E1.1 Circuit Analysis

Problem Sheet 3 - Solutions

1. (a) Thévenin voltage equals the open circuit voltage is 4 V (from potential divider). To obtain the Thévenin/Norton we set the voltage source to 0 (making it a short circuit) and find the resistance of the network to be $1||4 = 0.8\,\Omega$. From this the Norton current is $\frac{4}{0.8} = 5\,\text{A}$. This may also be found directly from the short-circuit current of the original circuit.

   (b) The open-circuit voltage is $-8\,\text{V}$ (since the 2 A current flows anticlockwise). To obtain the Thévenin/Norton resistance, we set the current source to zero (zero current implies an open circuit), so the resultant network has a resistance of $4\,\Omega$. The Norton current is $\frac{-8}{4} = -2\,\text{A}$; this may also be found by observing that the short-circuit current (flowing into node A) is $+2\,\text{A}$.

2. KCL at node A gives $\frac{A-5}{1} + \frac{A}{4} - I = 0$ from which $5A - 20 - 4I = 0$ which we can rearrange to give $A = 4 + 0.8I = V_{Th} + R_{Th}I$. We can also rearrange to give $I = -5 + \frac{1}{0.8}A = -I_{Nor} + \frac{1}{R_{Nor}}A$.

3. (a) When $U = 5$, the diode is on and so $X = U - 0.7 = 4.3$. To check our assumption about the diode operating region, we need to calculate the current through the diode; this equals 4.3 mA which is indeed positive.

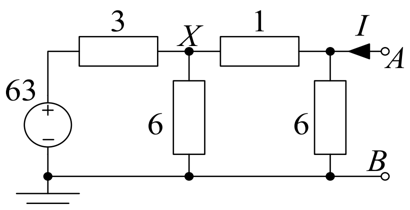
   (b) When $U = -5$, the diode is off, so the current through the resistor is zero and $X = 0$. As a check, the voltage across the diode is $U - X = -5$ which confirms our assumption that it is off.

4. As in part (a) of the previous question, $I = 4.3\,\text{mA}$. The power dissipation in the diode is therefore $V_D I = 0.7 \times 4.3 = 3.01\,\text{mW}$. The power dissipation in the resistor is $I^2R = 18.5\,\text{mW}$.

5. Circuit (a) is an inverting amplifier with gain $\frac{X}{U} = -\frac{10}{1} = -10$. Circuit (b) is a non-inverting amplifier with gain $\frac{Y}{U} = 1 + \frac{10}{1} = +11$. Another way to see this is to notice that, since the opamp inputs draw no current, the potential divider means that the -ve opamp input is at $\frac{Y}{11}$ and, since the negative feedback ensures the opamp terminals are at the same voltage, $U = \frac{Y}{11}$.

6. [Method 1 - circuit manipulation] To calculate the Thévenin equivalent, we want to determine the open-circuit voltage and the Thévenin resistance. To determine the open-circuit voltage, we assume that $I = 0$ and calculate V_{AB}. Since $I = 0$, we can combine the $1\,\Omega$ and $6\,\Omega$ resistors to give $7\,\Omega$ and then combine this with the $6\,\Omega$ resistor in parallel to give $\frac{42}{13}\,\Omega$. We now have a potential divider so the voltage at point X is $63 \times \frac{42/13}{3+42/13} = \frac{98}{3}$. This is then divided by the $1\,\Omega$ and $6\,\Omega$ resistors to give an open-circuit voltage of $\frac{98}{3} \times \frac{6}{7} = 28\,\text{V}$. The Thévenin/Norton resistance can be found by short-circuiting the voltage source to give $3\,\Omega$ in parallel with $6\,\Omega$ which equals $2\,\Omega$. This is then in series with $1\,\Omega$(to give $3\,\Omega$) and finally in parallel with $6\,\Omega$ to give $2\,\Omega$.

