E1.1 Circuit Analysis

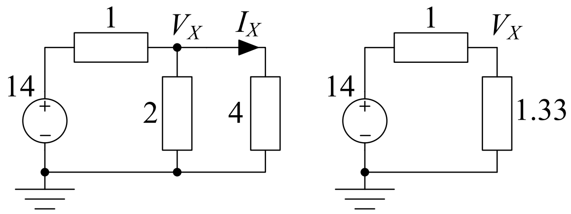
Problem Sheet 2 - Solutions

Note: In many of the solutions below I have written the voltage at node X as the variable X instead of V_X in order to save writing so many subscripts.

1. [Nodal analysis] KCL at node V_X gives $\frac{V_X-14}{1}+\frac{V_X}{2}+\frac{V_X}{4}=0$ which simplifies to $7V_X-56=0$ from which $V_X=8$.

   [Parallel resistors] We can merge the $2\,\Omega$ and $4\,\Omega$ resistors to make one of $\frac{2\times 4}{2+4}=\frac{4}{3}\,\Omega$ as shown below. Now we have a *potential divider*, so $V_X=14\times\frac{1.33}{2.33}=8\,\text{V}$.

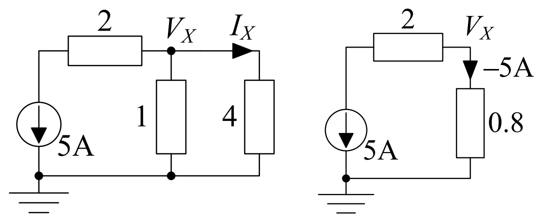
   In both cases, we can now calculate $I_X=\frac{V_X}{4}=2\,\text{A}$. Note that when we merge the two resistors, I_X is no longer a distinct current on the diagram.



Original Simplified

2. [Nodal Analysis] KCL at node V_x gives $5+\frac{V_X}{1}+\frac{V_X}{4}=0$ which simplifies to $20+5V_X=0$ from which $V_X=-4$.

   [Parallel Resistors] We can combine the $1\,\Omega$ and $4\,\Omega$ resistors to make one of $\frac{1\times 4}{1+4}=\frac{4}{5}\,\Omega$ as shown below. Now we have $V_X=-5\times 0.8=-4\,\text{V}$.



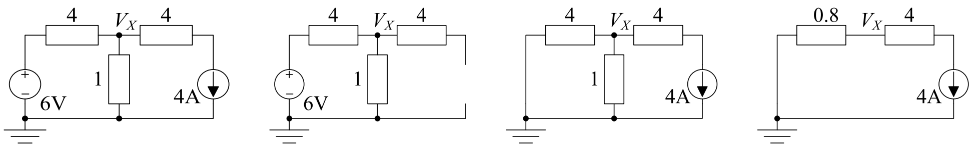
Original Simplified

   In both cases, we can now calculate $I_X=\frac{V_X}{4}=-1\,\text{A}$.

   In this question, you have to be a bit careful about the sign used to represent currents. Whenever you use Ohm's law, you must be sure that you use the passive sign convention (with the current arrow in the opposite direction to the voltage arrow); this is why the current through the $0.8\,\Omega$ resistor is $-5\,\text{A}$ rather than $+5\,\text{A}$.

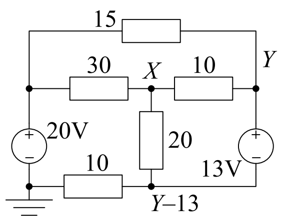
3. [Nodal Analysis] KCL at node V_x gives $\frac{V_X-6}{4}+\frac{V_X}{1}+4=0$ which simplifies to $5V_X+10=0$ from which $V_X=-2$.

   [Superposition] (i) If we set the current source to zero, then the $4\,\Omega$ resistor connected to it plays no part in the circuit and we have a potential divider giving $V_X=6\times\frac{1}{5}=1.2$. (ii) We now set the voltage source to zero and then simplify the resultant circuit as shown below. Being careful with signs, we now get $V_X=-4\times 0.8=-3.2$. Adding these two values together gives a final answer of $V_X=-2$.

