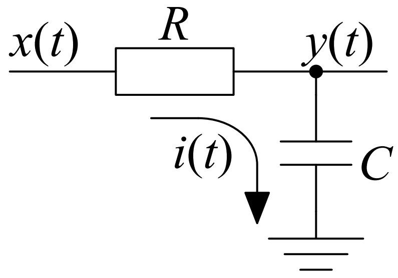
15: Transients (A)

To find $y(t)$:
- $x(t)$ constant: Nodal analysis
- $x(t)$ sinusoidal: Phasors + nodal analysis
- $x(t)$ anything else: Differential equation



$i(t) = C\frac{dy}{dt} = \frac{x-y}{R} \Rightarrow RC\frac{dy}{dt} + y = x$

General Solution: Particular Integral + Complementary Function

Particular Integral: Any solution to $RC\frac{dy}{dt} + y = x$
If $x(t)$ is piecewise constant or sinusoidal, we will use nodal/phasor analysis to find the *steady state solution* for $y(t)$.

Complementary Function: Solution to $RC\frac{dy}{dt} + y = 0$
Does not depend on $x(t)$, only on the circuit.
Solution is $y(t) = Ae^{-t/\tau}$
where $\tau = RC$ is the *time constant* of the circuit.

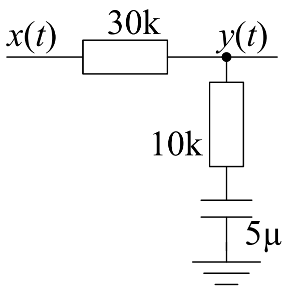
The amplitude, A, is determined by the initial conditions at $t = 0$.

Piecewise steady state inputs

We will consider input signals that are sinusoidal or constant for a particular time interval and then suddenly change in amplitude, phase or frequency.

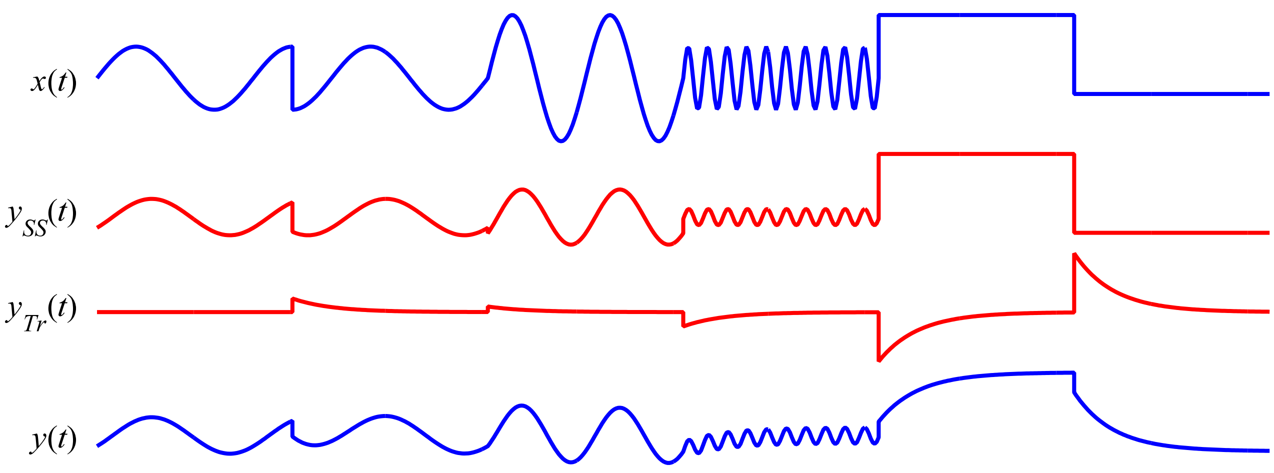
Output is the sum of the steady state and a transient:

