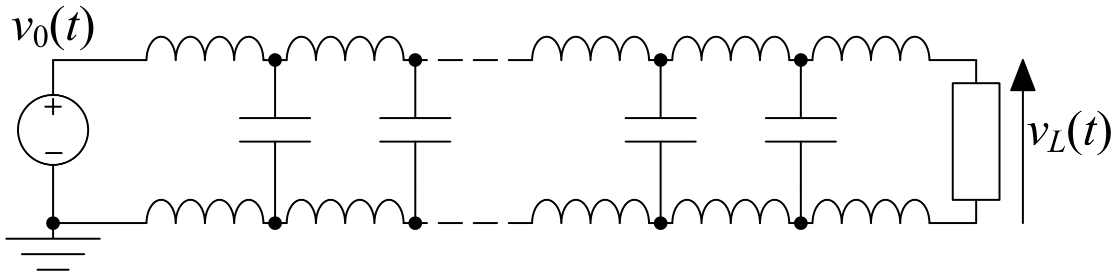
17: Transmission Lines



Previously assume that any change in $v_0(t)$ appears instantly at $v_L(t)$.

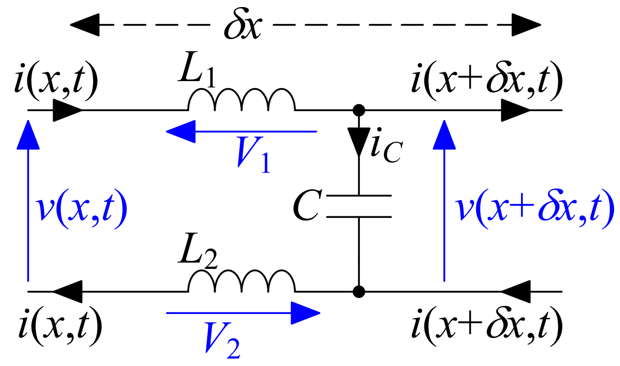
This is not true.

If fact signals travel at around half the speed of light ($c = 30$ cm/ns).

Reason: all wires have capacitance to ground and to neighbouring conductors and also self-inductance. It takes time to change the current through an inductor or voltage across a capacitor.

A *transmission line* is a wire with a uniform goemetry along its length: the capacitance and inductance of any segment is proportional to its length.
We represent as a large number of small inductors and capacitors spaced along the line.
The signal speed along a transmisison line is predictable.

A short section of line δx long:



$v(x,t)$ and $i(x,t)$ depend on both position and time.

Small $\delta x \Rightarrow$ ignore 2nd order derivatives:

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial v(x+\delta x,t)}{\partial t} \triangleq \frac{\partial v}{\partial t}.$$

Basic Equations

KVL: $v(x,t) = V_2 + v(x+\delta x,t) + V_1$

KCL: $i(x,t) = i_C + i(x+\delta x,t)$

Capacitor equation: $C\frac{\partial v}{\partial t} = i_C = i(x,t) - i(x+\delta x,t) = -\frac{\partial i}{\partial x}\delta x$

Inductor equation (L_1 and L_2 have the same current):

$$(L_1 + L_2)\frac{\partial i}{\partial t} = V_1 + V_2 = v(x,t) - v(x+\delta x,t) = -\frac{\partial v}{\partial x}\delta x$$

Transmission Line Equations

$$C_0\frac{\partial v}{\partial t} = -\frac{\partial i}{\partial x}$$
$$L_0\frac{\partial i}{\partial t} = -\frac{\partial v}{\partial x}$$

where $C_0 = \frac{C}{\delta x}$ is the capacitance per unit length (Farads/m) and $L_0 = \frac{L_1+L_2}{\delta x}$ is the total inductance per unit length (Henries/m).