   [Method 2 - Nodal Analysis]. We can do KCL at node X (see diagram below) to get $\frac{X-63}{3} + \frac{X}{6} + \frac{X-A}{1} = 0$ which simplifies to $9X - 6A = 126$ or $3X - 2A = 42$. We now do KCL at A but include an additional input current I as shown in the diagram. This gives $\frac{A-X}{1} + \frac{A}{6} - I = 0$ from which $7A - 6X = 6I$. Substituting for $6X = 4A + 84$ gives $3A = 84 + 6I$ or $A = 28 + 2I$. This gives the Thévenin voltage as 28 and the Thévenin/Norton resistance as $2\,\Omega$.Hence the Norton current is 14 A.

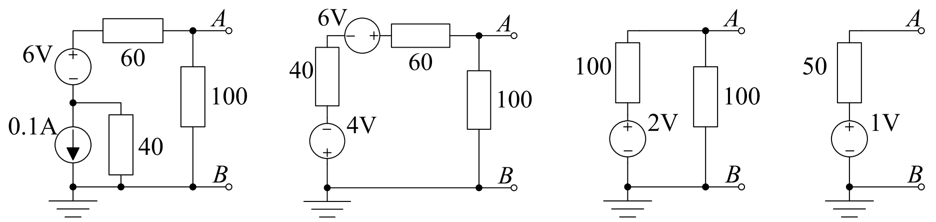


7. (a) KCL @ X gives $\frac{X-14}{1} + \frac{X}{4} + \frac{X}{2} = 0$ from which $7X = 56 \Rightarrow X = 8 \Rightarrow I = \frac{X}{2} = 4\,\text{mA}$.

   (b)Finding the Thévenin equivalent of the left three components: we consider the two resistors as a potential divider to give $V_{Th} = 14 \times \frac{4}{5} = 11.2\,\text{V}$. Setting the source to zero (short circuit) gives $R_{Th} = 1||4 = 800\,\Omega$. Hence $I = \frac{V_{Th}}{R_{Th}+2000} = \frac{11.2}{2.8} = 4\,\text{mA}$.

8. From question 7, the left three components have a Thévenin equivalent: $V_{Th} = 11.2\,\text{V}$ and $R_{Th} = 800\,\Omega$. It follows that the maximum power will be dissipated in R when $R = R_{Th} = 800\,\Omega$ (see notes page 5-8). Since the voltage across R will then be $\frac{1}{2}V_{Th}$ the power dissipation will be $\frac{1}{4R_{Th}}V_{Th}^2 = 39.2\,\text{mW}$.

9. (a) As shown in the sequence below, we first combine the current source with the $40\,\Omega$resistor, then combine the four series components into a single Thévenin equivalent and finally find the Thévenin equivalent of the simple network (using parallel resistors for R_{Th} and a potential divider for V_{Th}).

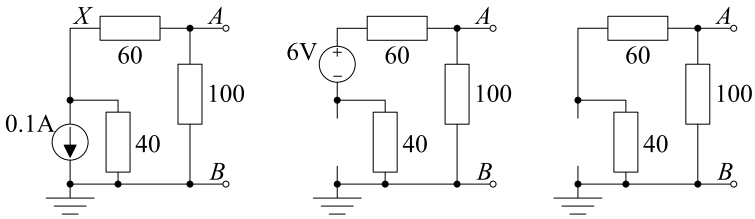


(b) Setting the voltage source to zero gives us the first diagram. Combining $40||(60 + 100) = 32$ so $X = -0.1 \times 32 = -3.2\,\text{V}$. it follows (potential divider) that $A = -3.2 \times \frac{100}{160} = -2\,\text{V}$.

Now setting the current source to zero gives the second diagram and we have a potential divider giving $V_{AB} = 6 \times \frac{100}{100+60+40} = 3\,\text{V}$.

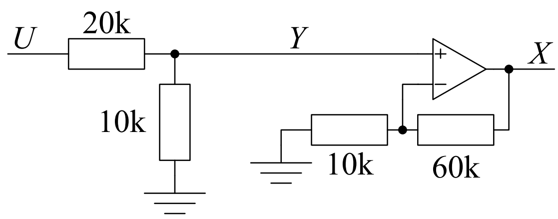
Superposition now gives us $V_{AB} = V_{Th} = -2 + 3 = 1\,\text{V}$.

