18: Phasors and Transmission Lines

Phasors and transmision lines

For a transmission line: $v(t,x) = f\left(t - \frac{x}{u}\right) + g\left(t + \frac{x}{u}\right)$ and
$i(t,x) = Z_0^{-1}\left(f(t - \frac{x}{u}) - g(t + \frac{x}{u})\right)$

We can use phasors to eliminate t from the equations if $f()$ and $g()$ are sinusoidal with the same ω: $f(t) = A\cos(\omega t + \phi) \Rightarrow F = Ae^{j\phi}$.

Then $f_x(t) = f(t - \frac{x}{u}) = A\cos\left(\omega\left(t - \frac{x}{u}\right) + \phi\right)$

$$\Rightarrow F_x = Ae^{j\left(-\frac{\omega}{u}x + \phi\right)} = Ae^{j\phi}e^{-j\frac{\omega}{u}x} = F_0 e^{-jkx}$$

where the *wavenumber* is $k \triangleq \frac{\omega}{u}$.
Units: ω is "radians per second", k is "radians per metre" (note $k \propto \omega$).

Similarly $G_x = G_0 e^{+jkx}$.

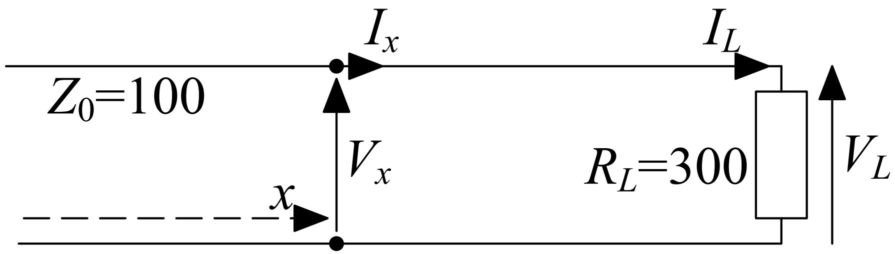
Everything is time-invariant: phasors do not depend on t.

Nice things about sine waves:
(1) a time delay is just a phase shift
(2) sum of delayed sine waves is another sine wave

Phasor Relationships

Time Domain	Phasor	Notes
$f(t) = A\cos(\omega t + \phi)$	$F = Ae^{j\phi}$	F indep of t
$f_x(t) = f\left(t - \frac{x}{u}\right)$ $= A\cos\left(\omega t + \phi - \frac{\omega}{u}x\right)$	$F_x = Ae^{j\left(\phi - \frac{\omega}{u}x\right)}$ $= Fe^{-jkx}$	$\|F_x\| \equiv \|F\|$ indep of x
$f_y(t) = f_x\left(t - \frac{(y-x)}{u}\right)$	$F_y = F_x e^{-jk(y-x)}$	Delayed by $\frac{y-x}{u}$
$g_y(t) = g_x\left(t + \frac{(y-x)}{u}\right)$	$G_y = G_x e^{+jk(y-x)}$	Advanced by $\frac{y-x}{u}$
$v_x(t) = f_x(t) + g_x(t)$	$V_x = F_x + G_x$	
$i_x(t) = \frac{f_x(t) - g_x(t)}{Z_0}$	$I_x = \frac{F_x - G_x}{Z_0}$	

Phasor Reflection



Phasors obey Ohm's law: $\frac{V_L}{I_L} = R_L = \frac{F_L + G_L}{Z_0^{-1}(F_L - G_L)}$

So $G_L = \rho_L F_L$ where $\rho_L = \frac{R_L - Z_0}{R_L + Z_0}$

At any x, $\frac{G_x}{F_x} = \frac{G_L e^{-jk(L-x)}}{F_L e^{+jk(L-x)}} = \rho_L e^{-2jk(L-x)}$

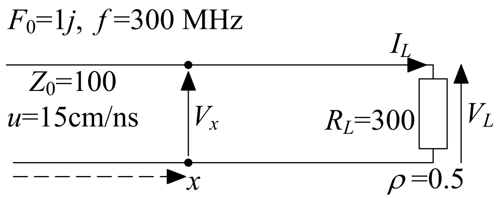
Ohm's law at the load determines the ratio $\frac{G_x}{F_x}$ everywhere on the line.

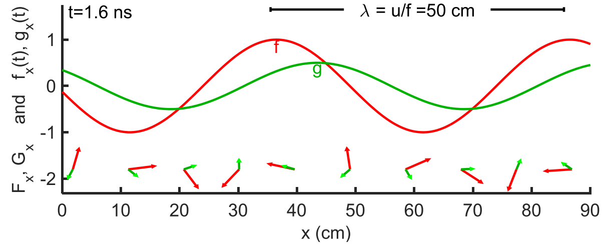
Note that $\left|\frac{G_x}{F_x}\right| \equiv |\rho_L|$ has the same value for all x.

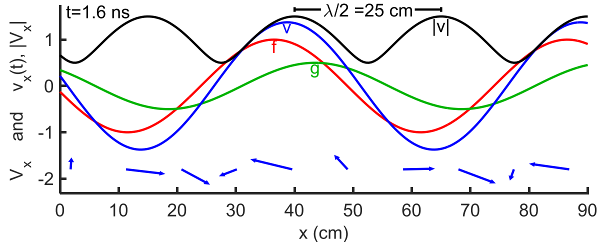
$$V_x = F_x + G_x = F_x \left(1 + \rho_L e^{-2jk(L-x)}\right)$$

$$I_x = Z_0^{-1}\left(F_x - G_x\right) = Z_0^{-1} F_x \left(1 - \rho_L e^{-2jk(L-x)}\right)$$

The exponent $-2jk\left(L - x\right)$ is the phase delay from travelling from x to L and back again (hence the factor 2).







Forward wave phasor: $F_x = Fe^{-jkx}$
Backward wave phasor: $G_x = \rho_L F_x e^{-2jk(L-x)} = \rho_L F e^{-2jkL} e^{+jkx}$

Line Voltage phasor: $V_x = F_x + G_x = Fe^{-jkx}\left(1 + \rho_L e^{-2jk(L-x)}\right)$
Line Voltage Amplitude: $|V_x| = |F|\left|1 + \rho_L e^{-2jk(L-x)}\right|$ varies with x but not t

Max amplitude equals $1 + |\rho_L|$ at values of x where F_x and G_x are in phase. This occurs every $\frac{\lambda}{2}$ away from L where λ is the *wavelength*, $\lambda = \frac{2\pi}{k} = \frac{u}{f}$.

Min amplitude equals $1 - |\rho_L|$ at values of x where F_x and G_x are out of phase.

Standing waves arise whenever a periodic wave meets its reflection: e.g. ponds, musical instruments, microwave ovens.

- Use phasors if forward and backward waves are sinusoidal with the same ω.
  - $f_x(t) = f\left(t - \frac{x}{u}\right) \quad \rightarrow \quad F_x = F_0 e^{-jkx}$
  - $g_x(t) = g\left(t + \frac{x}{u}\right) \quad \rightarrow \quad G_x = G_0 e^{+jkx}$
    - ▷ $k = \frac{\omega}{u}$ is the wavenumber in "radians per metre"
- Time delays $\simeq$ phase shifts: $F_y = F_x e^{-jk(y-x)}$
- When a periodic wave meets its reflection you get a standing wave:
  - Oscillation amplitude varies with x: $\propto \left|1 + \rho_L e^{-2jk(L-x)}\right|$
  - Max amplitude of $(1 + |\rho_L|)$ occurs every $\frac{\lambda}{2}$

Merry Xmas