[Partial Derivatives]

When we differentiate a function of two variables, we keep one of the variables fixed while differentiating with respect to the other; this is called a partial derivative and is written with a curly version of the letter "d". Thus

$$\frac{\partial v}{\partial x} \triangleq \lim_{\delta x \to 0} \frac{v(x+\delta x, t) - v(x,t)}{\delta x} \quad \text{and} \quad \frac{\partial v}{\partial t} \triangleq \lim_{\delta t \to 0} \frac{v(x, t+\delta t) - v(x,t)}{\delta t}.$$

Higher order derivatives may be obtained by differentiating the partial derivatives again to give

$$\frac{\partial^2 v}{\partial x^2} \triangleq \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right), \quad \frac{\partial^2 v}{\partial t^2} \triangleq \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial t}\right) \quad \text{and} \quad \frac{\partial^2 v}{\partial x \partial t} \triangleq \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right).$$

Provided the second order partial derivatives are continuous, the order of differentiation doesn't matter so that $\frac{\partial^2 v}{\partial x \partial t} = \frac{\partial^2 v}{\partial t \partial x}$.

If we take the normal Taylor series with respect to x, $v(x+\delta x,\, t) = v(x,t) + \frac{\partial v(x,t)}{\partial x}\delta x + O\left(\delta x^2\right)$, and differentiate each term with respect to t, we get

$$\frac{\partial v(x+\delta x,\, t)}{\partial t} = \frac{\partial v(x,t)}{\partial t} + \frac{\partial^2 v(x,t)}{\partial t \partial x}\delta x + O\left(\delta x^2\right).$$

If $\delta x \to 0$, then we get $\frac{\partial v(x+\delta x,\, t)}{\partial t} \to \frac{\partial v(x,t)}{\partial t}$ as assumed on the previous slide.

[Deriving the Transmission Line Equations]

This note provides slightly more detail about how we derive the transmission line equations. By expanding $v(x+\delta x, t)$ and $i(x+\delta x, t)$ as Taylor Series in x, we can write

$$v(x+\delta x, t) = v(x,t) + \delta x \frac{\partial v}{\partial x}(x,t) + O(\delta x^2) \quad \text{and} \quad i(x+\delta x, t) = i(x,t) + \delta x \frac{\partial i}{\partial x}(x,t) + O(\delta x^2).$$

From the diagram on the previous page, the voltage across the capacitor is $v(x+\delta x, t)$ and so the capacitor equation is

$$C\frac{\partial v}{\partial t}(x+\delta x, t) = i(x,t) - i(x+\delta x, t).$$

Substituting in the Taylor series expansions for $v(x+\delta x, t)$ and $i(x+\delta x, t)$ and also substituting $C = C_0 \delta x$ results in

$$\begin{aligned} C_0 \delta x \left(\frac{\partial v}{\partial t}(x,t) + \delta x \frac{\partial^2 v}{\partial x \partial t}(x,t) + O(\delta x^2) \right) &= -\delta x \frac{\partial i}{\partial x}(x,t) - O(\delta x^2) \\ \Rightarrow \quad C_0 \left(\frac{\partial v}{\partial t}(x,t) + \delta x \frac{\partial^2 v}{\partial x \partial t}(x,t) + O(\delta x^2) \right) &= -\frac{\partial i}{\partial x}(x,t) - O(\delta x). \end{aligned}$$

Finally, we let $\delta x \to 0$ and so all the terms that are $O(\delta x)$ or smaller disappear which leaves

$$C_0 \frac{\partial v}{\partial t}(x,t) = -\frac{\partial i}{\partial x}(x,t).$$

The inductor equation, $L_0 \frac{\partial i}{\partial t} = -\frac{\partial v}{\partial x}$, can be derived in a similar way.

Transmission Line Equations: $C_0\frac{\partial v}{\partial t} = -\frac{\partial i}{\partial x}$ $\quad L_0\frac{\partial i}{\partial t} = -\frac{\partial v}{\partial x}$

General solution:

$$v(t,x) = f(t - \tfrac{x}{u}) + g(t + \tfrac{x}{u})$$

$$i(t,x) = \frac{f(t-\frac{x}{u}) - g(t+\frac{x}{u})}{Z_0}$$

where $u = \sqrt{\frac{1}{L_0C_0}}$ and $Z_0 = \sqrt{\frac{L_0}{C_0}}$.

u is the *propagation velocity* and Z_0 is the *characteristic impedance*.