$$y(t) = y_{SS}(t) + y_{Tr}(t)$$

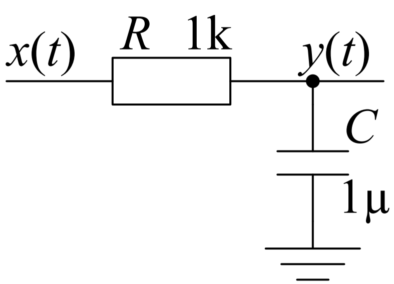


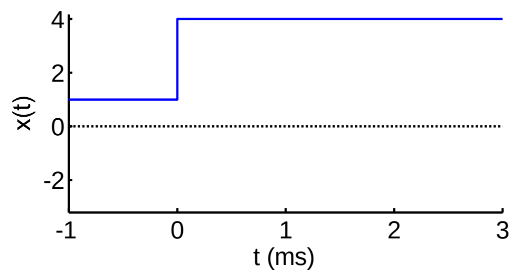
Steady state, $y_{SS}(t)$, is the same frequency as the input; use phasors + nodal analysis.

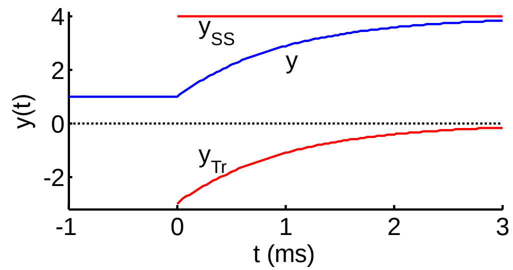
Transient is always $y_{Tr}(t) = Ae^{-\frac{t}{\tau}}$ at each change.



For $t < 0$, $y(t) = x(t) = 1$
For $t \geq 0$, $RC\frac{dy}{dt} + y = x = 4$
Time Const: $\tau = RC = 1\,\text{ms}$







Steady State (Particular Integral)
$y_{SS}(t) = x(t) = 4$ for $t \geq 0$

Transient (Complementary Function)
$y_{Tr}(t) = Ae^{-t/\tau}$

Steady State + Transient
$y(t) = y_{SS} + y_{Tr} = 4 + Ae^{-t/\tau}$

To find A, use capacitor property:
Capacitor voltage never changes abruptly

$y(0+) = 4 + A$ and $y(0-) = 1 \Rightarrow 4 + A = 1 \Rightarrow A = -3$

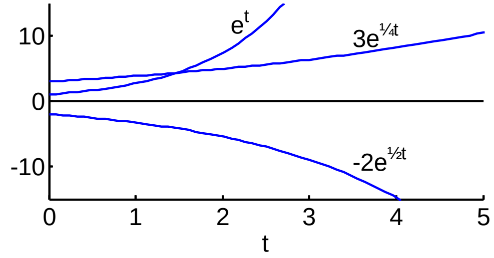
So transient: $y_{Tr}(t) = -3e^{-t/\tau}$ and total $y(t) = 4 - 3e^{-t/\tau}$

Transient amplitude $\Leftarrow$ capacitor voltage continuity: $v_C(0+) = v_C(0-)$

Negative exponentials

Positive exponentials grow to $\pm\infty$:

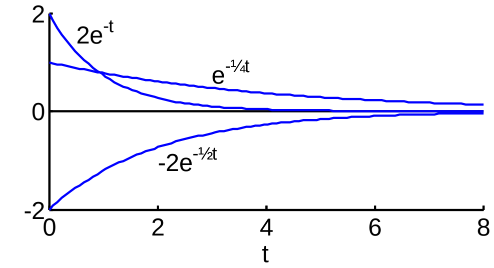
e^{t}, $3e^{t/4}$, $-2e^{t/2}$



Negative exponentials decay to 0:

$2e^{-t}$, $e^{-t/4}$, $-2e^{-t/2}$

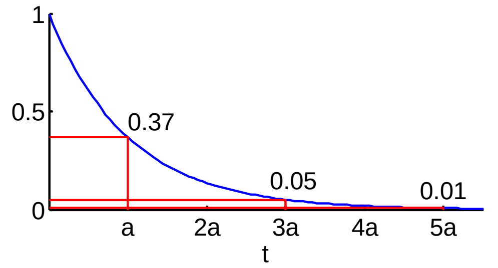
Transients are negative exponentials.