To find R_{Th} we set both sources to zero and find the resultant resistance of $100||(60{+}40) = 100||100 = 50\,\Omega$.



10. (a) Negative, (b) Positive, (c) Negative, (d) Positive. In simple circuits like these, you can just see which terminal the output feeds back to.

11. The best way to think of this circuit is as a potential divider with gain $\frac{Y}{U} = \frac{1}{3}$ followed by a non-inverting opamp circuit with gain $\frac{X}{Y} = 1 + \frac{60}{10} = 7$. The combined gain is then $\frac{X}{U}{=}\frac{1}{3} \times 7 = \frac{7}{3}$.



12. (a) You can either recognise this a a standard inverting summing amplifier with gain $X = -\left(\frac{40}{20}U_1 + \frac{40}{10}U_2\right) = -2U_1 - 4U_2$ or else apply KCL at the +ve input terminal with the assumption that negative feedback will ensure that this terminal is at the same voltage as the –ve terminal i.e. 0 V. This gives: $\frac{0-U_1}{20} + \frac{0-U_2}{10} + \frac{0-X}{40} = 0$ from which $X = -2U_1 - 4U_2$.

    (b) The network connected to the +ve terminal is a weighted averaging circuit (page 3-7 of the notes) so $V_+ = \frac{1}{3}U_1 + \frac{2}{3}U_2$. The opamp circuit itself is a non-inverting amplifier with a gain of $1 + \frac{50}{10} = 6$. So, $Y = 6 \times \left(\frac{1}{3}U_1 + \frac{2}{3}U_2\right) = 2U_1 + 4U_2$.

    (c) [Superposition method] Following the method of part (b) above, if $U_3 = 0$, we have $Z = 5 \times \left(\frac{1}{5}U_1 + \frac{4}{5}U_2\right) = U_1 + 4U_2$. If, on the other hand, $U_1 = U_2 = 0$, then $V_+ = 0$ and so we have an inverting amplifier with a gain of $-\frac{40}{10} = -4$. Hence $Z = -4U_3$.

    Combining these gives $Z = U_1 + 4U_2 - 4U_3$.

    [Nodal analysis method] The top two resistors are a weighted average circuit so $V_+ = \frac{1}{5}U_1 + \frac{4}{5}U_2$. Now, assuming that $V_- = V_+$, we do KCL at V_- to give $\frac{\frac{1}{5}U_1 + \frac{4}{5}U_2 - U_3}{10} + \frac{\frac{1}{5}U_1 + \frac{4}{5}U_2 - Z}{40} = 0$ from which $U_1 + 4U_2 - 4U_3 - Z = 0$ giving $Z = U_1 + 4U_2 - 4U_3$.

13. We can use superposition. If $U_2 = 0$,then $V_+ = 0$ and we have an inverting amplifier with a gain $\frac{X}{U_1} = -\frac{60}{R_1}$. The question tells us that this must equal -3 so we must have $R_1 = 20$. Now, if $U_1 = 0$, the circuit consists of a potential divider with a gain of $\frac{60}{R_2+60}$ followed by a non-inverting amplifier with a gain of $1 + \frac{60}{R_1} = 4$. The combined gain must equal 2 (from the question) so the potential divider must have a gain of $\frac{1}{2}$ which means $R_2 = 60\,\text{k}\Omega$.

14. The first opamp is non-inverting with a gain $\frac{X}{U_2} = 1 + \frac{10}{50} = 1.2$. We can use superposition to find Y: If $U_2 = 0$,then $X = 0$ and we have a non-inverting amplifier with a gain of $\frac{Y}{U_1} = 1 + \frac{50}{10} = 6$. If, on the other hand, $U_1 = 0$, then we have an inverting amplifier and $\frac{Y}{X} = -\frac{50}{10} = -5$. It follows that $\frac{Y}{U_2} = \frac{Y}{X} \times \frac{X}{U_2} = -5 \times 1.2 = -6$. Combining both portions of the superposition, $Y = 6U_1 - 6U_2 = 6(U_1 - U_2)$. This is therefore a differential amplifier (whose output is $\propto (U_1 - U_2)$) that draws almost no current from either if its inputs. A better circuit is given in question 22.