$f()$ and $g()$ can be *any* differentiable functions.

Verify by substitution:

$$-\frac{\partial i}{\partial x} = -\left(\frac{-f'(t-\frac{x}{u}) - g'(t+\frac{x}{u})}{Z_0} \times \frac{1}{u}\right)$$

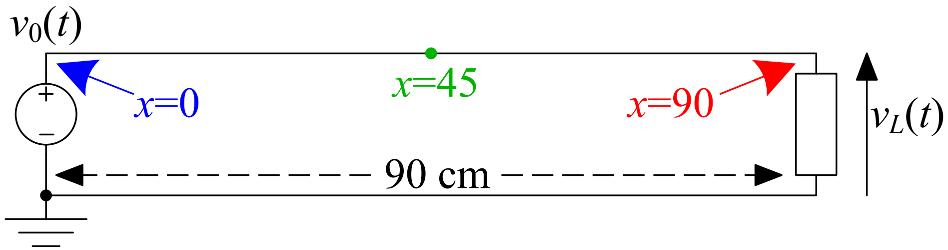
$$= C_0\left(f'(t-\tfrac{x}{u}) + g'(t+\tfrac{x}{u})\right) = C_0\frac{\partial v}{\partial t}$$

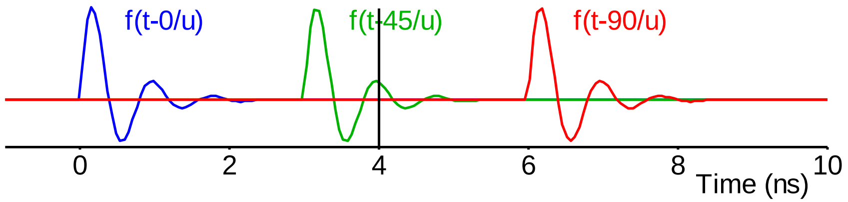
Suppose:

$u = 15\,\mathrm{cm/ns}$

and $g(t) \equiv 0$

$\Rightarrow v(x,t) = f\left(t - \frac{x}{u}\right)$



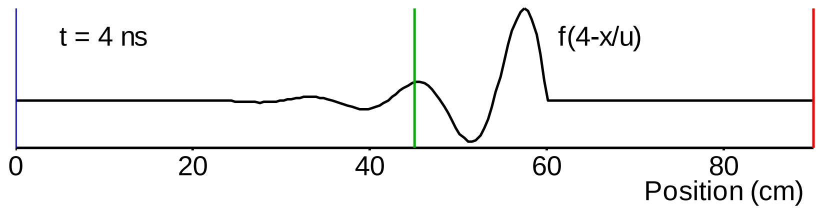


- At $x = 0$ cm [▲], $v_S(t) = f(t - \frac{0}{u})$
- At $x = 45$ cm [▲], $v(45, t) = f(t - \frac{45}{u})$
  $f(t - \frac{45}{u})$ is exactly the same as $f(t)$ but delayed by $\frac{45}{u} = 3$ ns.
- At $x = 90$ cm [▲], $v_R(t) = f(t - \frac{90}{u})$; now delayed by $6\,\mathrm{ns}$.

Waveform at $x = 0$ completely determines the waveform everywhere else.

Snapshot at $t_0 = 4\,\mathrm{ns}$:

the waveform has just arrived at the point $x = ut_0 = 60$ cm.

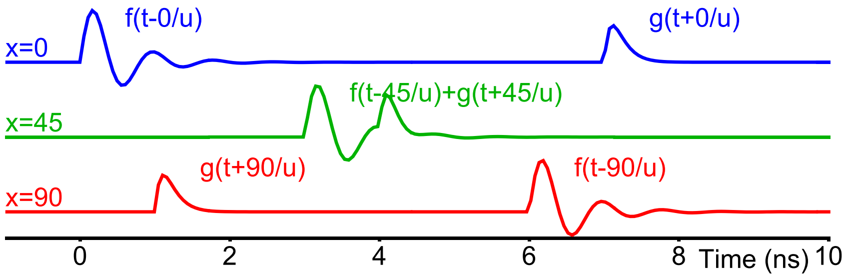


$f(t - \frac{x}{u})$ is a wave travelling forward (i.e. towards +x) along the line.