Decay rate of $e^{-t/a}$

37% after 1 time constant

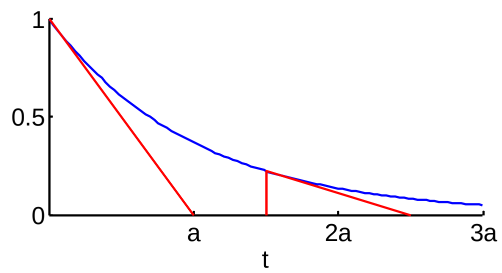
5% after 3, <1% after 5



Gradient of $e^{-t/a}$

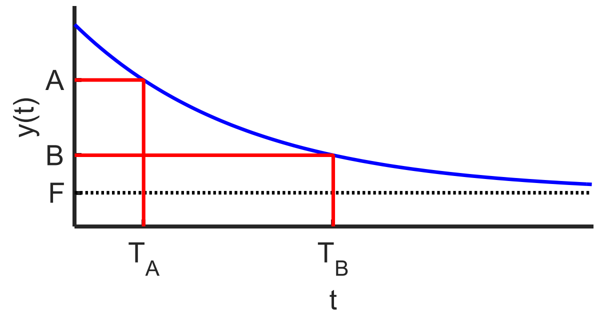
Gradient at t hits zero at $t+a$.

True for any t.



Negative exponential with a final value of F.

$$y(t) = F + (A - F)\, e^{-(t-T_A)/\tau}$$



How long does it take to go from A to B ?

At $t = T_B$:

$$y(T_B) = B = F + (A - F)\, e^{-(T_B - T_A)/\tau}$$

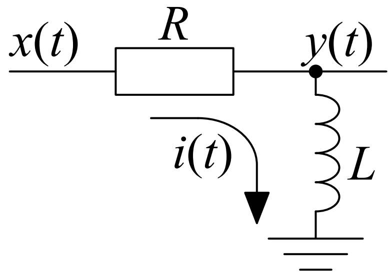
$$\frac{B-F}{A-F} = e^{-(T_B - T_A)/\tau}$$

Hence $T_B - T_A = \tau \ln\left(\frac{A-F}{B-F}\right) = \tau \ln\left(\frac{\text{initial distance to } F}{\text{final distance to } F}\right)$

Useful formula - worth remembering.

Inductor Transients

We know $i = \frac{x-y}{R}$
$y(t) = L\frac{di}{dt} = \frac{L}{R} \times \frac{d(x-y)}{dt} = \frac{L}{R}\frac{dx}{dt} - \frac{L}{R}\frac{dy}{dt}$
$\Rightarrow \frac{L}{R}\frac{dy}{dt} + y = \frac{L}{R}\frac{dx}{dt}$



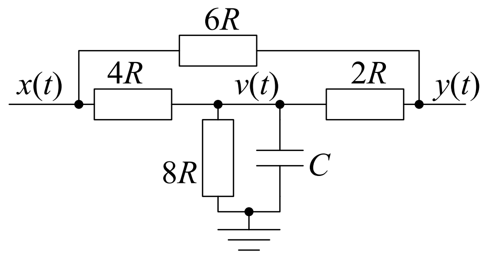
Solution: Particular Integral + Complementary Function

Particular Integral: Any solution to $\frac{L}{R}\frac{dy}{dt} + y = \frac{L}{R}\frac{dx}{dt}$
If $x(t)$ is piecewise constant or sinusoidal, we will use nodal/phasor analysis to find the *steady state solution*, $y_{SS}(t)$.

Complementary Function: Solution to $\frac{L}{R}\frac{dy}{dt} + y = 0$
Does not depend on $x(t)$, only on the circuit.
Solution is $y_{Tr}(t) = Ae^{-t/\tau}$
where $\tau = \frac{L}{R}$ is the *time constant* of the circuit.