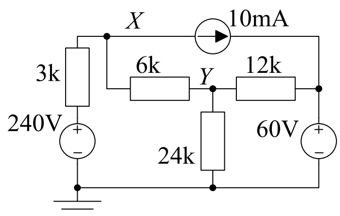


Original | $I=0$ | $V=0$ | $V=0$ simplified

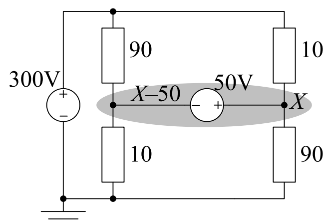
4. We first label the nodes; we only need two variables because of the floating voltage source. KCL at X gives $\frac{X-20}{30}+\frac{X-(Y-13)}{20}+\frac{X-Y}{10}=0$ which gives $11X-9Y=1$. KCL at the supernode $\{Y,Y-13\}$gives $\frac{Y-20}{15}+\frac{Y-X}{10}+\frac{(Y-13)-X}{20}+\frac{(Y-13)}{10}=0$ which gives $-9X+19Y=197$. Solving these two simultaneous equations gives $X=14$ and $Y=17$.



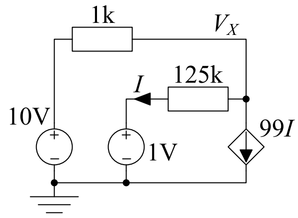
5. We first label the nodes; there are only two whose voltage is unknown. Working in mA and kΩ, KCL at X gives $\frac{X-240}{3}+\frac{X-Y}{6}+10=0$ which gives $3X-Y=420$. KCL at Y gives $\frac{Y-X}{6}+\frac{Y}{24}+\frac{Y-60}{12}=0$ which gives $-4X+7Y=120$. Solving these simultaneous equations gives $X=180$ and $Y=120$.



6. We first label the nodes; since the two nodes having unknown voltages are joined by a fixed voltage source, we only need one variable. We write down KCL for the supernode $\{X,X-50\}$ (shaded in the diagram) which gives $\frac{(X-50)-300}{90}+\frac{(X-50)}{10}+\frac{X-300}{10}+\frac{X}{90}=0$ which simplifies to $20X=3500$ or $X=175\,\mathrm{V}$.

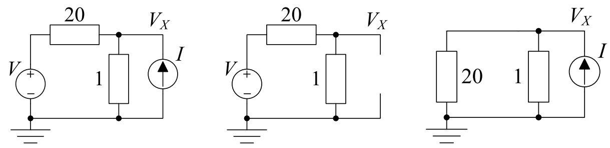


7. There is only one node with an unknown voltage, namely X. However, there is a dependent current source, so we need to express its value in terms of the node voltages: $99I = 99 \times \frac{X-1}{125}$ where we are expressing currents in mA. So now we can apply KCL to node X to obtain $\frac{X-10}{1} + \frac{X-1}{125} + 99 \times \frac{X-1}{125} = 0$ which simplifies to $225X = 1350$ from which $X = 6$.



8. [Nodal Analysis] Using KCL at node X gives $\frac{X-V}{20} + \frac{X}{1} - I = 0$ which we can rearrange to give $X = \frac{1}{21}V + \frac{20}{21}I$.

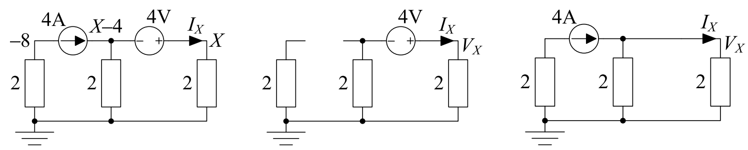
   [Superposition] If we set $I = 0$ then we have a voltage divider in which $X = \frac{1}{21}V$ (see middle diagram). If we set $V = 0$ then (see right diagram) we can combine the two parallel resistors as $\frac{20\times 1}{20+1} = \frac{20}{21}\,\Omega$ and it follows that $X = \frac{20}{21}I$. By superposition, we can add these two expression together to give $X = \frac{1}{21}V + \frac{20}{21}I$.



Original $I = 0$ $V = 0$

9. [Nodal Analysis] We can easily see that the 4 A current flowing through the leftmost resistor means the top left node has a voltage of -8 (although actually we do not need to calculate this because of the isolating effect of the current source). Using KCL at the supernode $\{X-4, X\}$ gives $-4 + \frac{X-4}{2} + \frac{X}{2} = 0$ which we can rearrange to give $2X = 12$ or $X = 6$. It follows that $I_X = \frac{6}{2} = 3\,\text{A}$.