Similarly $g(t+\frac{x}{u})$ is a wave travelling backwards, i.e. in the $-x$ direction.

$v(x,t) =$
$f(t-\frac{x}{u}) + g(t+\frac{x}{u})$

At $x = 0$ cm [▲],
$v_S(t) = f(t) + g(t)$

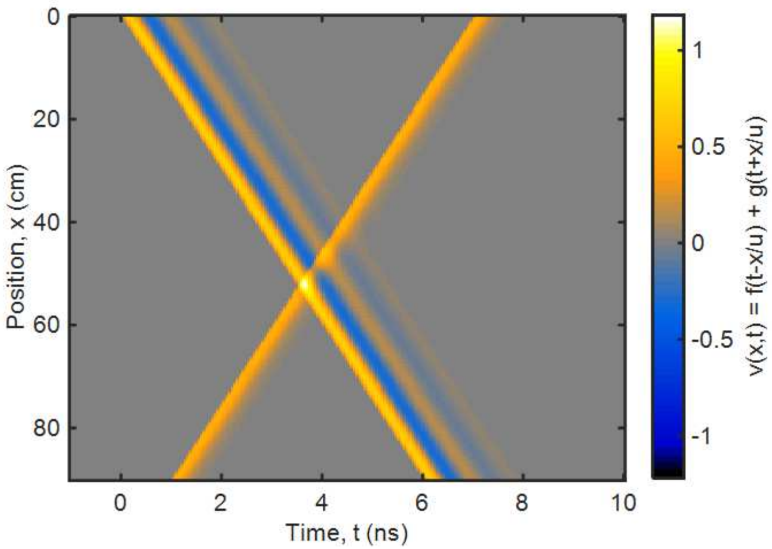


At $x = 45$ cm [▲], g is only 1 ns behind f and they add together.
At $x = 90$ cm [▲], g starts at $t = 1$ and f starts at $t = 6$.

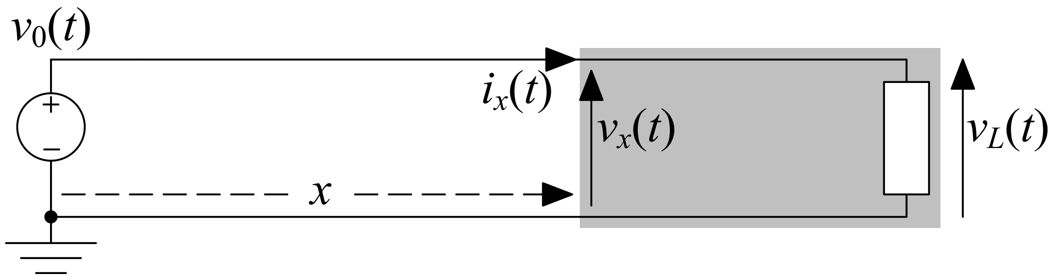
A vertical line on the diagram gives a snapshot of the entire line at a time instant t.

f and g first meet at $t = 3.5$ and $x = 52.5$.

Magically, f and g pass through each other entirely unaltered.



Define $f_x(t) = f\left(t - \frac{x}{u}\right)$ and $g_x(t) = g\left(t + \frac{x}{u}\right)$ to be the forward and backward waveforms at any point, x.



i is always measured in the $+$ve x direction.

Then $\quad v_x(t) = f_x(t) + g_x(t) \quad$ and $\quad i_x(t) = Z_0^{-1}\left(f_x(t) - g_x(t)\right)$.

Note: Knowing the waveform $f_x(t)$ or $g_x(t)$ at any position x, tells you it at all other positions: $f_y(t) = f_x\left(t - \frac{y-x}{u}\right)$ and $g_y(t) = g_x\left(t + \frac{y-x}{u}\right)$.