1st order transient is *always* $y_{Tr}(t) = Ae^{-t/\tau}$ where $\tau = RC$ or $\frac{L}{R}$
Amplitude $A \Leftarrow$ no abrupt change in capacitor voltage or inductor current.

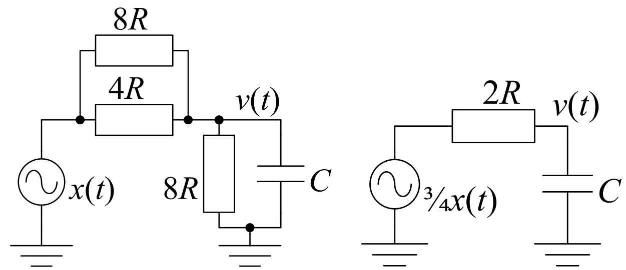
1st order circuit has only one C or L.
Make a Thévenin equivalent of the network connected to the terminals of C. Remember x is a voltage source but y is not.



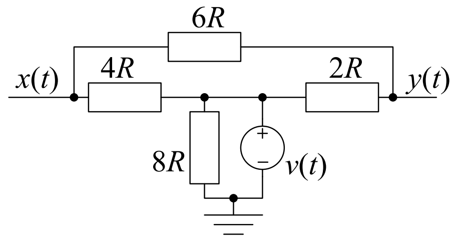
Now $v(t) = v_{SS}(t) + v_{Tr}(t)$

$= v_{SS}(t) + Ae^{-t/\tau}$

Time constant is $\tau = R_{Th}C$
where R_{Th} is the Thévenin resistance.



Replace the capacitor with a voltage source $v(t)$; all voltages and currents in the circuit will remain unchanged.



Linearity: $y = ax + bv = ax + bv_{SS} + bv_{Tr} = y_{SS} + bv_{Tr}$

All voltages and currents in a circuit have the same transient (but scaled).

The *circuit's time constant* is $\tau = R_{Th}C$ or $\frac{L}{R_{Th}}$ where R_{Th} is the Thévenin resistance of the network connected to C or L.

Find Steady State (DC $\Rightarrow Z_L = 0$)

Potential divider: $y_{SS} = \frac{1}{2}x$

$y_{SS}(0-) = 1$, $y_{SS}(0+) = 3$

Inductor Current Continuity

$i_{SS}(0-) = 1\,\text{mA} \Rightarrow i_L(0+) = 1\,\text{mA}$

At $t = 0+$

$x - y = 1\,\text{mA} \times 1\,\text{k} = 1$

$y(0+) = x(0+) - 1 = 5$

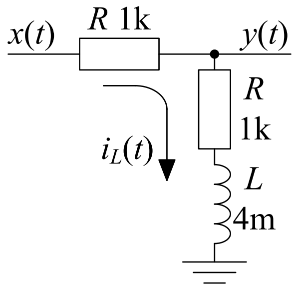
Time Constant

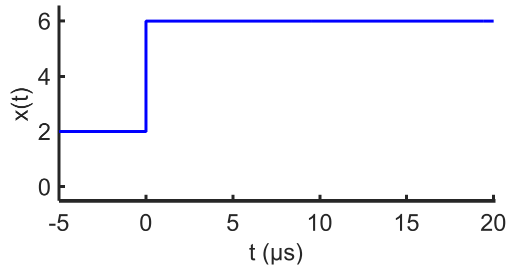
Set $x \equiv 0 \rightarrow R_{Th} = 2\,\text{k}$

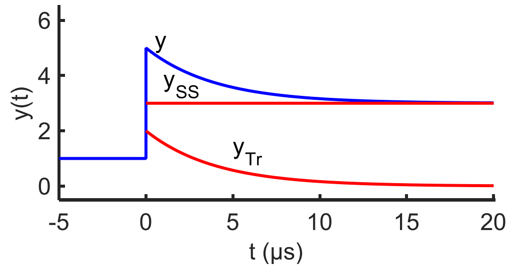
$\tau = \frac{L}{R_{Th}} = 2\,\mu\text{s}$