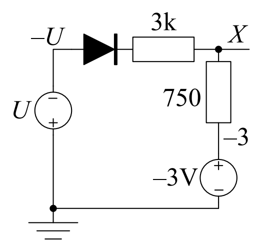
15. The potentiometer is a potential divider and so $V_+ = aU$. Assuming that $V_- = V_+$, we can do KCL at V_- to get $\frac{aU-U}{40} + \frac{aU-X}{40} = 0$ from which $X = (2a-1)\,U$. For $a = 0$, the gain is -1 and when $a = 1$, the gain is $+1$. So the circuit can generate any gain between these two extremes.

16. The Thévenin voltage is $-3\,\text{V}$ (potential divider) and the Thévenin resistance is $1||3 = 0.75\,\Omega$ as shown in the diagram below. Note that if you drawn the Thévenin voltage source the other way around (with "+" at the bottom) then the Thévenin voltage will be $+3\,\text{V}$; this is an equally valid solution.

    (a) If $U = 0\,\text{V}$, the diode is forward biassed and KCL @ X gives $\frac{X-(-3)}{0.75} + \frac{X-(0-0.7)}{3} = 0$ from which $X = -2.54$. The current through the diode is $\frac{X-(-3)}{0.75} = 613\,\text{mA}$ which is > 0 confirming our guess about the diode operating region.

    (b) If $U = 5\,\text{V}$, the diode is off and so $X = -3$ (since no current flows through the $750\,\Omega$ resistor). The diode forward voltage is $(-U) - X = -2$. This is < 0.7 confirming our operating region guess.

    The diode switches regions when both operating region equations are true: $I_D = 0$ and $V_D = 0.7$. The current equation implies $X = -3$ while the voltage equation (and the zero current through the 3 k resistor) implies $X = -U - 0.7$. Combining these gives $U = 2.3$. Extreme care with signs is needed in this question.



17. There is only one feedback loop in this circuit: from W back to the input adder. We can write $W = FG(X - W)$ from which $W = \frac{FG}{1+FG}X$. Then $Y = F(X - W) + HW$. From the first equation $(X - W) = \frac{W}{FG}$ so we can substitute this in to get $Y = \frac{W}{G} + HW = \left(\frac{1}{G} + H\right)\frac{FG}{1+FG}X = \frac{F+FGH}{1+FG}X$.

18. (a) [Nodal analysis including I] We have $J = \frac{U-A}{100}$, so the current source value (expressed in terms of node voltages) is $49J = 0.49(U - A)$. The KCL equation @ A is $\frac{A-U}{100} - 0.49(U - A) - I = 0$ from which $50(A - U) = 100I$ which gives $A = 2I + U = R_{Th}I + V_{Th}$. So the Thévenin voltage is U and the Thévenin resistance is $2\,\text{k}\Omega$.

    (b) In this method, we assume $I = 0$ and calculate the open-circuit voltage and short-circuit current. For the open-circuit voltage, we do KCL at A and obtain $\frac{A-U}{100} - 0.49(U - A) = 0$ from which $50(A - U) = 0$ so $A = V_{Th} = U$. For the short-circuit current, we join A and B and the current is then $\frac{V_{Th}}{R_{Th}} = \frac{U}{100} + 0.49U = 0.5U$. Hence $R_{Th} = 2\,\text{k}\Omega$.