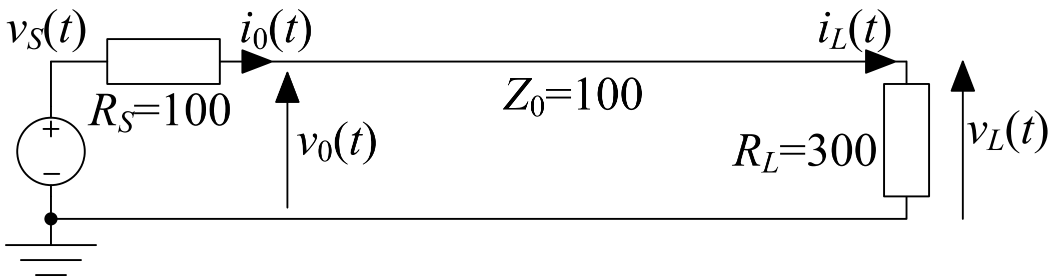
Power Flow

The power transferred into the shaded region across the boundary at x is

$$P_x(t) = v_x(t) i_x(t) = Z_0^{-1}\left(f_x(t) + g_x(t)\right)\left(f_x(t) - g_x(t)\right) = \frac{f_x^2(t)}{Z_0} - \frac{g_x^2(t)}{Z_0}$$

f_x carries power into shaded area and g_x carries power out independently.
Power travels in the same direction as the wave.
The same power as would be absorbed by a [ficticious] resistor of value Z_0.

Reflections



$$v_x = f_x + g_x$$
$$i_x = Z_0^{-1}\left(f_x - g_x\right)$$

From Ohm's law at $x = L$, we have $v_L(t) = i_L(t)R_L$

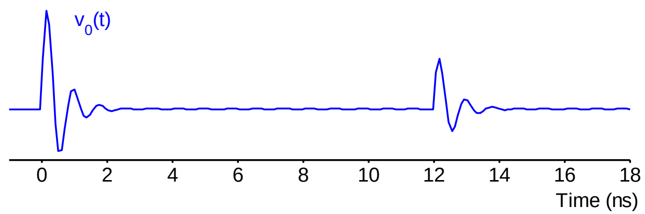
Hence $\left(f_L(t) + g_L(t)\right) = Z_0^{-1}\left(f_L(t) - g_L(t)\right)R_L$

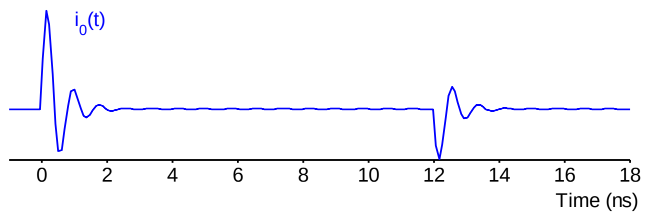
From this: $g_L(t) = \frac{R_L - Z_0}{R_L + Z_0} \times f_L(t)$

We define the *reflection coefficient*: $\rho_L = \frac{g_L(t)}{f_L(t)} = \frac{R_L - Z_0}{R_L + Z_0} = +0.5$

Substituting $g_L(t) = \rho_L f_L(t)$ gives

$$v_L(t) = (1 + \rho_L)\, f_L(t) \text{ and } i_L(t) = (1 - \rho_L)\, Z_0^{-1} f_L(t)$$





At source end: $g_0(t) = \rho_L f_0\left(t - \frac{2L}{u}\right)$ i.e. delayed by $\frac{2L}{u} = 12$ ns.

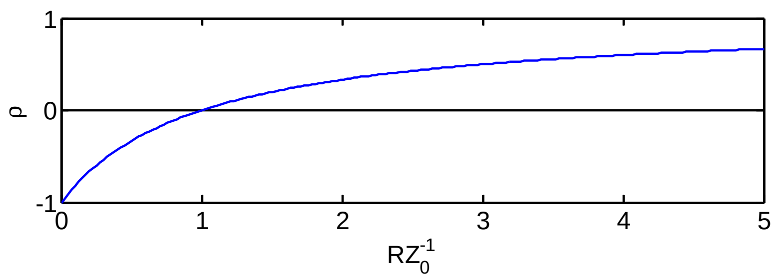
Note that the reflected current has been multiplied by $-\rho$.