   [Superposition] If we set $I = 0$ then we have a voltage divider (since the resistors are in series) in which $X = 4 \times \frac{2}{2\times 2} = 2$ and $I_X = 1\,\text{A}$ (see middle diagram). If we set $V = 0$ then (see right diagram) we can combine the two parallel resistors as $\frac{2\times 2}{2+2} = 1\,\Omega$ and it follows that $X = 4$ and, by current division, that $I_X = 2\,\text{A}$. By superposition, we can add these two expression together to give $X = 6\,\text{V}$ and $I_X = 3\,\text{A}$.

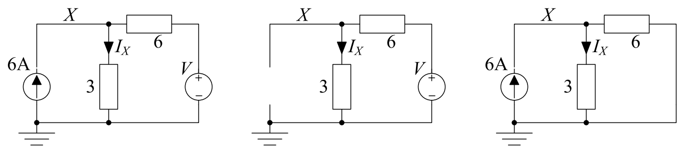


Original $I = 0$ $V = 0$

10. [Nodal Analysis] We first label the unknown node as X. Now, KCL at this node gives $-6+\frac{X}{3}+\frac{X-V}{6}=0$ which rearranges to give $3X=V+36$ from which $X=\frac{1}{3}V+12$. Now $I_X=\frac{X}{3}$ so $I_X=\frac{1}{9}V+4$.

    [Superposition] If we set $I=0$ (see middle diagram) then $I_X=\frac{V}{9}$. If we set $V=0$ then (see right diagram) we have a current divider in which the 6 A current divides in proportion to the conductances. So $I_X=6\times\frac{1/3}{1/3+1/6}=6\times\frac{2}{3}=4$. Adding these results together gives $I_X=\frac{1}{9}V+4$.

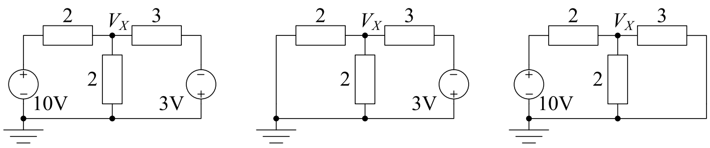
    If $V=-36$ then $I_X=0$.



Original $\qquad$ $I=0$ $\qquad$ $V=0$

11. [Nodal Analysis] KCL at node X gives $\frac{X-10}{2}+\frac{X}{2}+\frac{X-(-3)}{3}=0$ which rearranges to give $8X=24$ from which $X=3$.

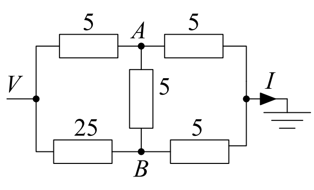
    [Superposition] If we set the left source to zero (see middle diagram) then, the two parallel $2\,\Omega$ resistors are equivalent to $1\,\Omega$ and so we have a potential divider and $X=-3\times\frac{1}{4}=-0.75$. If we set the other source to zero (see right diagram) we can combine the $2\,\Omega$ and $3\,\Omega$ parallel resistors to obtain $\frac{2\times3}{2+3}=1.2\,\Omega$. We again have a potential divider giving $X=10\times\frac{1.2}{2+1.2}=3.75$. Adding these together gives $X=-0.75+3.75=3\,\text{V}$.



Original $\qquad$ $U_1=0$ $\qquad$ $U_2=0$

12. KCL at node Y gives $\frac{Y-4}{1}+\frac{Y-X}{5}=0$ which rearranges to $-X+6Y=20$. We also have the equation of the dependent voltage source: $X=-6Y$. We can conveninetly eliminate $6Y$ between these two to give $-2X=20$ and so $X=-10$.

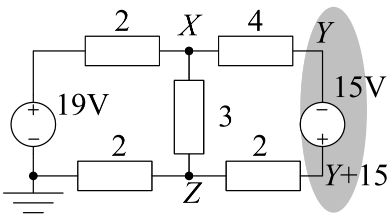
13. We first pick a ground reference at one end of the network and label all the other nodes. The equivalent resistance is now $\frac{V}{I}$. We assume that we know V and then calculate I. KCL at node A gives $\frac{A-V}{5}+\frac{A-B}{5}+\frac{A}{5}=0$ from which $3A-B=V$ or $B=3A-V$. KCL at node B gives $\frac{B-V}{25}+\frac{B-A}{5}+\frac{B}{5}=0$ from which $11B-5A=V$. Substituting $B=3A-V$ into this equation gives $33A-5A=12V$ or $A=\frac{12}{28}V$ which in turn gives $B=3A-V=\frac{8}{28}V$. The current I is the sum of the currents throught the rightmost two $5\,\Omega$resistors: $I=\frac{A}{5}+\frac{B}{5}=\frac{1}{7}V$. So the equivalent resistance is $\frac{V}{I}=7\,\Omega$.



