E1.1 Circuit Analysis

Problem Sheet 4 - Solutions

1. (a) 8, (b) $3 - j4 = 5\angle - 0.93\ (-53°)$, (c) $1.4 + j1.4 = 2\angle 0.79\ (45°)$, (d) $-j8 = 8\angle - 1.57\ (90°)$, (e) $-2 = 2\angle 3.14\ (180°)$, (f) 4, (g) $2.12 + j2.12 = 3\angle 0.79\ (45°)$.

2. (a) $\cos \omega t$, (b) $-2\cos \omega t = 2\cos(\omega t + \pi)$, (c) $-3\sin \omega t = 3\cos\left(\omega t + \frac{\pi}{2}\right)$, (d) $4\sin \omega t = 4\cos\left(\omega t - \frac{\pi}{2}\right)$, (e) $-\cos \omega t - \sin \omega t = 1.4\cos\left(\omega t + \frac{3\pi}{4}\right)$, (f) $3\cos \omega t + 4\sin \omega t = 5\cos(\omega t - 0.93)$, (g) $-2\sin \omega t = 2\cos\left(\omega t + \frac{\pi}{2}\right)$, (h) $3.46\cos \omega t + 2\sin \omega t = 4\cos\left(\omega t - \frac{\pi}{6}\right)$.

3. (a) $\cos \omega t$ by $\frac{\pi}{2}$, (b) $\sin(\omega t + \pi)$ by $\frac{\pi}{2}$, (c) $\sin(\omega t - \pi)$ by $\frac{\pi}{2}$: note that $\sin(\omega t + \pi)$ and $\sin(\omega t - \pi)$ are actually the same waveform, (d) $(1 + j)$ by 0.322 rad, (e) $(1 + j)$ by $\frac{\pi}{2}$, (f) $(-1 - j)$ by $\frac{\pi}{2}$, (g) 1 by 10°. Because angles are only defined to within a multiple of 360°, you always need to be careful when comparing them. To find out which is leading, you need to take the difference in phase angles and then add or subtract multiples of 360° to put the answer into the range ±180°. Note that a sine wave is defined for all values of t (not just for $t > 0$) and so there is no such thing as the "first peak" of a sine wave.

4. $i = C\frac{dv}{dt}$. $\frac{dv}{dt}$ is 3000 V/s for the first 4 ms and −6000 V/s for the next 2 ms. So $i = +15$ or −30 mA. (see Fig. 4)

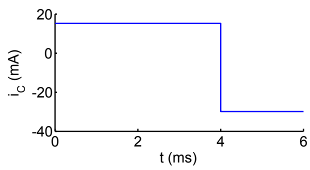

Fig. 4

5. The average value of $x(t)$ is $X = 4$ (note that we use capital letters for quantities that do not vary with time). For averages (or equivalently for DC or $\omega = 0$) capacitors act as open circuit and inductors as short circuits; this gives the simplified circuits shown below. So this gives (a) $Y = 3$, (b) $Y = X = 4$, (c) $Y = X = 4$, (d) $Y = \frac{1}{2}X = 2$.

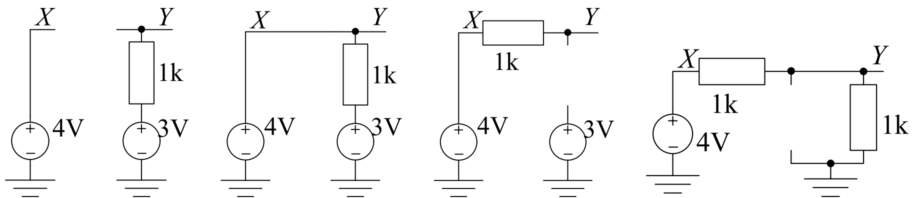

Fig. 5(a) Fig. 5(b) Fig. 5(c) Fig. 5(d)

6. $4 + 4 = 8$. $8||8 = 4$. $4 + 4 = 8$. $24||8 = 6$ mH.

7. C_9 and C_{10} are short-circuted and play no part in the circuit. We can merge series and parallel capacitors as follows: $C_{4,5} = 0.5$, $C_{7,8} = 2$, $C_{2,3} = 2$. Now merge $C_{4,5}$with C_6 to give $C_{4,5,6} = 1.5$ and merge this with $C_{7,8} = 2$ to give $C_{4,5,6,7,8} = \frac{6}{7}$. Now merge this with $C_{2,3} = 2$ to give $C_{2,3,4,5,6,7,8} = \frac{20}{7}$. Finally merge this with $C_1 = 1$ to give $C = \frac{20}{27}\,\mu$F.

8. To determine average values, we can treat C as open circuit and L as short circuit. The original circuit simplifies to that shown in Fig. 8. So we have a simple potential divider and $\bar{v} = \frac{1}{2}\bar{u} = 1$ V where the overbar denotes "average value".