Reflection Coefficients

$$\rho = \frac{R-Z_0}{R+Z_0} = \frac{\frac{R}{Z_0}-1}{\frac{R}{Z_0}+1}$$

$$\frac{v_L(t)}{f(t)} = 1+\rho$$

$$\frac{i_L(t)Z_0}{f(t)} = 1-\rho$$

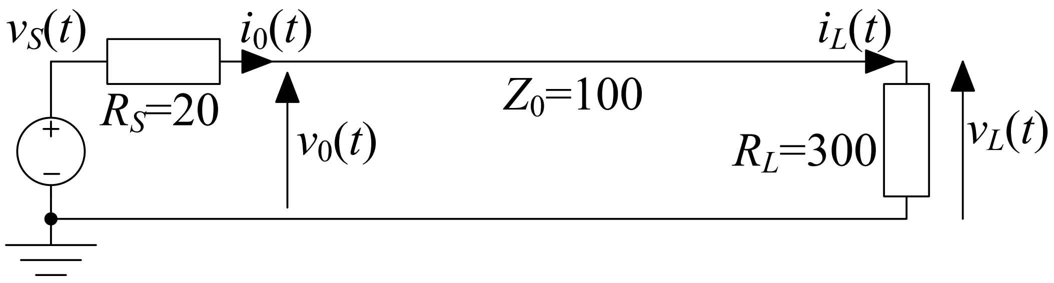


ρ depends on the ratio $\frac{R}{Z_0}$.

$\frac{R}{Z_0}$	ρ	$\frac{v_L(t)}{f(t)}$	$\frac{i_L(t)Z_0}{f(t)}$	Comment
∞	$+1$	2	0	Open circuit: $v_L = 2f$, $i_L \equiv 0$
3	$+0.5$	1.5	0.5	$R > Z_0 \Rightarrow \rho > 0$
1	0	1	1	Matched: No reflection at all
$\frac{1}{3}$	-0.5	0.5	1.5	$R < Z_0 \Rightarrow \rho < 0$
0	-1	0	2	Short circuit: $v_L \equiv 0$, $i_L = \frac{2f}{Z_0}$

Note: Reverse mapping is $R = \frac{v_L}{i_L} = \frac{1+\rho}{1-\rho} \times Z_0$

Remember: $\rho \in \{-1, +1\}$ and increases with R.



$$v_x = f_x + g_x$$
$$i_x = \frac{f_x - g_x}{Z_0}$$

From Ohm's law at $x = 0$, we have $v_0(t) = v_S(t) - i_0(t)R_S$ where R_S is the Thévenin resistance of the voltage source.

Substituting $v_0(t) = f_0 + g_0$ and $i_0(t) = \frac{f_0 - g_0}{Z_0}$ leads to:

$$f_0(t) = \frac{Z_0}{R_S + Z_0} v_S(t) + \frac{R_S - Z_0}{R_S + Z_0} g_0(t) \triangleq \tau_0 v_S(t) + \rho_0 g_0(t)$$

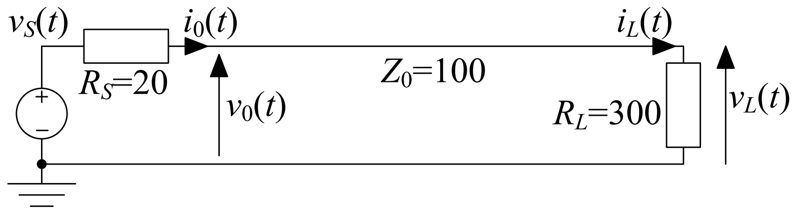
So $f_0(t)$ is the superposition of two terms:

(1) Input $v_S(t)$ multiplied by $\tau_0 = \frac{Z_0}{R_S + Z_0}$ which is the same as a potential divider if you replace the line with a [ficticious] resistor Z_0.

(2) The incoming backward wave, $g_0(t)$, multiplied by a reflection coefficient: $\rho_0 = \frac{R_S - Z_0}{R_S + Z_0}$.