19. From the notes (page 3-7) $X = \frac{U_1G_1+U_2G_2+U_3G_3}{G_1+G_2+G_3}$. The equations are made much easier to solve because we know that the Thévenin resistance must be $50\,\Omega$. The Thévenin resistance is just the parallel combination $R_1||R_2||R_3 = \frac{1}{G_1+G_2+G_3} = \frac{1}{20\,\text{mS}}$. Hence $X = \frac{1}{2}U_3 + \frac{1}{3}U_2 + \frac{1}{6}U_1 = 50(U_1G_1 + U_2G_2 + U_3G_3)$ where the first expression is given in the question and the second comes from substituting for $G_1 + G_2 + G_3$. Identifying the coefficients in this equation gives $G_1 = \frac{1}{300}$, $G_2 = \frac{1}{150}$ and $G_3 = \frac{1}{100}$ from which $R_1 = 300$, $R_2 = 150$ and $R_3 = 100$. In a weighted average circuit, the coefficients must sum to 1; thus $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$ but $\frac{1}{2} + \frac{1}{3} + \frac{1}{3} \neq 1$ so the latter set of coefficients is inadmissible.

20. From the notes (page 3-7) $X = \frac{U_2G_2+U_3G_3+5G_4}{G_1+G_2+G_3+G_4}$. where U_2 and U_3 can equal 0 or 5. As in the previous question, we are told that $G_1 + G_2 + G_3 + G_4 = 20\,\text{mS}$. Hence $X = 50(U_2G_2 + U_3G_3 + 5G_4)$. The only way that we can obtain the required voltages is if switching U_2 causes X to change by 1 V and switching U_3 causes X to change by 2 V (or vice versa). Thus we need $X = \frac{1}{5}U_2 + \frac{2}{5}U_3 + 1$; it is easy to see that this satisfies the required output voltages. Now equating coefficients, we get $50G_2 = \frac{1}{5}$, $50G_3 = \frac{2}{5}$ and $250G_4 = 1$ from which $R_2 = \frac{1}{G_2} = 250$, $R_3 = \frac{1}{G_3} = 125$ and $R_4 = \frac{1}{G_4} = 250$. Finally, $G_1 = 0.02 - G_2 - G_3 - G_4 = 0.004$ so $R_1 = \frac{1}{G_1} = 250$.

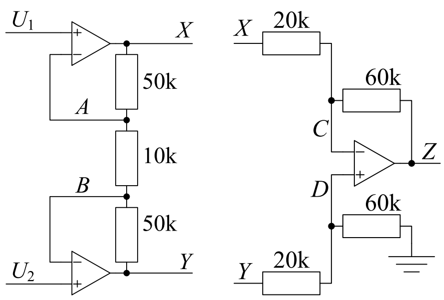
21. We need to determine how the differential input to the opamp $(V_+ - V_-)$depends on X. We are not interested in how it depends on U so it is convenient to set $U = 0$. So, with this assumption, we get potential divider equations:

    (a) $V_+ = \frac{20}{10+20}X = \frac{2}{3}X$ and $V_- = \frac{10}{20+10}X = \frac{1}{3}X$. Hence $V_+ - V_- = \frac{1}{3}X$ which, since the coefficient is positive means positive feedback.

    (b) $V_+ = \frac{10}{10+20}X = \frac{1}{3}X$ and $V_- = \frac{20}{20+10}X = \frac{2}{3}X$. Hence $V_+ - V_- = -\frac{1}{3}X$ which, since the coefficient is negative means negative feedback.

22. We can split the circuit up into two independent parts because the opamp outputs are voltage sources whose voltage is not affected by how many other things are connected to them. Note that all three opamps have negative feedback and so we can assume that $V_+ = V_-$.

    For the first part, we have $A = U_1$ and $B = U_2$. KCL @ A therefore gives $\frac{U_1-X}{50} + \frac{U_1-U_2}{10} = 0$ which gives $X = 6U_1 - 5U_2$. Similarly, KCL @ B gives $\frac{U_2-Y}{50} + \frac{U_2-U_1}{10} = 0$ which gives $Y = 6U_2 - 5U_1$. The second part of the circuit is a differential amplifier (see page 6-9 of the notes) for which $Z = \frac{60}{20}(Y - X)$. Substituting in the expressions for X and Y gives $Z = 3((6U_2 - 5U_1) - (6U_1 - 5U_2)) = 3(11U_2 - 11U_1) = 33(U_2 - U_1)$.