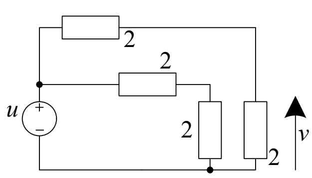

Fig. 8

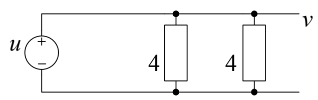

Fig. 9

9. To determine average values, we can treat C as open circuit and L as short circuit. The original circuit simplifies to that shown in Fig. 9 and so $\bar{v} = \bar{u} = 8\,\text{V}$ where the overbar denotes "average value".

10. $Z = R_S + R_P||j\omega L$. (a) $Z = R_S = 10$, (b) $Z = 10 + 10000||100j = 11 + 100j = 100.6\angle 83.7°$, (c) $Z = 10 + 10000||200j = 14 + 200j = 200.4\angle 86°$, (d) $Z = 10010$.

11. We denote the impedance by Z and the admittance by $Y = \frac{1}{Z}$. (a) $Z = 1.44 + 1.92j$ and $Y = 0.25 - 0.33j$, (b) $Z = 4 - 3j$ and $Y = 0.16 + 0.12j$, (c) $Z = 4$ and $Y = 0.25$. Notice that the imaginary part of the impedance (the reactance) is positive for inductive circuits and negative for capacitive circuits, but that the imaginary part of the admittance (the susceptance) has the opposite sign. In part (c), the impedances of the inductor and the capacitor have cancelled out leaving an overall impedance that is purely real; because the impedances are frequency dependent, this cancellation will only happen at one particular frequency which is called the network's "resonant frequency". I strongly advise you to learn how to do these complex arithmetic manipulations using the built-in capabilities of the Casio fx-991.

12. (a) $\omega = 6283$ so the impedance is $Z = 100 + 314j$. Taking the reciprocal gives $Y = \frac{1}{Z} = 0.92 - 2.89j\,\text{mS}$. Since parallel admittances add, the parallel component values must be $R_P = \frac{1000}{0.92} = 1087\,\Omega$ and $L_P = \frac{1000}{2.89\omega} = 55.1\,\text{mH}$. For case (b), we now have $\omega = 62832$ and, following the same argument, we get $R_P = 98.8\,\text{k}\Omega$ and $L_P = 50.05\,\text{mH}$. As the frequency goes up, the series resistor becomes a less significant part of the total impedance and its effect becomes less. This means that L_P becomes approximately equal to the original inductance and R_P becomes larger.

13. $v = L\frac{di}{dt}$. $\frac{di}{dt}$ is $3\,\text{A/s}$ for the first $4\,\text{ms}$ and $-6\,\text{A/s}$ for the next $2\,\text{ms}$. So $v = +6$ or $-12\,\text{mV}$. (see Fig. 13)

14. If we define the voltage phase to be ϕ will be $\angle i_1 = \phi + 0$, $\phi - \frac{\pi}{2} < \angle i_2 < \phi + 0$, $\angle i_3 = \phi + \frac{\pi}{2}$ as the CIVIL mnemonic reminds us. Thus i_3 will have the most positive phase shift. It follows that $i_1 = 2\cos\left(\omega t + \frac{\pi}{4}\right)$, $i_2 = \sqrt{8}\cos\omega t$ and $i_3 = 5\cos\left(\omega t + \frac{3\pi}{4}\right)$ and that $\phi = \frac{\pi}{4}$. As phasors these are $I_1 = 1.4 + j1.4 = 2\angle 45°$, $I_2 = 2.8$, $I_3 = -3.5 + j3.5 = 5\angle 135°$. Adding these together gives $I = 0.71 + j4.95 = 5\angle 1.43\,(82°)$. So $i(t) = 0.71\cos\omega t - 4.95\sin\omega t = 5\cos\left(\omega t + 1.43\right)$ A.

15. $i(t) = i(0) + \frac{1}{L}\int_{\tau=0}^{t} v(\tau)d\tau$ where $v(\tau) = 3000\tau$ for $0 \le \tau \le 4\,\text{ms}$ and $v(\tau) = 36 - 6000\tau$ for $4\,\text{ms} \le \tau \le 6\,\text{ms}$ (obtain this formula by finding the straight line equalling 12 at $\tau = 4\,\text{ms}$ and 0 at $\tau = 6\,\text{ms}$). So, for the first segment, $i = \frac{1}{L} \times 1500t^2$ which reaches $12\,\text{A}$ at $t = 0.004$. For the second segment, $i = \frac{1}{L} \times \left(36t - 3000t^2\right) + c$. To find c, we force $i = 12$ at $t = 0.004$. This gives $c = -36\,\text{A}$. When $t = 0.006$ we then get $i = 18\,\text{A}$. For part (b), we just add $2\,\text{A}$ to the curve. (see Fig. 15)