For $R_S = 20$: $\tau_0 = \frac{100}{20+100} = 0.83$ and $\rho_0 = \frac{20-100}{20+100} = -0.67$.

Multiple Reflections



$$\rho_0 = -\frac{2}{3}$$
$$\rho_L = \frac{1}{2}$$
$$v_x = f_x + g_x$$

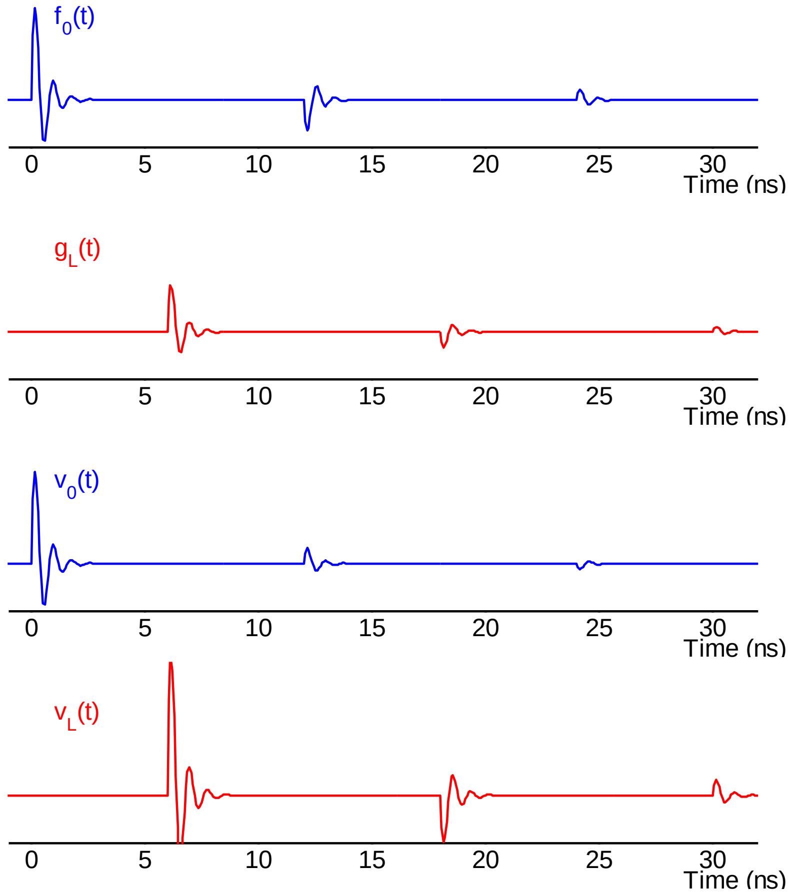
Each extra bit of f_0 is delayed by $\frac{2L}{u}$ $(=12\,\text{ns})$ and multiplied by $\rho_L \rho_0$:

$$f_0(t) = \sum_{i=0}^{\infty} \tau_0 \rho_L^i \rho_0^i v_S\left(t - \frac{2Li}{u}\right)$$

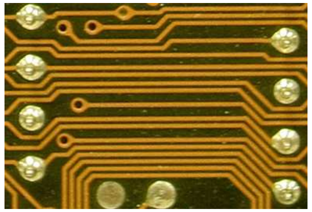
$$g_L(t) = \rho_L f_0\left(t - \frac{L}{u}\right)$$

$$v_0(t) = f_0(t) + g_L\left(t - \frac{L}{u}\right)$$

$$v_L(t) = f_0\left(t - \frac{L}{u}\right) + g_L(t)$$

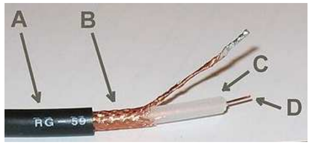


Integrated circuits & Printed circuit boards

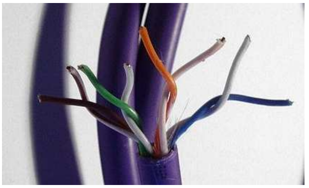


High speed digital or high frequency analog interconnections
$Z_0 \approx 100\,\Omega$, $u \approx 15\,\text{cm/ns}$.