23. We can verify that the opamp has negative feedback since (if we set the 10 V source to zero), we have $V_- = \frac{1}{2}Y$ but $V_+ = \alpha Y$ where $\alpha = \frac{2||R}{2+2||R} < \frac{1}{2}$. So, we can assume that $V_+ = V_- = X$. KCL @ V_- gives $\frac{X-(-10)}{10} + \frac{X-Y}{10} = 0$ which gives $X - Y = -X - 10$. KCL @ V_+ gives $\frac{X}{2} + \frac{X-Y}{2} + I = 0$. Substituting $X - Y = -X - 10$ gives $\frac{X}{2} + \frac{-X-10}{2} + I = 0$ which simplifies to $I = 5$. Notice that this does not depend on R, so we have constructed a current source.

24. From the block diagram, we can deduce $Y = \sqrt{X} = \sqrt{AU - AY}$ from which $Y^2 + AY - AU = 0$. Solving this quadratic equation gives $Y(U) = 0.5\left(-A + \sqrt{A^2 + 4AU}\right)$. Notice that only one of the two roots will result in Y being positive. Provided that $A \gg 4U$, we can use the Taylor series approximation to give $Y(U) \approx 0.5\left(-A + A\left(1 + \frac{2U}{A} - \frac{2U^2}{A^2}\right)\right) = U - \frac{U^2}{A}$. From this, $Y(1) = 1 - \frac{1}{A}$ and $Y(0.5) = 0.5 - \frac{0.25}{A} = 0.5Y(1) + \frac{0.25}{A}$. We would like the error, $\frac{0.25}{A}$ to be 1% of $Y(1)$, i.e. $\frac{0.25}{A} \le 0.01\left(1 - \frac{1}{A}\right)$. Solving this gives $A \ge 26$.

25. (a) The $-$ terminal of the opamp is a virtual earth so the current through the resistor is $\frac{X}{R}$. All the current flows through the diode whose voltage is $-U$. Therefore we have $\frac{X}{R} = k \exp \frac{-U}{V_T}$ from which $U = -V_T \ln \frac{X}{kR}$.

    (b) The second opamp is a non-inverting amplifier with a gain of 3 so $W = 3U = -3V_T \ln \frac{X}{kR}$. For the third opamp, the current thorough the diode is $\frac{Y}{2R} = k \exp \frac{-W}{V_T} = k \exp\left(3 \ln \frac{X}{kR}\right) = k\left(\frac{X}{kR}\right)^3$. From this we find that $Y = \frac{2}{k^2R^2}X^3$. This we have made a circuit that cubes its input voltage (times a scale factor).

26. (a) Despite the current I_B, it is still the case that $V_+ = 0$ and so, because of the negative feedback, $P = 0$ also. KCL @ P gives $\frac{0-U}{10} + I_B + \frac{0-Y}{40} = 0$ from which $-4U + 40I_B - Y = 0$ or $Y = -4U + 40I_B$. Substituting $I_B = 0.0001\,\text{mA}$ gives $Y = -4U + 0.004$ so there is an output error of 4 mV.

    (b) This time, the current I_B flows through the $8\,\text{k}\Omega$ resistor so $V_+ = -8I_B$. As before, we assume that $Q = V_+$ also. Then KCL @ Q gives $\frac{Q-U}{10} + I_B + \frac{Q-Y}{40} = 0$ from which $-4U + 40I_B - Y + 5Q = 0$. Substituting $Q = -8I_B$ gives $-4U + 40I_B - Y - 40I_B = 0$ which simplifies to $Y = -4U$. Thus in this circuit, the bias currents do not cause any error.

    The moral is that when designing opamp circuits, you should try to make the Thévenin resistance seen by the two input terminals the same. If you achieve this, and if the bias currents are the same at both inputs (usually approximately true) there will be no resultant errors.