16. $v = \frac{1}{C}\int i dt$ where $i = 3t$ or $i = 36 \times 10^{-3} - 6t$. So, for the first segment, $v = \frac{1}{C} \times 1.5t^2$ which reaches $4.8\,\text{V}$ at $t = 0.004$. For the second segment, $v = \frac{1}{C} \times \left(36 \times 10^{-3}t - 3t^2\right) + a$. To find a, we force $v = 4.8$ at $t = 0.004$. This gives $a = -14.4\,\text{V}$. When $t = 0.006$ we then get $v(t) = 7.2\,\text{V}$. For part (b), we just subtract $5\,\text{V}$ from the curve. (see Fig. 16)

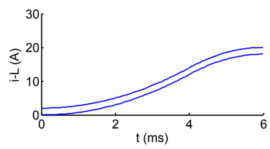

Fig. 15

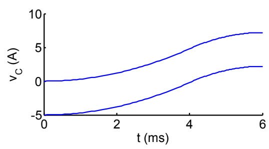

Fig. 16

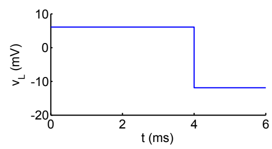

Fig. 13

17. (a) The duty cycle is $0.25 = 25\%$, (b) Since the average voltage across an inductor is always zero, $\bar{x} = \bar{v} = \frac{1}{4} \times 20 = 5\,\text{V}$ (where an overbar denotes the time-average), (c) $\bar{i}_R = \frac{\bar{x}}{R} = 5\,\text{mA}$, (d) Since the average current through a capacitor is always zero, $\bar{i}_C = 0$, and, from KCL, $\bar{i}_L = \bar{i}_R = 5\,\text{mA}$, (e) The voltage across the inductor is $v - x = L\frac{di_L}{dt}$. So when $v = 20$, $\frac{di_L}{dt} = \frac{20-5}{L} = 7.5\,\text{kA/s}$. So the total change in i_L over the $1\,\mu\text{s}$ interval is $7.5\,\text{kA/s} \times 1\,\mu\text{s} = 7.5\,\text{mA}$. Similarly, during the $3\,\mu\text{s}$ interval, $\frac{di_L}{dt} = \frac{0-5}{L} = -2.5\,\text{kA/s}$, so the total change in i_L over the $3\,\mu\text{s}$ interval is $-2.5\,\text{kA/s} \times 3\,\mu\text{s} = -7.5\,\text{mA}$. It follows that $i_L$varies from its average value of $5\,\text{mA}$by $\pm 3.75\,\text{mA}$and has minimum and maximum values of 1.25 and $8.75\,\text{mA}$ (see Fig. 17(a)). An alternative way to see this is to assume the the initial value of i_L equals C at the start of the $1\,\mu\text{s}$ interval and so the value at the end of the $1\,\mu\text{s}$

interval will be $C + 7.5$ and the average value during this interval will be $\bar{i}_L = C + 3.75$. Since we know $\bar{i}_L = \bar{i}_R = 5\,\text{mA}$, it follows that $C = 5 - 3.75 = 1.25\,\text{mA}$. (f) Average powers are $P_R = 25\,\text{mW}$, $P_L = P_C = 0$. Max powers are $P_R = 25\,\text{mW}$, $P_L = v_L i_L = 15 \times 8.75 = 131.25\,\text{mW}$, $P_C = v_C i_C = 5 \times 3.75 = 18.75\,\text{mW}$. Min powers are $P_R = 25\,\text{mW}$, $P_L = v_L i_L = -5 \times 8.75 = -43.75\,\text{mW}$ (see Fig. 17(b)), $P_C = v_C i_C = 5 \times -3.75 = -18.75\,\text{mW}$ (see Fig. 17(c)). Note that during the time that it is positive ($0.5 < t < 2.5\,\text{ms}$), the average value of i_C is $\overline{i_C} = 1.375\,\text{mA}$ and so the total rise in v_C will be $\Delta v_C = \frac{\overline{i_C}\Delta t}{C} = \frac{1.375 \times 2}{10} = 275\,\text{mV}$ (i.e. ± 138mVaround its mean) which is small compared to its mean value of $5\,\text{V}$; this justifies the assumption that it is constant.

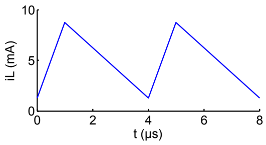

Fig. 17(a)

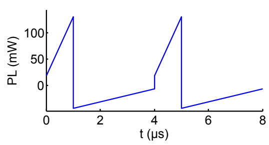

Fig. 17(b)

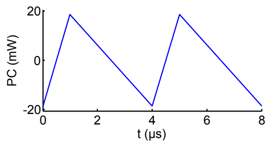

Fig. 17(c)

18. When x changes from low to high, y will change from high to low. The maximum current is $2\,\text{mA}$ so $\frac{dy}{dt} = -\frac{i}{C} = -50\,\text{MV/s}$. So the time to fall from $5\,\text{V}$ to $1.5\,\text{V}$ is $\frac{3.5}{50} \times 10^{-6} = 70\,\text{ns}$.