Long Cables



Coaxial cable ("coax"): unaffacted by external fields; use for antennae and instrumentation.
$Z_0 = 50$ or $75\,\Omega$, $u \approx 25\,\text{cm/ns}$.



Twisted Pairs: cheaper and thinner than coax and resistant to magnetic fields; use for computer network and telephone cabling. $Z_0 \approx 100\,\Omega$, $u \approx 19\,\text{cm/ns}$.

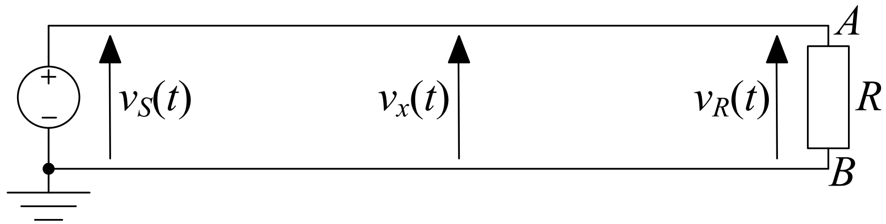
When do you have to bother?

Answer: long cables or high frequencies. You can completely ignore transmission line effects if $\text{length} \ll \frac{u}{\text{frequency}} = \text{wavelength}$.

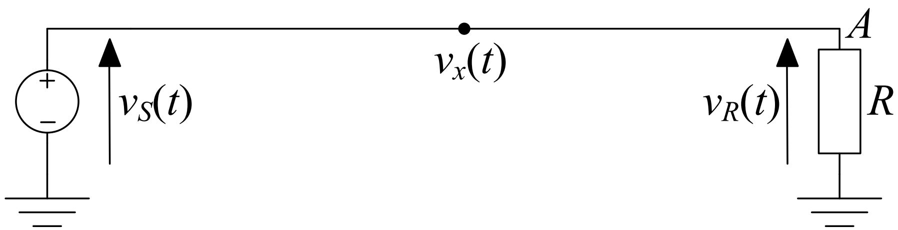
- Audio ($< 20\,\text{kHz}$) never matters.
- Computers ($1\,\text{GHz}$) usually matters.
- Radio/TV usually matters.

[Transmission Line Grounds]

For long coaxial or twisted pair cables, the "ground" wire has significant inductance and so its two ends are not necessarily at the same voltage. This means that $v_x(t)$, $f_x(t)$ and $g_x(t)$ are measured relative to the "ground" at position x as shown. It follows that potential differences like $v_R(t) = v_A(t) - v_B(t)$ make sense but talking about $v_A(t)$ on its own is meaningless.



Integrated circuits and printed circuit boards normally have a low impedance "ground plane" covering the entire circuit; in a multilayer printed circuit board this typically forms one entire layer. In this case we have a single ground reference for the whole circuit and it now makes sense to talk about the voltage "at" a node and to say $v_R(t) = v_A(t)$.



- Signals travel at around $u \approx \frac{1}{2}c = 15\,\text{cm/ns}$.
  Only matters for high frequencies or long cables.
- Forward and backward waves travel along the line:
  $$f_x(t) = f_0\left(t - \frac{x}{u}\right) \quad \text{and} \quad g_x(t) = g_0\left(t + \frac{x}{u}\right)$$
  - Knowing f_x and g_x at any single x position tells you everything
- Voltage and current are: $v_x = f_x + g_x$ and $i_x = \frac{f_x - g_x}{Z_0}$
- Terminating line with R at $x = L$ links the forward and backward waves:
  - backward wave is $g_L = \rho_L f_L$ where $\rho_L = \frac{R-Z_0}{R+Z_0}$
  - the reflection coefficient, $\rho_L \in \{-1, +1\}$ and increases with R
  - $R = Z_0$ avoids reflections: *matched* termination.
  - Reflections go on for ever unless one or both ends are matched.
  - f is infinite sum of copies of the input signal delayed successively by the round-trip delay, $\frac{2L}{u}$, and multiplied by $\rho_L \rho_0$.