Result

$$y = y_{SS} + \left(y\left(0+\right) - y_{SS}\left(0+\right)\right) e^{-t/\tau}$$
$$= 3 + (5-3)\, e^{-t/\tau}$$
$$= 3 + 2e^{-t/\tau}$$







Capacitor Voltage Continuity

Find Steady State (DC $\Rightarrow Z_C = \infty$)

KCL @ V: $\frac{v-x}{4R} + \frac{v}{8R} + \frac{v-y}{2R} = 0$

KCL @ Y: $\frac{y-v}{2R} + \frac{y-x}{6R} = 0$

$v_{SS} = \frac{3}{4}x,\ y_{SS} = \frac{13}{16}x$

Capacitor Voltage Continuity

$v_{SS}(0-) = -3 \Rightarrow v(0+) = -3$

At $t = 0+$: $x = 4$ and $v = -3$

KCL @ Y: $\frac{y-(-3)}{2R} + \frac{y-4}{6R} = 0$

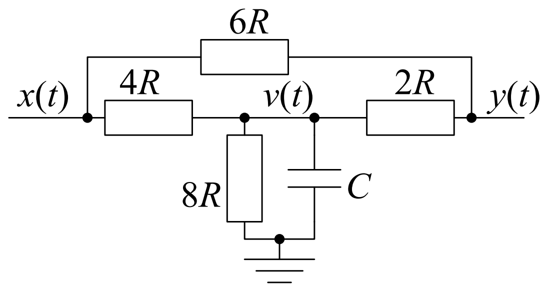
$y(0+) = \frac{-9+4}{4} = -\frac{5}{4}$

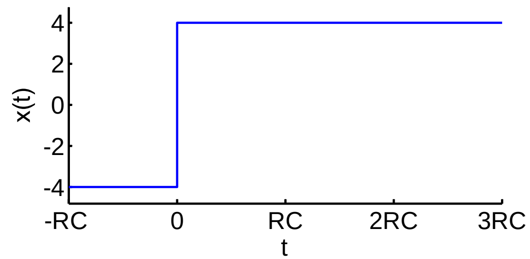
Time Constant

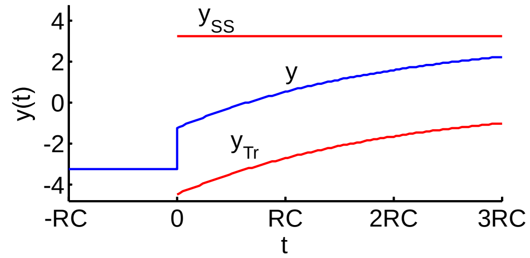
$\tau = R_{Th}C = 2RC$ (from earlier slide)

Result

$$y = y_{SS} + \left(y\left(0+\right) - y_{SS}\left(0+\right)\right)e^{-t/\tau}$$
$$= \frac{13}{4} + \left(-\frac{5}{4} - \frac{13}{4}\right)e^{-t/\tau}$$
$$= \frac{13}{4} - \frac{18}{4}e^{-t/\tau} = 3\frac{1}{4} - 4\frac{1}{2}e^{-t/2RC}$$







Summary

- 1st order circuits: include one C or one L.
  - v_C or i_L never change abruptly. The output, y, is not necessarily continuous unless it equals v_C.
- Circuit time constant: $\tau = R_{Th}C$ or $\frac{L}{R_{Th}}$
  - R_{Th} is the Thévenin resistance seen by C or L.
  - Same τ for all voltages and currents.
- Output = Steady State + Transient
  - Steady State: use nodal/Phasor analysis when input is piecewise constant or piecewise sinusoidal. The steady state has the same frequency as the input signal.
  - Transient: Find $v_C(0-)$ or $i_L(0-)$: unchanged at $t = 0+$
    Find $y(0+)$ assuming source of $v_C(0+)$ or $i_L(0+)$
    Amplitude never complex, never depends on t.
  - $y(t) = y_{SS}(t) + (y(0+) - y_{SS}(0+))\, e^{-t/\tau}$

See Hayt Ch 8 or Irwin Ch 7.