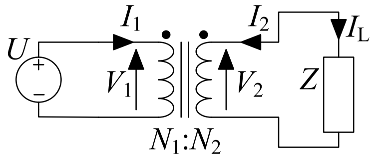


Since Φ is the same for all windings, $\frac{V_1}{N_1} = \frac{V_2}{N_2} = \frac{V_3}{N_3}$.

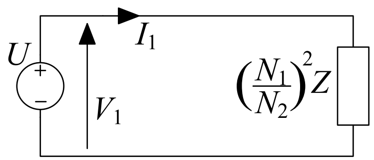
Assume $\mu \to \infty \Rightarrow N_1 I_1 + N_2 I_2 + N_3 I_3 = 0$

These two equations allow you to solve circuits and also imply that $\sum S_i = 0$.

Special Case:



For a 2-winding transformer this simplifies to
$V_2 = \frac{N_2}{N_1} V_1$ and $I_L = -I_2 = \frac{N_1}{N_2} I_1$



Hence $\frac{V_1}{I_1} = \left(\frac{N_1}{N_2}\right)^2 \frac{V_2}{I_L} = \left(\frac{N_1}{N_2}\right)^2 Z$

Equivalent to a *reflected impedance* of $\left(\frac{N_1}{N_2}\right)^2 Z$

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Power Transmission

Suppose a power transmission cable has $1\,\Omega$ resistance.
100 kVA@ 1 kV $= 100$ A $\Rightarrow \widetilde{I}^2 R = 10$ kW losses.
100 kVA@ 100 kV $= 1$ A $\Rightarrow \widetilde{I}^2 R = 1$ W losses.

Voltage Conversion

Electronic equipment requires ≤ 20 V but mains voltage is 240 V $\sim$.

Interference protection

Microphone on long cable is susceptible to interference from nearby mains cables. An $N : 1$ transformer reduces the microphone voltage by N but reduces interference by N^2.

Isolation

There is no electrical connection between the windings of a transformer so circuitry (or people) on one side will not be endangered by a failure that results in high voltages on the other side.

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- Complex Power: $S = \widetilde{V}\widetilde{I}^* = P + jQ$ where $\widetilde{V} = V_{rms} = \frac{1}{\sqrt{2}}V$.
    - For an impedance Z: $S = \left|\widetilde{I}\right|^2 Z = \frac{\left|\widetilde{V}\right|^2}{Z^*}$
    - Apparent Power: $|S| = \left|\widetilde{V}\right|\left|\widetilde{I}\right|$ used for machine ratings.
    - Average Power: $P = \Re(S) = \left|\widetilde{V}\right|\left|\widetilde{I}\right|\cos\phi$ (in Watts)
    - Reactive Power: $Q = \Im(S) = \left|\widetilde{V}\right|\left|\widetilde{I}\right|\sin\phi$ (in VARs)
    - Power engineers *always* use $\widetilde{V}$ and $\widetilde{I}$ and omit the $\tilde{}$.
- Tellegen: In any circuit $\sum_b S_b = 0 \Rightarrow \sum_b P_b = \sum_b Q_b = 0$
- Power Factor Correction: add parallel C to generate extra VARs
- Ideal Transformer: $V_i \propto N_i$ and $\sum N_i I_i = 0$ (implies $\sum S_i = 0$)

For further details see Hayt Ch 11 or Irwin Ch 9.