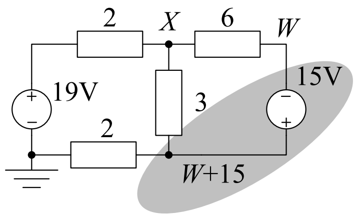
14. If $V_{AB} = 0$ then no current flows through the $2\,\text{k}$ resistor, so the two vertical resistor chains form potential dividers. In the leftmost chain, $V_A = 100 \times \frac{4}{4+4} = 50\,\text{V}$. Since $V_{AB} = 0$, $V_B = V_A = 50 = 100 \times \frac{R}{4+R}$ which implies that $R = 4\,\text{k}$.

    KCL at nodes A gives $\frac{A-100}{4} + \frac{A}{4} + \frac{A-B}{2} = 0$ which gives $4A - 2B = 100$ or $B = 2A - 50$. KCL at node B now gives $\frac{B-100}{4} + \frac{B}{R} + \frac{B-A}{2} = 0$ into which we can substitute the expression for B to get $\frac{2A-150}{4} + \frac{2A-50}{R} + \frac{A-50}{2} = 0$ from which $A = \frac{100+125R}{4+2R}$. Substituting this into $B = 2A - 50$ gives $B = \frac{150R}{4+2R}$ and hence $A - B = \frac{100-25R}{4+2R}$. If we can detect a value of $A - B = 10\,\text{mV} = 0.01$ then the corresponding value of R is the solution to $\frac{100-25R}{4+2R} = 0.01$ which gives $25.02R = 99.96$ from which $R = 3995.2\,\Omega$ which is a change of $4.8\,\Omega$ or 0.12%.

15. We label the nodes as shown below (using X instead of V_X for ease of writing). Note that when we have labelled the upper node of the floating voltage source as Y we can label the lower node as $Y + 13$ and do not need another variable. KCL at node X gives $\frac{X-19}{2} + \frac{X-Z}{3} + \frac{X-Y}{4} = 0$ which gives $13X - 3Y - 4Z = 114$. KCL at node Z gives $\frac{Z}{2} + \frac{Z-X}{3} + \frac{Z-Y-15}{2} = 0$ which gives $-2X - 3Y + 8Z = 45$. Finally, KCL at the supernode $\{Y, Y+15\}$ gives $\frac{Y-X}{4} + \frac{Y+15-Z}{2} = 0$ from which $X - 3Y + 2Z = 30$. Solving these three simultaneous equations gives $X = 11$, $Y = -1$ and $Z = 8$.

    Original

    Simplified

    An alternative approach is to notice that the three rightmost components are in series and so you can reorder them without affecting the rest of the circuit to give the simplified circuit shown above. Now, we only have two unknowns and hence only two simultaneous equations to solve. KCL at node X gives $\frac{X-19}{2} + \frac{X-W-15}{3} + \frac{X-W}{6} = 0$ which gives $6X - 3W = 87$. KCL at the supernode $\{W, W+15\}$ gives $\frac{W-X}{6} + \frac{W+15-X}{3} + \frac{W+15}{2} = 0$ from which $3X - 6W = 75$. These equations are easily solved to give $X = 11$ and $W = -7$.

16. Since the floating voltage source is a dependent voltage source, we need to label its two ends with separate variables (see below). We now write down a KCL equation for the supernode shown shaded: $\frac{Y-48}{4} + \frac{Y}{4} + \frac{X-48}{9} + \frac{X}{6} = 0$ which simplifies to $10X + 18Y = 624$.

    We also need to express the voltage source value in terms of nodal voltages: $Y - X = 8I = 8 \times \frac{X}{6} = \frac{4}{3}X$ which rearranges to give $Y = \frac{7}{3}X$. Substituting this in the previous equation gives $10X + 18 \times \frac{7}{3}X = 624$ which simplifies to $52X = 624$ from which $X = 12$.